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Certain ranks of some ideals in symmetric inverse semigroups contains S_n or A_n

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Abstract

Let I_n, S_n and A_n be the symmetric inverse semigroup, the symmetric group and the alternating group on $X_n = \{1, \dots, n\}$, for $n \geq 2$, respectively. Also let $I_{n,r}$ be the subsemigroup consists of all partial injective maps with height less than or equal to r for $1 \leq r \leq n-1$, and let $SI_{n,r} = I_{n,r} \cup S_n$ and $AI_{n,r} = I_{n,r} \cup A_n$. A non-idempotent element whose square is an idempotent is called a quasi-idempotent. In this paper we obtain the rank and the quasi-idempotent rank of $SI_{n,r}$ (of $AI_{n,r}$). Also we obtain the relative rank and the relative quasi-idempotent rank of $SI_{n,r}$ modulo S_n (of $AI_{n,r}$ modulo A_n).

Keywords: *Symmetric inverse semigroup, quasi-idempotent, rank.*

Simetrik inverse yarıgrubun S_n veya A_n i içeren bazı ideallerinin rankları

Öz

$n \geq 2$ için I_n, S_n ve A_n, sırasıyla, $X_n = \{1, \dots, n\}$ üzerindeki simetrik inverse yarıgrup, simetrik grup ve alterne grup olsun. Ayrıca, $1 \leq r \leq n-1$ için $I_{n,r}$, yüksekliği en fazla r olan tüm kısmi bire-bir dönüşümlerden oluşan altyarıgrup, $SI_{n,r} = I_{n,r} \cup S_n$ ve $AI_{n,r} = I_{n,r} \cup A_n$ olsun. Karesi idempotent olan fakat kendisi idempotent olmayan bir elemana quasi-idempotent denir. Bu calışmada $SI_{n,r}$ ($AI_{n,r}$) nin rankını elde ettik. Ayrıca, modulo S_n e göre $SI_{n,r}$ nin (modulo A_n e göre $AI_{n,r}$ nin) ilişkili rankını ve quasi-ilişkili rankını elde ettik.

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Anahtar kelimeler: *Simetrik inverse yarıgrup, quasi-idempotent, rank.*

1. Introduction

For $n \in \mathbb{Z}^+$ let $X_n = \{1, \dots, n\}$. Also let I_n be the semigroup of all partial injective maps on X_n, called *symmetric inverse semigroup*, let S_n be the group of all permutations on X_n, called *symmetric group*, and let A_n be the group of all even permutations on X_n, called *alternating group*. Clearly, $A_n \leq S_n \leq I_n$. For escape from triviality throughout this paper we consider the case $n \geq 2$ unless otherwise stated. It is well known that I_n is an inverse semigroup and that every finite inverse semigroup S is embeddable in I_n for a suitable $n \in \mathbb{N}$. Thus, investigating the structure of I_n is an important research topic in inverse semigroup theory, like as investigating the structure of symmetric group S_n in group theory.

An element $\alpha \in I_n$ is called an *idempotent* if $\alpha^2 = \alpha$, and, as introduced in [6] that an element $\alpha \in I_n$ is called a *quasi-idempotent* if $\alpha \neq \alpha^2 = \alpha^4$, that is, α is a non-idempotent element whose square is an idempotent. We denote the set of all quasi-idempotents in any subset U of any semigroup by $Q(U)$.

Let S be a semigroup, and let A be a non-empty subset of S. Then the subsemigroup generated by A is defined as the smallest subsemigroup of S containing A and denoted by $\langle A \rangle$. If there exists a non-empty subset A of S such that $S = \langle A \rangle$, then A is called a *generating set of* S. Also, the *rank* of a semigroup S is defined by

$$\operatorname{rank}(S) = \min\{\, |A| : \langle A \rangle = S, |A| < \infty\}. \tag{1}$$

In particular, if there exists a generating set A of S consists of some quasi-idempotents, then A is called *quasi-idempotent generating set* of S and the *quasi-idempotent rank* of S is defined by

$$\operatorname{qrank}\,(S) = \min\{\, |A| : \langle A \rangle = S, A \subseteq Q(S), |A| < \infty\}. \tag{2}$$

For a fixed subset U of a semigroup S, if there exists a non-empty subset A of S such that $\langle A \cup U \rangle = S$, then A is called a *relative generating set* of S *modulo* U and the *relative rank* of S modulo U is defined by

$$\operatorname{rerank}(S : U) = \min\{|A| : \langle A \cup U \rangle = S, |A| < \infty\}. \tag{3}$$

Similarly, if there exists a non-empty subset A of $Q(S)$ such that $\langle A \cup U \rangle = S$, then A is called a *relative quasi-idempotent generating set* of S *modulo* U, and *relative quasi-idempotent rank* of S modulo U is defined by

$$\operatorname{reqrank}\,(S : U) = \min\{|A| : \langle A \cup U \rangle = S, A \subseteq Q(S), |A| < \infty\}. \tag{4}$$

For more studies about various ranks of a semigroup, we refer [2, 5, 9, 10] for example.
The *height*, *fix* and *shift* of $\alpha \in I_n$ are defined by

$$\mathrm{h}(\alpha) = |\mathrm{im}(\alpha)| \quad (5)$$
$$\mathrm{fix}\,(\alpha) = \{x \in \mathrm{dom}(\alpha) : x\alpha = x\} \quad \textit{and} \quad (6)$$
$$\mathrm{shift}\,(\alpha) = \{x \in \mathrm{dom}(\alpha) : x\alpha \neq x\} = \mathrm{dom}(\alpha) \backslash \mathrm{fix}\,(\alpha), \quad (7)$$

respectively. A permutation $\alpha \in S_n$ with shift $(\alpha) = \{a_1, \ldots, a_k\}$ $(2 \leq k \leq n)$ is called a *cycle* of size k (k −cycle) and denoted by $\alpha = (a_1 \ldots a_k)$ if

$$a_i\alpha = a_{i+1} \quad (1 \leq i \leq k-1) \quad \textit{and} \quad a_k\alpha = a_1. \quad (8)$$

In particular, a 2 −cycle (a_1a_2) is called a *transposition*. The identity permutation ε on X_n is expressible as (a), for any $1 \leq a \leq n$, and (a) is called a 1-cycle. Also, a map $\alpha \in I_n$ with $\mathrm{dom}(\alpha) = X_n \backslash \{a_k\}$ and shift $(\alpha) = \{a_1, \ldots, a_{k-1}\}$ $(2 \leq k \leq n)$ is called a *chain* of size k (k −chain) and denoted by $[a_1 \ldots a_k]$ if

$$a_i\alpha = a_{i+1} \quad (1 \leq i \leq k-1). \quad (9)$$

Moreover, a map $\alpha \in I_n$ with $\mathrm{dom}(\alpha) = \mathrm{fix}\,(\alpha) = X_n \backslash \{a_k\}$ called a 1-*chain* and denoted by $[a_k]$. Two cycles $(a_1 \ldots a_k)$ and $(b_1 \ldots b_t)$ (and similarly two chains $[a_1 \ldots a_k]$ and $[b_1 \ldots b_t]$, or a cycle $(a_1 \ldots a_k)$ and a chain $[b_1 \ldots b_t]$), for $1 \leq k, t \leq n$, are said to be *disjoint* if the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_t\}$ are disjoint.

It is well known that every map in I_n can be written as a product of disjoint cycles (1-cycles are neglected in general) and chains, and every permutation in S_n can be written as a product of disjoint cycles (1-cycles are neglected in general), more particularly, as a product of transpositions. Moreover, it is also well known that $S_2 = \langle (12) \rangle$, $S_3 = \langle (13), (23) \rangle$, $S_n = \langle (12), (12 \ldots n) \rangle$ for $n \geq 3$, and that $A_3 = \langle (123) \rangle$ and A_n is generated by two elements:

$$(123) \quad \textit{and} \quad \begin{cases} (12\ldots n) & \textit{if} \;\; n \;\; \textit{is odd} \\ (23\ldots n) & \textit{if} \;\; n \;\; \textit{is even} \end{cases} \quad (10)$$

for $n \geq 4$. Furthermore,

$$\mathrm{rank}(S_n) = \begin{cases} 1 & \textit{for} \;\; n = 2 \\ 2 & \textit{for} \;\; n \geq 3 \end{cases} \quad \textit{and} \quad (11)$$

$$\mathrm{rank}(A_n) = \begin{cases} 1 & \textit{for} \;\; n = 3 \\ 2 & \textit{for} \;\; n \geq 4 \end{cases}. \quad (12)$$

(For unexplained terms in semigroup theory see for example [4, 7].)

Let P_n and T_n be the partial transformations semigroup and the full transformations semigroup on X_n, respectively. Moreover, let $PK(n,r) = \{\alpha \in P_n : |\mathrm{im}(\alpha)| \leq r\}$ and $K(n,r) = \{\alpha \in T_n : |\mathrm{im}(\alpha)| \leq r\}$ for $1 \leq r \leq n-1$. Yiğit et al. showed in [9] that

$$\text{rerank}(T_{n,r}: S_n) = p_r(n) \quad (as\ shown\ in\ [1,8]\ before)\,, \tag{13}$$

$$\text{rerank}(PT_{n,r}: S_n) = \sum_{s=0}^{n-r} p_r(n-s), \tag{14}$$

$$\text{rerank}(A_{n,r}: A_n) = p_r(n), \tag{15}$$

$$\text{rerank}(PA_{n,r}: A_n) = \sum_{s=0}^{n-r} p_r(n-s) \tag{16}$$

for $1 \leq r \leq n-1$, where

$$T_{n,r} = K_{n,r} \cup S_n, \quad PT_{n,r} = PK_{n,r} \cup S_n, \tag{17}$$

$$A_{n,r} = K_{n,r} \cup A_n, \quad PA_{n,r} = PK_{n,r} \cup A_n, \tag{18}$$

and also $p_r(n)$ is the cardinality of the set $P_r(n)$, the set of all integer solutions of the equation

$$x_1 + x_2 + \cdots + x_r = n \quad with \quad x_1 \geq x_2 \geq \cdots \geq x_r \geq 1. \tag{19}$$

Recall from [6, Lemma 2.1] that a non-idempotent map $\alpha \in I_n$ is a quasi-idempotent if and only if all its orbits are of size at most 2, and so, $\alpha \in Q(I_n)$ if and only if α can be written as a product of some disjoint 1-cycles (1-cycles are neglected in general), 1-chains and at least one 2-cycle and/or 2-chain. In particular, it is easy to see that $\alpha \in Q(S_n)$ if and only if α can be written as a product of some disjoint 2-cycles, and that $\alpha \in Q(A_n)$ if and only if α can be written as a product of positive even number of disjoint 2-cycles. In addition to these results recently it is shown in [3] that

$$\text{qrank}\,(S_n) = \begin{cases} 1 & for \quad n = 2 \\ 2 & for \quad n = 3, \\ 3 & for \quad n \geq 4 \end{cases} \tag{20}$$

$$\text{qrank}\,(I_n) = \begin{cases} 2 & for \quad n = 2 \\ 3 & for \quad n = 3 \\ 4 & for \quad n \geq 4 \end{cases} \tag{21}$$

and $\text{qrank}\,(A_n) = 3$ for $n \geq 5$. Now let

$$I_{n,r} = \{\alpha \in I_n : |\text{im}(\alpha)| \leq r\} \tag{22}$$

$$SI_{n,r} = I_{n,r} \cup S_n \tag{23}$$

for $n \geq 2$ and $1 \leq r \leq n-1$, and let

$$AI_{n,r} = I_{n,r} \cup A_n \tag{24}$$

for $n \geq 3$ and $1 \leq r \leq n-1$. Clearly each one of the sets $I_{n,r}$, $SI_{n,r}$ and $AI_{n,r}$ is an ideal of I_n. Moreover, $I_{n,n-1} = I_n \backslash S_n$ and so $SI_{n,n-1} = I_n$.

In this paper we obtain the rank and the quasi-idempotent rank of $SI_{n,r}$ (of $AI_{n,r}$), and then we immediately obtain the relative rank and the relative quasi-idempotent rank of $SI_{n,r}$ modulo S_n (of $AI_{n,r}$ modulo A_n).

2. Certain ranks of $SI_{n,r}$

For any α, β in $I_{n,r}$ it is easy to see that

$$
\begin{aligned}
&(\alpha,\beta) \in \mathcal{L} \Leftrightarrow \mathrm{im}(\alpha) = \mathrm{im}(\beta)\\
&(\alpha,\beta) \in \mathcal{R} \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta)\\
&(\alpha,\beta) \in \mathcal{D} \Leftrightarrow \mathrm{h}(\alpha) = \mathrm{h}(\beta)\\
&(\alpha,\beta) \in \mathcal{H} \Leftrightarrow \mathrm{dom}(\alpha) = \mathrm{dom}(\beta) \;\; and \;\; \mathrm{im}(\alpha) = \mathrm{im}(\beta)
\end{aligned} \tag{25}
$$

where $\mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$ denotes the Green's equivalences. Hence, there exist $r+1$ $\mathcal{D}$-classes in $I_{n,r}$ as follows:

$$D_k = \{\alpha \in I_{n,r} : \mathrm{h}(\alpha) = k\} \;\; for \;\; 0 \le k \le r. \tag{26}$$

Let $\alpha \in D_k$ with $\mathrm{dom}(\alpha) = \{a_1 < \cdots < a_k\}$ $(1 \le k \le r-1)$. Then, as usual, α can be written in the following tabular form:

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k & X_n \backslash \mathrm{dom}(\alpha) \\ a_1\alpha & \cdots & a_k\alpha & - \end{pmatrix} \; (shortly \;\; \alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ a_1\alpha & \cdots & a_k\alpha \end{pmatrix}). \tag{27}$$

Since $1 \le k \le r-1 \le n-2$, there exist two distinct elements $a, a' \in X_n \backslash \{a_1, \dots, a_k\}$ and there exists $b \in X_n \backslash \{a_1\alpha, \dots, a_k\alpha\}$. Then consider the maps

$$\beta = \begin{pmatrix} a_1 & \cdots & a_k & a \\ a_1 & \cdots & a_k & a \end{pmatrix} \;\; and \tag{28}$$

$$\gamma = \begin{pmatrix} a_1 & \cdots & a_k & a' \\ a_1\alpha & \cdots & a_k\alpha & b \end{pmatrix}. \tag{29}$$

Then we have $\beta, \gamma \in D_{k+1}$ and $\alpha = \beta\gamma$, that is $D_k \subseteq \langle D_{k+1} \rangle$. Thereby, $I_{n,r} = \langle D_r \rangle$. Furthermore, it is easy to see that a non-empty subset A of $I_{n,r}$ is a generating set of $I_{n,r}$ if and only if $D_r \subseteq \langle A \rangle$ for $1 \le r \le n-1$. Moreover, it is well known that $\mathrm{h}(\rho\sigma) \le \min\{\mathrm{h}(\rho), \mathrm{h}(\sigma)\}$ for $\rho, \sigma \in I_n$, and so we may consider only the subsets of D_r to generate $I_{n,r}$.

Theorem 2.1. *For* $1 \le r \le n-1$ $D_r \subseteq \langle S_n \cup \{\xi\} \rangle$ *where*

$$\xi = \begin{cases} [12] & for \;\; n = 2 \;\; and \;\; r = 1 \\ [12][3] \cdots [n] & for \;\; n \geq 3 \;\; and \;\; r = 1 \\ (1\,2)[r+1] \cdots [n] & for \;\; n \geq 3 \;\; and \;\; 2 \leq r \leq n-1. \end{cases} \tag{30}$$

Proof. Let $\alpha \in D_r$ for $n \geq 2$ and $1 \leq r \leq n-1$ and suppose that $\mathrm{dom}(\alpha) = \{a_1, \dots, a_r\}$, $X_n \backslash \mathrm{dom}(\alpha) = \{a_{r+1}, \dots, a_n\}$, and that $X_n \backslash \mathrm{im}(\alpha) = \{b_1, \dots, b_{n-r}\}$. Then we have $\alpha = \beta\xi\gamma$ where

$$\beta = \begin{pmatrix} a_1 & \cdots & a_\mathrm{r} & a_{\mathrm{r}+1} & \cdots & a_\mathrm{n} \\ 1 & \cdots & r & r+1 & \cdots & n \end{pmatrix} \in S_n \tag{31}$$

for $1 \leq r \leq n-1$;

$$\gamma = \begin{cases} \begin{pmatrix} 1 & 2 \\ b_1 & a_1\alpha \end{pmatrix} \in S_2 & for \;\; n = 2 \;\; and \;\; r = 1 \\ \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & a_1\alpha & b_2 & \cdots & b_{n-1} \end{pmatrix} \in S_n & for \;\; n \geq 3 \;\; and \;\; r = 1 \end{cases} \tag{32}$$

and

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & \cdots & r & r+1 & \cdots & n \\ a_2\alpha & a_1\alpha & a_3\alpha & \cdots & a_r\alpha & b_1 & \cdots & b_{n-r} \end{pmatrix} \in S_n \tag{33}$$

for $n \geq 3$ and $2 \leq r \leq n-1$. ∎

Corollary 2.2. *For* $1 \leq r \leq n-1$ $SI_{n,r} = \langle (12), (12 \dots n), \xi \rangle$ *where* $\xi \in D_r$ *is the map defined in Theorem 2.1.*

Proof. The result follows from the facts $I_{n,r} = \langle D_r \rangle$, $D_r \subseteq \langle S_n \cup \{\xi\} \rangle$ and $S_n = \langle (12), (12 \dots n) \rangle$. ∎

Recall the following well-known property: Let S be a finite semigroup and let T be a subsemigroup of S such that $S \backslash T$ is an ideal of S. It is well-known that if $S = \langle A \rangle$, for any $\emptyset \neq A \subseteq S$, then $T = \langle T \cap A \rangle$, and so any generating set of S must contain at least one extra element in addition to any generating set of T. Therefore, $\mathrm{rank}(S) \geq \mathrm{rank}(T) + 1$. Similarly, $\mathrm{qrank}\,(S) \geq \mathrm{qrank}\,(T) + 1$ when S and T are generated by their own quasi-idempotents.

Corollary 2.3. *For* $n \geq 2$ *and* $1 \leq r \leq n-1$ $rank(SI_{n,r}) = \begin{cases} 2, & n = 2 \\ 3, & n \geq 3 \end{cases}$.

Proof. Clearly $SI_{n,r} \backslash S_n = I_{n,r}$ is an ideal of $SI_{n,r}$, and so $\mathrm{rank}(SI_{n,r}) \geq \mathrm{rank}(S_n) + 1$. Then the result follows from Corollary 2.2 since $\mathrm{rank}(S_2) = 1$ and $\mathrm{rank}(S_n) = 2$ for $n \geq 3$. ∎

As in [3], for any m-tuple $(b_1, b_2, \dots, b_m)$ $(2 \le m \le n)$ let

$$[[b_1, \dots, b_m]] = \begin{cases} (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m}{2}} b_{\frac{m}{2}+1}) & if\ m\ is\ an\ even\ number \\ (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-1}{2}} b_{\frac{m+3}{2}}) & if\ m\ is\ an\ odd\ number \end{cases} \quad (34)$$

where $(b_i\, b_j)$ denotes a 2-cycle for $1 \le i, j \le k$, also let $\sigma, \rho \in Q(S_n)$ be the maps with one of the following n-many forms:

- $\sigma = [[1, \dots, k+1]][[k+2, \dots, n]]$,
  $\rho = [[1, \dots, k+2]]\ [[k+3, \dots, n]]$ $(1 \le k \le n-4\ \ and\ \ n \ge 5)$;
- $\sigma = [[1, \dots, n-2]](n-1\ n)$,
  $\rho = [[1, \dots, n-1]]$;
- $\sigma = [[1, \dots, n-1]]$,
  $\rho = [[1, \dots, n]]$;
- $\sigma = [[1, \dots, n]]$,
  $\rho = [[2, \dots, n]]$;
- $\sigma = [[2, \dots, n]]$,
  $\rho = (12)[[3, \dots, n]]$.

Then recall from Theorem 1 and Corollary 2 given in [3] that, for $n \ge 4$, $S_n = \langle (12), \sigma, \rho \rangle$ for each $\sigma, \rho \in Q(S_n)$ with one of the n-many forms given above, and that

$$\text{qrank}\,(S_n) = \begin{cases} 1 & for\ \ n = 2 \\ 2 & for\ \ n = 3. \\ 3 & for\ \ n \ge 4 \end{cases} \quad (35)$$

Moreover, notice that the map ξ defined in Theorem 2.1 is a quasi-idempotent in D_r, say $\xi \in Q(D_r)$. Then we have the following corollary.

Corollary 2.4. *For $1 \le r \le n-1$* $\text{qrank}\,(SI_{n,r}) = \begin{cases} 2 & for\ \ n = 2 \\ 3 & for\ \ n = 3. \\ 4 & for\ \ n \ge 4 \end{cases}$

Proof. Clearly $SI_{2,1} = \langle (12), \xi \rangle$, $SI_{3,r} = \langle (13), (23), \xi \rangle$ for $1 \le r \le 2$ and $SI_{n,r} = \langle (12), \sigma, \rho, \xi \rangle$ for $n \ge 4$ and $1 \le r \le n-1$ where $\xi \in Q(D_r)$ is the map defined in Theorem 2.1 and $\sigma, \rho \in Q(S_n)$ are one of the n-many forms given above. Then the result follows from the fact $\text{qrank}\,(SI_{n,r}) \ge \text{qrank}\,(S_n) + 1$ since $SI_{n,r} \backslash S_n = I_{n,r}$ is an ideal of $SI_{n,r}$. ∎

Corollary 2.5. *For* $1 \le r \le n-1$ $\text{rerank}(SI_{n,r} : S_n) = \text{reqrank}\,(SI_{n,r} : S_n) = 1$. ∎

3. Certain ranks of $AI_{n,r}$

Theorem 3.1. *For* $n \geq 3$ *and* $1 \leq r \leq n-1$ $D_r \subseteq \langle A_n \cup \{\xi\} \rangle$ *where*

$$\xi = \begin{cases} [12][3]\cdots[n] & for \;\; n \geq 3 \;\; and \;\; r = 1 \\ (1\,2)[r+1]\cdots[n] & for \;\; n \geq 3 \;\; and \;\; 2 \leq r \leq n-1. \end{cases} \tag{36}$$

Proof. Let $\alpha \in D_r$ for $n \geq 3$ and $1 \leq r \leq n-1$. From the proof of Theorem 2.1 we have $\alpha = \beta\xi\gamma$ where β, γ are the permutations defined in the proof of Theorem 2.1. Then we have $\alpha = \beta'\xi\gamma'$ where

$$\beta' = \begin{cases} \beta & if \;\; \beta \in A_n \\ \beta(n-1n) & if \;\; \beta \notin A_n \end{cases} \tag{37}$$

$$\gamma' = \begin{cases} \gamma & if \;\; \gamma \in A_n \\ \gamma(b_1\, b_2) & if \;\; \gamma \notin A_n \end{cases} \tag{38}$$

for $n \geq 3$ and $r = 1$, and we have

$$\alpha = \begin{cases} \beta'\xi\gamma' & if \;\; \beta,\gamma \in A_n \;\; or \;\; \beta,\gamma \notin A_n \\ \beta'\xi^2\gamma' & otherwise, \end{cases} \tag{39}$$

where

$$\beta' = \begin{cases} \beta & if \;\; \beta \in A_n \\ \beta(12) & if \;\; \beta \notin A_n \end{cases} \tag{40}$$

$$\gamma' = \begin{cases} \gamma, & if \;\; \gamma \in A_n \\ (12)\gamma, & if \;\; \gamma \notin A_n \end{cases} \tag{41}$$

for $n \geq 3$ and $2 \leq r \leq n-1$. ■

Corollary 3.2. *For* $n \geq 3$

$$AI_{n,r} = \begin{cases} \langle (123), \xi \rangle & for \;\; n = 3 \;\; and \;\; 1 \leq r \leq 2 \\ \langle (123), (12\ldots n), \xi \rangle & for\; an \;\; odd \;\; number\; n \geq 4 \;\; and\; 1 \leq r \leq n-1 \\ \langle (123), (23\ldots n), \xi \rangle & for \;\; an \;\; even \;\; number\; n \geq 4 \;\; and\; 1 \leq r \leq n-1 \end{cases} \tag{42}$$

where $\xi \in D_r$ is the map defined in Theorem 3.1.

Proof. The result follows from the fact $D_r \subseteq \langle A_n \cup \{\xi\} \rangle$ since $A_3 = \langle (123) \rangle$ and A_n is generated by two elements:

$$(123) \quad and \quad \begin{cases} (12\ldots n) & if \quad n \quad is \quad odd \\ (23\ldots n) & if \quad n \quad is \quad even \end{cases}$$

for $n \geq 4$. ■

Similarly, notice that the map ξ defined in Theorem 3.1 is a quasi-idempotent, say $\xi \in Q(D_r)$. Then we have the following corollary.

Corollary 3.3. $\operatorname{rank}(AI_{n,r}) = \begin{cases} 2 & for \quad n = 3 \quad and \quad 1 \leq r \leq 2 \\ 3 & for \quad n \geq 4 \quad and \quad 1 \leq r \leq n-1 \end{cases}$.

Proof. Clearly $AI_{n,r} \backslash A_n = I_{n,r}$ is an ideal of $AI_{n,r}$ and so $\operatorname{rank}(AI_{n,r}) \geq \operatorname{rank}(A_n) + 1$. Then the result follows from Corollary 3.2 and the fact $\operatorname{rank}(A_n) = \begin{cases} 1, & n = 3 \\ 2, & n \geq 4 \end{cases}$. ■

As in [3], for any m-tuple $(b_1, b_2, \ldots, b_m)$ $(4 \leq m \leq n)$ let

$$[[b_1, b_2, \ldots, b_m]]^\sharp = \begin{cases} (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-2}{2}} b_{\frac{m+4}{2}}), & if \quad m \, is \quad an \quad even \quad number \\ (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-3}{2}} b_{\frac{m+5}{2}}), & if \, m \, is \quad an \quad odd \quad number \end{cases} \tag{43}$$

where $(b_i \, b_j)$ denotes a 2-cycle for $1 \leq i, j \leq m$. Also, recall from Theorem 3 given in [3] that $A_n = \langle \lambda, \mu, \psi \rangle$ where

$$\lambda = \begin{cases} (13) \, [[4, \ldots, n]]^\sharp & if \quad n \equiv 0 \mod 4 \\ (12) \, [[4, \ldots, n]] & if \quad n \equiv 1,2 \mod 4 \\ (12) \, [[4, \ldots, n]]^\sharp & if \quad n \equiv 3 \mod 4 \end{cases} \tag{44}$$

$$\mu = \begin{cases} (23) \, (\frac{n+2}{2} \, \frac{n+6}{2}) & if \quad n \equiv 0 \mod 4 \\ (13) \, [[4, \ldots, n]] & if \quad n \equiv 1,2 \mod 4 \\ (13) \, (\frac{n+3}{2} \, \frac{n+5}{2}) & if \quad n \equiv 3 \mod 4 \end{cases} \tag{45}$$

$$\psi = \begin{cases} (1n)(23) \, [[4, \ldots, n-1]] & if \quad n \equiv 0 \mod 4 \\ (14)(23) \, [[5, \ldots, n]] & if \quad n \equiv 1 \mod 4 \\ (24) \, [[5, \ldots, n]] & if \quad n \equiv 2 \mod 4 \\ (14) \, [[5, \ldots, n]] & if \quad n \equiv 3 \mod 4 \end{cases} \tag{46}$$

and that $\operatorname{qrank}(A_n) = 3$ for $n \geq 5$.

Corollary 3.4. *For* $n \geq 5$ *and* $1 \leq r \leq n-1$ $\operatorname{qran}k\,(AI_{n,r}) = 4$.

Proof. Clearly $AI_{n,r} = \langle \lambda, \mu, \psi, \xi \rangle$ for $n \geq 5$ and $1 \leq r \leq n-2$ where $\lambda, \mu, \psi \in Q(A_n)$ are quasi-idempotents given above. Then the result follows from the fact qrank $(AI_{n,r}) \geq$ qrank $(A_n) + 1$ since $AI_{n,r} \backslash A_n = I_{n,r}$ is an ideal of $AI_{n,r}$. ■

Corollary 3.5. *For* $n \geq 3$ *and* $1 \leq r \leq n-1$ $\text{rerank}(AI_{n,r}: A_n) = 1$; *and for* $n \geq 5$ *and* $1 \leq r \leq n-1$ $\text{reqrank}\left(AI_{n,r}: A_n\right) = 1$. ■

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