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# ON EXPONENTIAL TYPE $P$-FUNCTIONS

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Abstract. In this paper, we introduce and study the concept of exponential type $P$-function and establish Hermite-Hadamard's inequalities for this type of functions. In addition, we obtain some new Hermite-Hadamard type inequalities for functions whose first derivative in absolute value is exponential type $P$-function by using Hölder and power-mean integral inequalities. We also extend our initial results to functions of several variables. Next, we point out some applications of our results to give estimates for the approximation error of the integral the function in the trapezoidal formula and for some inequalities related to special means of real numbers.

## 1. Preliminaries

Let $\Psi : I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq \frac{\Psi(r)+\Psi(s)}{2} \tag{1}$$

for all $r, s \in I$ with $r < s$. Both inequalities hold in the reversed direction if the function $\Psi$ is concave. This double inequality is well known as the Hermite-Hadamard inequality [6]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping $\Psi$.

In [5], Dragomir et al. gave the following definition and related Hermite-Hadamard integral inequalities as follow:

**Definition 1.** *A nonnegative function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be $P$-function if the inequality*

$$\Psi(\theta r + (1-\theta)s) \leq \Psi(r) + \Psi(s)$$

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*holds for all* $r, s \in I$ *and* $\theta \in (0,1)$.

**Theorem 2.** *Let* $\Psi \in P(I)$, $r, s \in I$ *with* $r < s$ *and* $\Psi \in L[r,s]$. *Then*

$$\Psi\left(\frac{r+s}{2}\right) \leq \frac{2}{s-r}\int_r^s \Psi(u)du \leq 2\left[\Psi(r) + \Psi(s)\right]. \tag{2}$$

**Definition 3.** *[14] Let* $h : J \to \mathbb{R}$ *be a non-negative function,* $h \neq 0$. *We say that* $\Psi : I \to \mathbb{R}$ *is an* $h$*-convex function, or that* $\Psi$ *belongs to the class* $SX(h, I)$, *if* $\Psi$ *is non-negative and for all* $u, v \in I$, $\theta \in (0,1)$ *we have*

$$\Psi(\theta r + (1-\theta)s) \leq h(\theta)\Psi(r) + h(1-\theta)\Psi(s).$$

*If this inequality is reversed, then* $\Psi$ *is said to be* $h$*-concave, i.e.* $\Psi \in SV(h, I)$. *It is clear that, if we choose* $h(\theta) = \theta$ *and* $h(\theta) = 1$, *then the* $h$*-convexity reduces to convexity and definition of* $P$*-function, respectively.*

Readers can look at [1, 14] for studies on $h$-convexity.

In [11], Kadakal and İşcan gave the following definition and related Hermite-Hadamard integral inequalities as follow:

**Definition 4.** *A non-negative function* $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ *is called exponential type convex function if for every* $r, s \in I$ *and* $\theta \in [0,1]$,

$$\Psi(\theta r + (1-\theta)s) \leq \left(e^{\theta} - 1\right)\Psi(r) + \left(e^{1-\theta} - 1\right)\Psi(s).$$

We note that every nonnegative convex function is exponential type convex function.

**Theorem 5** ( [11]). *Let* $\Psi : [r,s] \to \mathbb{R}$ *be a exponential type convex function. If* $r < s$ *and* $\Psi \in L[r,s]$, *then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{2\left[\sqrt{e} - 1\right]}\Psi\left(\frac{r+s}{2}\right) \leq \frac{1}{s-r}\int_r^s \Psi(u)du \leq (e-2)\left[\Psi(r) + \Psi(s)\right].$$

The main purpose of this paper is to introduce the concept of exponential type $P$-function which is connected with the concepts of $P$-function and exponential type convex function and establish some new Hermite-Hadamard type inequality for this class of functions. In recent years many authors have studied error estimations of Hermite-Hadamard type inequalities; for refinements, counterparts, generalizations, for some related papers see [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13].

## 2. The definition of exponential type $P$-function

In this section, we introduce a new concept, which is called exponential type $P$-function and we give by setting some algebraic properties for the exponential type $P$-function, as follows:

**Definition 6.** *A non-negative function* $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ *is called exponential type* $P$*-function if for every* $r, s \in I$ *and* $\theta \in [0,1]$,

$$\Psi(\theta r + (1-\theta)s) \leq \left(e^{\theta} + e^{1-\theta} - 2\right)\left[\Psi(r) + \Psi(s)\right]. \tag{3}$$

We will denote by $ETP(I)$ the class of all exponential type $P$-functions on interval $I$.

We note that, every exponential type $P$-function is a $h$-convex function with the function $h(\theta) = e^{\theta} + e^{1-\theta} - 2$. Therefore, if $\Psi, \Phi \in ETP(I)$, then

i.) $\Psi + \Phi \in ETP(I)$ *and for* $c \in \mathbb{R}$ *(*$c \geq 0$*)* $c\Psi \in ETP(I)$ *(see [14], Proposition 9).*

ii.) If $\Psi$ and $g$ be a similarly ordered functions on $I$ , then $\Psi.\Phi \in ETP(I)$ *.(see [14], Proposition 10).*

Also, if $\Psi : I \to J$ is a convex and $\Phi \in ETP(J)$ and nondecreasing, then $\Phi \circ \Psi \in ETP(I)$ *(see [14], Theorem 15).*

**Remark 7.** *We note that if* $\Psi$ *is satisfy (3), then* $\Psi$ *is a nonnegative function. Indeed, if we rewrite the inequality (3) for* $\theta = 0$ *and* $r = s$ *then*

$$0 \leq (2e - 3)\, \Psi(r)$$

*for every* $r \in I$*. Thus we have* $\Psi(r) \geq 0$ *for all* $r \in I$*.*

**Proposition 8.** *Every exponential type convex function is also a exponential type* $P$*-function.*

*Proof.* Let $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ be an arbitrary exponential type convex function. Then $\Psi$ is nonnegative and the following inequality holds

$$\Psi\left(\theta r + (1-\theta)\, s\right) \leq \left(e^{\theta} - 1\right) \Psi(r) + \left(e^{1-\theta} - 1\right) \Psi(s)$$

for every $r, s \in I$ and $\theta \in [0,1]$. By $\Psi(r) \leq \Psi(r) + \Psi(s)$ and $\Psi(s) \leq \Psi(r) + \Psi(s)$, we obtain desired result. □

**Proposition 9.** *Every* $P$*-function is also a exponential type* $P$*-function.*

*Proof.* The proof is clear from the following inequalities

$$\theta \leq e^{\theta} - 1 \text{ and } 1 - \theta \leq e^{1-\theta} - 1$$

for all $\theta \in [0,1]$. In this case, we can write

$$1 \leq e^{\theta} + e^{1-\theta} - 2.$$

Therefore, the desired result is obtained. □

We can give the following corollary for every nonnegative convex function is also a $P$-function.

**Corollary 10.** *Every nonnegative convex function is also a exponential type* $P$*-function.*

**Theorem 11.** *If* $\Psi : [r,s] \subset \mathbb{R} \to \mathbb{R}$ *is an exponential type* $P$*-function, then* $\Psi$ *is bounded on* $[r,s]$.

*Proof.* Let $M = \max\{\Psi(r), \Psi(s)\}$. For any $x \in [r,s]$, there exists a $\theta \in [0,1]$ such that $x = \theta r + (1-\theta)s$. Since $\Psi$ is an exponential type $P$-function on $[a,b]$, we have

$$\Psi(x) \leq \left(e^{\theta} + e^{1-\theta} - 2\right)[\Psi(r) + \Psi(s)] \leq 4M(e-1).$$

This shows that $\Psi$ is bounded from above. For any $x \in [r,s]$, there exists a $\theta \in [0,1]$ such that either $x = \frac{r+s}{2} + \theta$ or $x = \frac{r+s}{2} - \theta$. Since it will lose nothing generality we can assume $x = \frac{r+s}{2} + \theta$. Thus we can write

$$\begin{aligned} \Psi\left(\frac{r+s}{2}\right) &= \Psi\left(\frac{1}{2}\left[\frac{r+s}{2} + \theta\right] + \frac{1}{2}\left[\frac{r+s}{2} - \theta\right]\right) \\ &\leq 2(\sqrt{e}-1)\left[\Psi(x) + \Psi\left(\frac{r+s}{2} - \theta\right)\right] \end{aligned}$$

and from here we have

$$\begin{aligned} \Psi(x) &\geq \frac{1}{2(\sqrt{e}-1)}\Psi\left(\frac{r+s}{2}\right) - \Psi\left(\frac{r+s}{2} - \theta\right) \\ &\geq \frac{1}{2(\sqrt{e}-1)}\Psi\left(\frac{r+s}{2}\right) - 4M(e-1) = m. \end{aligned}$$

This completes the proof. □

**Theorem 12.** *Let $s > r$ and $\Psi_\alpha : [r,s] \to \mathbb{R}$ be an arbitrary family of exponential type $P$-function and let $\Psi(x) = \sup_\alpha \Psi_\alpha(x)$. If $J = \{u \in [r,s] : \Psi(u) < \infty\}$ is nonempty, then $J$ is an interval and $\Psi$ is a exponential type $P$-function on $J$.*

*Proof.* Let $\theta \in [0,1]$ and $r, s \in J$ be arbitrary. Then

$$\begin{aligned} & \Psi(\theta r + (1-\theta)s) \\ = \; & \sup_\alpha \Psi_\alpha(\theta r + (1-\theta)s) \\ \leq \; & \sup_\alpha \left\{\left(e^{\theta} + e^{1-\theta} - 2\right)[\Psi_\alpha(r) + \Psi_\alpha(s)]\right\} \\ \leq \; & \left(e^{\theta} + e^{1-\theta} - 2\right)\left[\sup_\alpha \Psi_\alpha(r) + \sup_\alpha \Psi_\alpha(s)\right] \\ = \; & \left(e^{\theta} + e^{1-\theta} - 2\right)[\Psi(r) + \Psi(s)] < \infty. \end{aligned}$$

This shows simultaneously that $J$ is an interval, since it contains every point between any two of its points, and that $\Psi$ is an exponential type $P$-function on $J$. This completes the proof of theorem. □

## 3. Hermite-Hadamard's inequality for exponential type $P$-functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for exponential type $P$-functions. In this section, we will denote by $L[r,s]$ the space of (Lebesgue) integrable functions on $[r,s]$.

**Theorem 13.** *Let $\Psi : [r,s] \to \mathbb{R}$ be a exponential type $P$-function. If $r < s$ and $\Psi \in L[r,s]$, then the following Hermite-Hadamard type inequalities hold:*

$$\frac{1}{4\left[\sqrt{e}-1\right]}\Psi\left(\frac{r+s}{2}\right) \le \frac{1}{s-r}\int_r^s \Psi(u)du \le (2e-4)\left[\Psi(r)+\Psi(s)\right]. \tag{4}$$

*Proof.* Since $\Psi$ is a exponential type $P$-function, we get

$$\begin{aligned} & \Psi\left(\frac{r+s}{2}\right) \\ = \;& \Psi\left(\frac{1}{2}\left[\theta r+(1-\theta)s\right]+\frac{1}{2}\left[\theta s+(1-\theta)r\right]\right) \\ \le \;& 2\left[\sqrt{e}-1\right]\left[\Psi\left(\theta r+(1-\theta)s\right)+\Psi\left(\theta s+(1-\theta)r\right)\right]. \end{aligned}$$

By taking integral in the last inequality with respect to $\theta \in [0,1]$, we deduce that

$$\Psi\left(\frac{r+s}{2}\right) \le \frac{4}{s-r}\left[\sqrt{e}-1\right]\int_r^s \Psi(u)du$$

By using the property of the exponential type $P$-function of $\Psi$, if the variable is changed as $u = \theta r + (1-\theta)s$, then

$$\begin{aligned} \frac{1}{s-r}\int_r^s \Psi(u)du &= \int_0^1 \Psi\left(\theta r+(1-\theta)s\right)d\theta \\ &\le \left[\Psi(r)+\Psi(s)\right]\int_0^1 \left(e^{\theta}+e^{1-\theta}-2\right)d\theta \\ &= (2e-4)\left[\Psi(r)+\Psi(s)\right]. \end{aligned}$$

This completes the proof of theorem. □

**Theorem 14.** *Let $r < s$ and $\Psi : [r,s] \to \mathbb{R}$ be a exponential type $P$-function. If $\Psi$ is symmetric with respect to $\frac{r+s}{2}$ (i.e. $\Psi(x) = \Psi(r+s-x)$ for all $x \in [r,s]$), then the following inequalities hold:*

$$\frac{1}{4\left[\sqrt{e}-1\right]}\Psi\left(\frac{r+s}{2}\right) \le \Psi(x) \le (e-1)\left[\Psi(r)+\Psi(s)\right]$$

*for all $x \in [r,s]$.*

*Proof.* Let $x \in [r,s]$ be arbitrary point. Since $e^{\theta}+e^{1-\theta}-2 \le e-1$ for all $\theta \in [0,1]$, we get

$$\begin{aligned} \Psi(x) &= \Psi\left(\frac{x-r}{s-r}s+\frac{s-x}{s-r}r\right) \\ &\le \left(e^{\frac{x-r}{s-r}}+e^{\frac{s-x}{s-r}}-2\right)\left[\Psi(r)+\Psi(s)\right] \\ &\le (e-1)\left[\Psi(r)+\Psi(s)\right] \end{aligned}$$

and

$$
\begin{aligned}
& \Psi\left(\frac{r+s}{2}\right) \\
= \; & \Psi\left(\frac{1}{2}x+\frac{1}{2}[r+s-x]\right) \\
\leq \; & 2\left[\sqrt{e}-1\right]\left[\Psi(x)+\Psi(r+s-x)\right] \\
= \; & 4\left[\sqrt{e}-1\right]\Psi(x).
\end{aligned}
$$

This completes the proof. □

## 4. Some new inequalities for exponential type $P$-functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value is exponential type $P$-function. Dragomir and Agarwal [4] used the following lemma:

**Lemma 15.** *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $r, s \in I^{\circ}$ with $r < s$. If $f' \in L[r, s]$, then*

$$\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx=\frac{s-r}{2}\int_0^1(1-2t)f'(tr+(1-t)s)\,dt.$$

**Theorem 16.** *Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $r, s \in I^{\circ}$ with $r < s$ and assume that $f' \in L[r, s]$. If $|f'|$ is exponential type $P$-function on interval $[r, s]$, then the following inequality holds*

$$\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right|\leq(s-r)\left[8\sqrt{e}-2e-7\right]A\left(\left|f'(r)\right|,\left|f'(s)\right|\right), \tag{5}$$

*where $A$ is the arithmetic mean.*

*Proof.* Using Lemma 15 and the inequality

$$\left|f'(tr+(1-t)s)\right|\leq\left(e^t+e^{1-t}-2\right)\left[|f'(r)|+|f'(s)|\right],$$

we get

$$
\begin{aligned}
& \left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right| \\
\leq \; & \frac{s-r}{2}\left[|f'(r)|+|f'(s)|\right]\int_0^1|1-2t|\left(e^t+e^{1-t}-2\right)dt \\
= \; & (b-a)\left[8\sqrt{e}-2e-7\right]A\left(\left|f'(r)\right|,\left|f'(s)\right|\right)
\end{aligned}
$$

where

$$\int_0^1|1-2t|\left(e^t+e^{1-t}-2\right)dt=8\sqrt{e}-2e-7$$

This completes the proof of theorem. □

**Theorem 17.** *Let $f : I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $r, s \in I^{\circ}$ with $r < s$ and assume that $f' \in L[r, s]$. If $|f'|^q$, $q > 1$, is an exponential type $P$-function on interval $[r, s]$, then the following inequality holds*

$$\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \tag{6}$$
$$\leq \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(4e-8)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(r)|^q, |f'(s)|^q\right),$$

*where $\frac{1}{p}+\frac{1}{q}=1$ and $A$ is the arithmetic mean.*

*Proof.* Using Lemma 15, Hölder's integral inequality and the following inequality

$$|f'(tr+(1-t)s)| \leq \left(e^t+e^{1-t}-2\right)[|f'(r)|+|f'(s)|]$$

which is the exponential type $P$-function of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \\
\leq\ & \frac{s-r}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tr+(1-t)s)|^q\,dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left[|f'(r)|^q+|f'(s)|^q\right]\int_0^1\left(e^t+e^{1-t}-2\right)dt\right)^{\frac{1}{q}} \\
=\ & \frac{s-r}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(4e-8)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(r)|^q, |f'(s)|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

**Theorem 18.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $r, s \in I^{\circ}$ with $r < s$ and assume that $f' \in L[r, s]$. If $|f'|^q, q \geq 1$, is an exponential type $P$-function on the interval $[r, s]$, then the following inequality holds*

$$\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \tag{7}$$
$$\leq \frac{s-r}{2^{2-\frac{2}{q}}}\left(\left[8\sqrt{e}-2e-7\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(r)|^q, |f'(s)|^q\right).$$

*Proof.* From Lemma 15, well known power-mean integral inequality and the property of exponential type $P$-function of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(r)+f(s)}{2} - \frac{1}{s-r}\int_r^s f(x)dx\right| \\
\leq\ & \frac{s-r}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,|f'(tr+(1-t)s)|^q\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{s-r}{2^{2-\frac{1}{q}}}\left(\left[|f'(r)|^q+|f'(s)|^q\right]\int_0^1|1-2t|\left(e^t+e^{1-t}-2\right)dt\right)^{\frac{1}{q}}\\
&= \frac{s-r}{2^{2-\frac{2}{q}}}\left(\left[8\sqrt{e}-2e-7\right]\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|f'(r)\right|^q,\left|f'(s)\right|^q\right).
\end{aligned}$$

This completes the proof of theorem. □

**Corollary 19.** *Under the assumption of Theorem 18, If we take $q=1$ in the inequality (7), then we get the following inequality:*

$$\left|\frac{f(r)+f(s)}{2}-\frac{1}{s-r}\int_r^s f(x)dx\right|\leq(s-r)\left[8\sqrt{e}-2e-7\right]A\left(\left|f'(r)\right|,\left|f'(s)\right|\right).$$

*This inequality coincides with the inequality (5).*

## 5. An extention of Theorem 16

In this section we will denote by $K$ an open and convex set of $\mathbb{R}^n$ $(n\geq 1)$.

We say that a function $f:K\to\mathbb{R}$ is exponential type $P$-function on $A$ if

$$f\left(tx+(1-t)y\right)\leq\left(e^t+e^{1-t}-2\right)\left[f(x)+f(y)\right]$$

for all $x,y\in K$ and $t\in[0,1]$.

**Lemma 20.** *Let $f:K\to\mathbb{R}$ be a function. Then $f$ is exponential type $P$-function on $K$ if and only if for all $x,y\in K$ the function $\Phi:[0,1]\to\mathbb{R}$, $\Phi(t)=f\left(tx+(1-t)y\right)$ is exponential type $P$-function on $[0,1]$.*

*Proof.* "$\Longleftarrow$"Let $x,y\in K$ be fixed. Assume that $\Phi:[0,1]\to\mathbb{R}$, $\Phi(t)=f\left(tx+(1-t)y\right)$ is exponential type $P$-function on $[0,1]$.

Let $t\in[0,1]$ be arbitrary, but fixed. Clearly, $t=(1-t).0+t.1$ and thus,

$$\begin{aligned}
f\left(tx+(1-t)y\right) &= \Phi(t)=\Phi(t.1+(1-t).0)\\
&\leq \left(e^t+e^{1-t}-2\right)\left[\Phi(0)+\Phi(1)\right]\\
&= \left(e^t+e^{1-t}-2\right)\left[f(x)+f(y)\right].
\end{aligned}$$

It follows that $f$ is exponential type $P$-function on $K$.

"$\Longrightarrow$" Assume that $f$ is exponential type $P$-function on $K$. Let $x,y\in K$ be fixed and define $\Phi:[0,1]\to\mathbb{R}$, $\Phi(t)=f\left(tx+(1-t)y\right)$. We must show that $\Phi$ is exponential type $P$-function on $[0,1]$.

Let $u_1,u_2\in[0,1]$ and $t\in[0,1]$. Then

$$\begin{aligned}
\Phi(tu_1+(1-t)u_2) &= f\left((tu_1+(1-t)u_2)x+(1-tu_1-(1-t)u_2)y\right)\\
&= f\left(t(u_1x+(1-u_1)y+(1-t)(u_2x+(1-u_2)y\right)\\
&\leq \left(e^t+e^{1-t}-2\right)\left[f(u_1x+(1-u_1)y)+f(u_2x+(1-u_2)y)\right]\\
&= \left(e^t+e^{1-t}-2\right)\left[\Phi(u_1)+\Phi(u_2)\right].
\end{aligned}$$

We deduce that $\Phi$ is exponential type $P$-function on $[0,1]$.

The proof of Lemma 20 is complete. □

Using the above lemma we will prove an extension of Theorem 16 to functions of several variables.

**Proposition 21.** *Assume $f : K \subseteq \mathbb{R}^n \to \mathbb{R}^+$ is a exponential type $P$-function on $K$. Then for any $x, y \in K$ and any $u, v \in (0,1)$ with $u < v$ the following inequality holds*

$$\begin{aligned} & \left| \frac{1}{2} \int_0^u f\left(sx + (1-s)y\right) ds + \frac{1}{2} \int_0^v f\left(sx + (1-s)y\right) ds \right. \\ & \left. - \frac{1}{v-u} \int_u^v \left( \int_0^\theta f\left(sx + (1-s)y\right) ds \right) d\theta \right| \\ \leq \quad & (v-u) \left[ 8\sqrt{e} - 2e - 7 \right] A\left( f\left(ux + (1-u)y\right), f\left(vx + (1-v)y\right) \right). \end{aligned} \tag{8}$$

*Proof.* We fix $x, y \in K$ and $u, v \in (0,1)$ with $u < v$. Since $f$ is exponential type $P$-function, by Lemma 20 it follows that the function

$$\Phi : [0,1] \to \mathbb{R}, \Phi(t) = f\left(tx + (1-t)y\right),$$

is exponential type $P$-function on $[0,1]$.

Define $\Psi : [0,1] \to \mathbb{R}$,

$$\Psi(t) = \int_0^t \Phi(s) ds = \int_0^t f\left(sx + (1-s)y\right) ds.$$

Obviously, $\Psi'(t) = \Phi(t)$ for all $t \in (0,1)$.

Since $f(K) \subseteq \mathbb{R}^+$ it results that $\Phi \geq 0$ on $[0,1]$ and thus, $\Psi' \geq 0$ on $(0,1)$.

Applying Theorem 16 to the function $\Psi$ we obtain

$$\left| \frac{\Psi(u) + \Psi(v)}{2} - \frac{1}{v-u} \int_u^v \Psi(\theta) d\theta \right| \leq (v-u) \left[ 8\sqrt{e} - 2e - 7 \right] A\left(\Psi'(u), \Psi'(v)\right),$$

and we deduce that relation (8) holds true. □

**Remark 22.** *We point out that a similar result as those of Proposition 21 can be stated by using Theorem 17 and Theorem 18 .*

## 6. Applications to the trapezoidal formula

Assume $\wp$ is a division of the interval $[r, s]$ such that

$$\wp : \quad r = x_0 < x_1 < \ldots < x_{n-1} < x_n = s.$$

For a given function $f : [r, s] \to \mathbb{R}$ we consider the trapezoidal formula

$$T\left(f, \wp\right) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \left(x_{i+1} - x_i\right).$$

It is well known that if $f$ is twice differentiable on $(r, s)$ and $M = \sup_{x \in (r,s)} |f''(x)| < \infty$ then

$$\int_r^s f(x) dx = T\left(f, \wp\right) + E\left(f, \wp\right),$$

where $E(f,\wp)$ is the approximation error of the integral $\int_r^s f(x)dx$ by the trapezoidal formula and satisfies,

$$|E(f,\wp)| \leq \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{9}$$

Clearly, if the function $f$ is not twice differentiable or the second derivative is not bounded on $(r,s)$, then (9) does not hold true. In that context, the following results are important in order to obtain some estimates of $E(f,\wp)$.

**Proposition 23.** *Assume $r,s\in\mathbb{R}$ with $r<s$ and $f:[r,s]\to\mathbb{R}$ is a differentiable function on $(r,s)$. If $|f'|$ is exponential type $P$-function on $[r,s]$ then for each division $\wp$ of the interval $[r,s]$ we have,*

$$|E(f,\wp)| \leq 2\left[8\sqrt{e}-2e-7\right](e-1)\,A(|f'(r)|,|f'(s)|)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{10}$$

*Proof.* We apply Theorem 16 on the sub-intervals $[x_i,x_{i+1}]$, $i=0,1,...,n-1$ given by the division $\wp$. Adding from $i=0$ to $i=n-1$ we deduce

$$\left|T(f,\wp)-\int_r^s f(x)dx\right| \leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[8\sqrt{e}-2e-7\right]A\left(|f'(x_i)|,|f'(x_{i+1})|\right). \tag{11}$$

On the other hand, for each $x_i\in[r,s]$ there exists $t_i\in[0,1]$ such that $x_i=t_ir+(1-t_i)s$. Since $|f'|$ is exponential type $P$-function and $e^t+e^{1-t}-2\leq e-1$ for all $t\in[0,1]$, we deduce

$$|f'(x_i)| \leq \left(e^{t_i}+e^{1-t_i}-2\right)[f(r)+f(s)] \leq 2(e-1)\,A(|f'(r)|,|f'(s)|) \tag{12}$$

for each $i=0,1,...,n-1$. Relations (11) and (12) imply that relation (10) holds true. Thus, Proposition 26 is completely proved. □

A similar method as that used in the proof of Proposition 23 but based on Theorem 17 and Theorem 18 shows that the following results are valid.

**Proposition 24.** *Assume $r,s\in\mathbb{R}$ with $r<s$ and $f:[r,s]\to\mathbb{R}$ is a differentiable function on $(r,s)$. If $|f'|^q$, $q>1$, is an exponential type $P$-function on interval $[r,s]$, then for each division $\wp$ of the interval $[r,s]$ we have,*

$$|E(f,\wp)| \leq (e-1)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(4e-8)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|f'(r)|^q,|f'(s)|^q\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2,$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

**Proposition 25.** *Assume $r,s\in\mathbb{R}$ with $r<s$ and $f:[r,s]\to\mathbb{R}$ is a differentiable function on $(r,s)$. If $|f'|^q$, $q>1$, is an exponential type $P$-function on interval*

$[r,s]$, *then for each division* $\wp$ *of the interval* $[r,s]$ *we have,*

$$|E(f,\wp)| \le \frac{e-1}{2^{1-\frac{2}{q}}}\left(\left[8\sqrt{e}-2e-7\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(|f'(r)|^q, |f'(s)|^q\right)\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2 .$$

## 7. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers $r,s$ with $s>r$:

1. The arithmetic mean

$$A := A(r,s) = \frac{r+s}{2}, \quad r,s \ge 0.$$

2. The geometric mean

$$G := G(r,s) = \sqrt{rs}, \quad r,s \ge 0.$$

3. The harmonic mean

$$H := H(r,s) = \frac{2rs}{r+s}, \quad r,s > 0.$$

4. The logarithmic mean

$$L := L(r,s) = \begin{cases} \frac{s-r}{\ln s - \ln r}, & r \ne s \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

5. The $p$-logarithmic mean

$$L_p := L_p(r,s) = \begin{cases} \left(\frac{s^{p+1}-r^{p+1}}{(p+1)(s-r)}\right)^{\frac{1}{p}}, & r \ne s, p \in \mathbb{R}\setminus\{-1,0\} \\ r, & r = s \end{cases} ; \quad r,s > 0.$$

6.The identric mean

$$I := I(r,s) = \frac{1}{e}\left(\frac{s^s}{r^r}\right)^{\frac{1}{s-r}}, \quad r,s > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that $L_p$ is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

**Proposition 26.** *Let* $r,s \in [0,\infty)$ *with* $r < s$ *and* $n \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. *Then, the following inequalities are obtained:*

$$\frac{1}{4\left[\sqrt{e}-1\right]}A^n(r,s) \le L_n^n(r,s) \le 4(e-2)\,A(r^n,s^n).$$

*Proof.* The assertion follows from the inequalities (4) for the function

$$f(x) = x^n, \quad x \in [0, \infty).$$

□

**Proposition 27.** *Let $r, s \in (0, \infty)$ with $r < s$ . Then, the following inequalities are obtained:*

$$\frac{1}{4\left[\sqrt{e}-1\right]} A^{-1}(r,s) \leq L^{-1}(r,s) \leq 4\,(e-2)\, H^{-1}(r,s).$$

*Proof.* The assertion follows from the inequalities (4) for the function

$$f(x) = x^{-1}, \quad x \in (0, \infty).$$

□

**Proposition 28.** *Let $r, s \in (0, 1]$ with $r < s$. Then, the following inequalities are obtained:*

$$4\,(e-2) \ln G(r,s) \leq \ln I(r,s) \leq \frac{1}{4\left[\sqrt{e}-1\right]} \ln A(r,s).$$

*Proof.* The assertion follows from the inequalities (4) for the function

$$f(x) = -\ln x, \quad x \in (0, 1].$$

□

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