İki SUR Model Altında Ön Tahmin Edicilerin Kovaryans Matrisleri Üzerine Bazı Notlar

Nesrin Güler¹*, Nevin Yüce²

ÖZ

Görününde ilişkisiz regresyon (SUR) modelleri denklemler arasında hataların ilişkili olduğu çoklu regresyon denklemlerinin ele alındığı lineer regresyon modellerinin uzantılandırır. Bu çalışmada, SUR modelleri altında ön tahmin problemi ele alınmıştır. İki SUR modeli altında tüm bilinmeyen vektörlerin en iyi lineer yansıtım ön tahmin edicilerinin (BLUP’ların) istatistiksel özellikleri üzerine çeşitli sonuçlar verilmiştir. Özellikle, matrislerin bazı rank formülleri kullanılarak iki model altında BLUP’ların kovaryans matrisleri üzerine bazı sonuçlar elde edilmiştir.

Anahtar Kelimeler: BLUP, Kovaryans matrisi, Rank, SUR model

¹*Sorumlu yazar iletişim: nesring@sakarya.edu.tr (https://orcid.org/0000-0003-3233-5377)
Department of Statistics, Sakarya University, Sakarya, 54187, Turkey
²İletişim: yuce.nevin@gmail.com (https://orcid.org/0000-0002-2967-2930)
Department of Mathematics, Sakarya University, Sakarya, 54187, Turkey
Some Notes on Covariance Matrices of Predictors under two SUR Models

ABSTRACT

Seemingly unrelated regression (SUR) models are extensions of linear regression models by considering multiple regression equations with correlated errors among equations. In this study, prediction problem under SUR models are considered. Several results are given on statistical properties of the best linear unbiased predictors (BLUPs) of all unknown vectors under two SUR models. Especially, some results established on covariance matrices of BLUPs under two models by using some rank formulas of matrices.

Keywords: BLUP, Covariance matrix, Rank, SUR model
I. INTRODUCTION

Several number of models with individual relations with each other can encounter in some statistical problems. These models can have correlated error terms among each other although they look like unrelated if the same data or an amount of same independent variables are used for the models. For example, we can consider the problem of investment in some companies across the same industry. Each company can be a set of model parameters. The models can have correlated error terms among each other although they look like unrelated if the correlation in error terms. They have attracted considerable interest in recent years; see e.g., [1-4], since these correlated. Seemingly unrelated regression (SUR) models are considered multiple regression equations having correlation in error terms. In this case, it can be expected that error terms of set of models may be simultaneously.

Consider a pair of SUR models formulated by

\[ M_1: y_1 = X_1 \beta_1 + \varepsilon_1, \quad E(\varepsilon_1) = 0, \quad D(\varepsilon_1) = \sigma_{11} I_n := \Sigma_1, \]

\[ M_2: y_2 = X_2 \beta_2 + \varepsilon_2, \quad E(\varepsilon_2) = 0, \quad D(\varepsilon_2) = \sigma_{22} I_n := \Sigma_2, \]

which are based on the equation

\[ y_{ij} = x_{ij1} \beta_{1i} + \cdots + x_{ijp_i} \beta_{pi} + \varepsilon_{ij}, \]

where \( y_i = (y_{ij}) \in \mathbb{R}^{n \times 1} \) is an observable random vector, \( X_i = (x_{ijt}) \in \mathbb{R}^{n \times p_i} \) is a known matrix of arbitrary rank, \( \beta_i = (\beta_{ij}) \in \mathbb{R}^{p_i \times 1} \) is an unknown parameter vector, and \( \varepsilon_i = (\varepsilon_{ij}) \in \mathbb{R}^{n \times 1} \) is an error vector with \( \text{cov}(\varepsilon_{ij}, \varepsilon_{ik}) = \sigma_{ik} I_t := \Sigma_{ik} \) for \( i, k = 1, 2, j = 1, 2, \ldots, n, \) and \( t = 1, \ldots, p_i. \)

In order to establish some results on predictions/estimations of all unknown parameters under models \( M_1 \) and \( M_2 \) we can consider the following general linear function

\[ \phi_i = K_i \beta_i + H_i \varepsilon_i \]

for given matrices \( K_i \in \mathbb{R}^{k \times p_i} \) and \( H_i \in \mathbb{R}^{k \times n}, \) \( i = 1, 2. \) According to the assumptions in (1) and (2),

\[ E(\phi_i) = K_i \beta_i, \quad D(\phi_i) = H_i \Sigma_i H_i', \quad \text{and} \quad \text{cov}(\phi_i, \varepsilon_i) = H_i \Sigma_{ii}, \]

\( i = 1, 2. \) The parameter vector \( \phi_i \) in (4) is said to be predictable under \( M_i \) if there exists a linear statistic \( L_i y_i \) with \( L_i \in \mathbb{R}^{k \times n} \) such that \( E(L_i y_i - \phi_i) = 0 \) holds; see [14], this is equivalently written as

\[ \mathcal{C}(K_i') \subseteq \mathcal{C}(X_i'), \]

\( i = 1, 2. \) (6) also corresponds the estimability condition of \( K_i \beta_i \) under \( M_i; \) see [15]. Let \( \phi_i \) be predictable under \( M_i. \) If there exists \( L_i y_i \) such that

\[ D(L_i y_i - \phi_i) = \min \text{ subject to } E(L_i y_i - \phi_i) = 0 \]

holds in the Löwner partial ordering, the linear statistic \( L_i y_i \) is defined to be the best linear unbiased predictor (BLUP) of \( \phi_i, \) a term introduced by Goldberger [16], and is denoted by \( L_i y_i = \text{BLUP}_{M_i}(\phi_i) = \text{BLUP}_{M_i}(K_i \beta_i + H_i \varepsilon_i), i = 1, 2. \) If \( H_i = 0, \) BLUP of \( \phi_i \) reduces to the best linear unbiased estimator (BLUE) of \( K_i \beta_i \) under \( M_i, \) expressed as \( L_i y_i = \text{BLUE}_{M_i}(\phi_i) = \text{BLUE}_{M_i}(K_i \beta_i). \)

Prediction problems under SUR models are one of the main subjects in the statistical inference of the models. The best known predictors/estimators of unknown vectors in the models are the BLUPs/BLUEs. In this
study, we consider two SUR models and give some results on covariance matrices between BLUPs of all unknown vectors under these models in theoretical point of view. Characterization of statistical properties of BLUPs under considered models involves some complex matrix expressions. Therefore, we use some matrix rank formulas for simplifying heavy matrix operations while establishing results.

Throughout this paper, the symbol \( \mathbb{R}^{m \times n} \) stands for the set of all \( m \times n \) real matrices. For \( A \in \mathbb{R}^{m \times n} \), the notations \( A', r(A), \) and \( C(A) \), denote the transpose, the rank, and the column space of \( A \), respectively. \( I_m \) denotes the identity matrix of order \( m \). The Moore-Penrose generalized inverse of \( A \) denoted by \( A^+ \), is defined to be the unique solution \( G \) satisfying the four matrix equation \( AGA = A \), \( GAG = G \), \( (AG)^T = AG \), \( (GA)^T = GA \). Furthermore, \( P_A = AA^+, E_A = A^+ = I_m - AA^+ \), and \( F_A = I_n - A^+A \) stand for the orthogonal projectors.

II. FORMULAS FOR BLUPS UNDER TWO SUR MODELS

In this section, firstly we give some rank formulas of matrices. Then we give fundamental BLUP equation of \( \phi_l \) and related properties of BLUPs. In what follows, it is assumed that the model \( \mathcal{M}_l \) is consistent, i.e., \( y_i \in \mathcal{C}[X_i, \Sigma_{ii}] \) with probability 1, \( i = 1, 2; \) see [17].

The well-known formulas for rank of matrices collected in the following lemma; see [18].

**Lemma 1** Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times k}, C \in \mathbb{R}^{l \times n}, \) and \( D \in \mathbb{R}^{l \times k} \). Then

(a) \( r[AB] = r(A) + r(E_AB) = r(B) + r(E_BA), \)

(b) \( r[A^C] = r(A) + r(CF_A) = r(C) + r(AF_C), \)

(c) If \( C(B) \subseteq C(A) \) and \( C(C') \subseteq C(A') \), then \( r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA'B). \)

The following lemma on BLUP for \( \phi_l \) under model \( \mathcal{M}_l \) is derived from Theorem 3.2 in [19].

**Lemma 2** Let \( \phi_l \) be predictable under \( \mathcal{M}_l \). Then

\[
L[y_l] = \text{BLUP}_{\mathcal{M}_l}(\phi_l) \iff L_l[X_i, \Sigma_{ii}, X_i^{-1}] = [K_l, H_l, \Sigma_{ii}^{-1}].
\]

General solution of this equation and corresponding BLUP of \( \phi_l \) is

\[
L[y_l] = \text{BLUP}_{\mathcal{M}_l}(\phi_l) = ([K_l, H_l, \Sigma_{ii}^{-1}][X_i, \Sigma_{ii}^{-1}] + U_l[X_i, \Sigma_{ii}^{-1}])y_l,
\]

where \( U_l \in \mathbb{R}^{k \times n} \) is arbitrary, \( i = 1, 2 \). Further, \( \text{BLUP}_{\mathcal{M}_l}(\phi_l) \) satisfies the following statements

\[
D[\text{BLUP}_{\mathcal{M}_l}(\phi_l)] = [K_l, H_l, \Sigma_{ii}^{-1}][X_i, \Sigma_{ii}^{-1}] + U_l[H_l, \Sigma_{ii}^{-1}][X_l, \Sigma_{ii}^{-1}]^T,
\]

(10)

\[
cov[\text{BLUP}_{\mathcal{M}_l}(\phi_l), \phi_l] = [K_l, H_l, \Sigma_{ii}^{-1}][X_l, \Sigma_{ii}^{-1}] + U_l[H_l, \Sigma_{ii}^{-1}],
\]

(11)

\[
D \left[ \phi_l - \text{BLUP}_{\mathcal{M}_l}(\phi_l) \right] = ([K_l, H_l, \Sigma_{ii}^{-1}][W_l^{-1} - H_l] \Sigma_{ii}([K_l, H_l, \Sigma_{ii}^{-1}][W_l^{-1} - H_l]),
\]

(12)

where \( W_l = [X_l, \Sigma_{ii}^{-1}], i = 1, 2 \). Furthermore, the following statements hold.

(a) \( r[X_i, \Sigma_{ii}^{-1}] = r[X_i, \Sigma_{ii}], \)

(13)

(b) \( \text{BLUP}_{\mathcal{M}_l}(\phi_l) \) is unique with probability 1 \( \iff y_l \in \mathcal{C}[X_l, \Sigma_{ii}] \) with probability 1,
(c) $L_i$ is unique $\iff r[X_i \Sigma_{ii}] = n, i = 1, 2.$

We note that BLUP of $\phi_i$ and ordinary least square predictor (OLSP) of $\phi_i$ coincide since $\Sigma_{ii} = \sigma_i I_n, i = 1, 2$. Then the results in Lemma 2 reduce the following results, for details, see; [18].

\[
L_i y_i = \text{BLUP}_{M_i}(\phi_i) = (K_i X_i^+ + H_i X_i^+) y_i,
\]

(13)

\[
D[\text{BLUP}_{M_i}(\phi_i)] = \sigma_i (K_i X_i^+ + H_i X_i^+) (K_i X_i^+ + H_i X_i^+)',
\]

(14)

\[
\text{cov}[\text{BLUP}_{M_i}(\phi_i), \phi_i] = \sigma_i (K_i X_i^+ + H_i X_i^+) H_i',
\]

(15)

\[
D[\phi_i - \text{BLUP}_{M_i}(\phi_i)] = \sigma_i (K_i X_i^+ - H_i P_{X_i}) (K_i X_i^+ - H_i P_{X_i})'.
\]

(16)

III. COVARIANCE MATRIX BETWEEN BLUPS UNDER TWO SUR MODELS

In this section, by presenting general approach to SUR models, we give the main result on the covariance matrices between BLUPS of unknown vectors under models $M_1$ and $M_2$. Then, we give some consequences which correspond special cases.

**Theorem 1** Assume that $\phi_1$ and $\phi_2$ is predictable under models $M_1$ and $M_2$, respectively, i.e., (6) holds. Let denote

\[
M = \begin{bmatrix}
\Sigma_{12} & \Sigma_{11} & X_1 & 0 & 0 \\
\Sigma_{22} & 0 & 0 & X_2 & 0 \\
X_2' & 0 & 0 & 0 & K_2' - X_2' H_2' \\
0 & X_1' & 0 & 0 & 0 \\
0 & 0 & K_1 - H_1 X_1 & 0 & 0
\end{bmatrix}
\]

(17)

Then

\[
r(\text{cov}\{\phi_1 - \text{BLUP}_{M_1}(\phi_1), \phi_2 - \text{BLUP}_{M_2}(\phi_2)\}) = r(M) - r[X_1 \Sigma_{11}] - r[X_2 \Sigma_{22}] - r(X_1) - r(X_2).
\]

(18)

Furthermore, $\phi_1 - \text{BLUP}_{M_1}(\phi_1)$ and $\phi_2 - \text{BLUP}_{M_2}(\phi_2)$ are uncorrelated if and only if

\[
r(M) = r[X_1 \Sigma_{11}] + r[X_2 \Sigma_{22}] + r(X_1) + r(X_2).
\]

(19)

**Proof.** From (10) - (12), we can write

\[
r(\text{cov}\{\phi_1 - \text{BLUP}_{M_1}(\phi_1), \phi_2 - \text{BLUP}_{M_2}(\phi_2)\})
\]

\[
= r\left( ([K_1 H_1 \Sigma_{11} X_1^+] W_1^+ - H_1) \Sigma_{12} ([K_2 H_2 \Sigma_{22} X_2^+] W_2^+ - H_2) \right).
\]

(20)

where $W_i = [X_i \Sigma_{ii} X_i^+]$, $i = 1, 2$. We can apply Lemma 1(c) to (20). Then we obtain

\[
r(\text{cov}\{\phi_1 - \text{BLUP}_{M_1}(\phi_1), \phi_2 - \text{BLUP}_{M_2}(\phi_2)\})
\]
Lemma 1 and simplifying elementary block matrix operations, we obtain

\[
\begin{align*}
&= r \left( [K_1 \ H_1 \Sigma_{11} \Sigma_{11}^{-1}]W_1 + H_2 \Sigma_{12}([K_2 \ H_2 \Sigma_{22} \Sigma_{22}^{-1}]W_2 + H_2) \right] - r(\Sigma_{12}) \\
&= r \left( \left[ \begin{array}{c}
\Sigma_{12} \\
-H_1 \Sigma_{12}
\end{array} \right] - r(\Sigma_{12}) \right)
\end{align*}
\] (21)

The following result is an immediate consequences of Theorem 1.

Assume that

\[
\begin{align*}
&\begin{bmatrix}
X_1 \\
-S_1 X_1 \\
0
\end{bmatrix} \\
&\begin{bmatrix}
0 \\
X_2 \\
0
\end{bmatrix} \\
&\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

are estimable under models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), respectively, i.e., (6) holds. Then

\[
\begin{align*}
&\begin{bmatrix}
\Sigma_{12} \\
\Sigma_{22}
\end{bmatrix} \\
&\begin{bmatrix}
X_1 \\
0
\end{bmatrix} \\
&\begin{bmatrix}
0 \\
X_1
\end{bmatrix}
\end{align*}
\]

are estimable under models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), respectively, i.e., (6) holds. Then

\[
r(\text{cov} \{ \text{BLUE}_{\mathcal{M}_1}(K_1\beta_1), \text{BLUE}_{\mathcal{M}_2}(K_2\beta_2) \})
\]

(23)
Furthermore, $\text{BLUE}_{M_1}(K_1\beta_1)$ and $\text{BLUE}_{M_2}(K_2\beta_2)$ are uncorrelated if and only if

$$\begin{bmatrix}
\Sigma_{12} & \Sigma_{11} & X_1 & 0 \\
\Sigma_{22} & 0 & 0 & X_2 \\
X_{1}' & 0 & 0 & K_1' \end{bmatrix} = r[X_1 \Sigma_{11}] - r[X_2 \Sigma_{22}] - r(X_1) - r(X_2).$$

(24)

$X_1\beta_1$ and $X_2\beta_2$ are always estimable under models $M_1$ and $M_2$, respectively. Then

$$r(\text{cov}\{\text{BLUE}_{M_1}(X_1\beta_1), \text{BLUE}_{M_2}(X_2\beta_2)\}) = r \begin{bmatrix}
\Sigma_{12} & \Sigma_{11} & 0 \\
\Sigma_{22} & 0 & X_2 \\
0 & 0 & X_1' \\
0 & 0 & 0 \\
0 & 0 & K_1 & 0 \end{bmatrix} = r[X_1 \Sigma_{11}] + r[X_2 \Sigma_{22}] + r(X_1) + r(X_2).$$

(25)

In particular, the following statements are equivalent.

(a) $\text{BLUE}_{M_1}(X_1\beta_1)$ and $\text{BLUE}_{M_2}(X_2\beta_2)$ are uncorrelated.

(b) $\varepsilon_1 - \text{BLUP}_{M_1}(\varepsilon_1)$ and $\varepsilon_2 - \text{BLUP}_{M_2}(\varepsilon_2)$ are uncorrelated.

(c) $r \begin{bmatrix}
\Sigma_{12} & \Sigma_{11} & 0 \\
\Sigma_{22} & 0 & X_2 \\
0 & 0 & X_1' \\
0 & 0 & 0 \\
0 & 0 & K_1 & 0 \end{bmatrix} = r[X_1 \Sigma_{11}] + r[X_2 \Sigma_{22}].$

We represent the general approach to SUR models in Theorem 1 and Corollary 1. The results in (18), (23), and (25) can also be expressed as follows since $\Sigma_{ik} = \sigma_{ik}I_n$, $i, k = 1, 2$.

$$r(\text{cov}\{\phi_1 - \text{BLUP}_{M_1}(\phi_1), \phi_2 - \text{BLUP}_{M_2}(\phi_2)\}) = r \begin{bmatrix}
\sigma_{12}X_1'X_2 & X_1'X_1 & 0 \\
X_2'X_2 & 0 & K_2 - X_2'X_2' \\
0 & K_1 - H_1X_1 & 0 \end{bmatrix} = r(X_1) - r(X_2).$$

(26)

$$r(\text{cov}\{\text{BLUE}_{M_1}(K_1\beta_1), \text{BLUE}_{M_2}(K_2\beta_2)\}) = r \begin{bmatrix}
\sigma_{12}X_1'X_2 & X_1'X_1 & 0 \\
X_2'X_2 & 0 & K_2' \end{bmatrix} = r(X_1) - r(X_2).$$

(27)

$$r(\text{cov}\{\text{BLUE}_{M_1}(X_1\beta_1), \text{BLUE}_{M_2}(X_2\beta_2)\}) = r(X_1'X_2).$$

(28)
IV. CONCLUSION

In this study, we present a general approach to SUR models by giving some statistical properties of BLUPs under two SUR models by using some rank formulas of matrices. In order to establish general results on predictor/estimator, we consider general linear function of all unknown vectors under models. Especially, we establish some results on covariance matrices between BLUPs of unknown vectors under two SUR models by addressing the subject theoretically. Although some results related to the predictors under general linear regression models can be applied to SUR models, algebraic properties of BLUPs under SUR models need to be clearly expressed and the formulations of statistical properties are worth to be considered to obtain more statistical inference of the models.

REFERENCES


