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Numerical Solutions of Second Order Initial Value Problems of Bratu-Type Equations using Predictor-Corrector Method

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Abstract

In this paper, numerical solutions of second order initial value problems of Bratu-type equation using predictor-corrector method is considered. The stability and convergence analysis are investigated. To validate the applicability of the scheme, two model problems are considered for numerical experimentation. In a nutshell, the present method improves the findings of some existing numerical methods reported in the literature.

Keywords: Predictor-corrector method, Initial value problem, quasi linearization method, Bratu-Type equation

1. Introduction

In this paper we presented a problem of the form

$$u''(x) + \lambda e^{u(x)} = 0, \quad 0 \le x \le l \tag{1}$$

subject to the initial conditions

$$u(0) = \alpha, \quad u'(0) = \gamma \tag{2}$$

where λ, α and γ are constant numbers for $u(x)$ is unknown function.
In numerical analysis, predictor-corrector methods belong to a class of algorithms designed to integrate ordinary differential equations to find an unknown function that satisfies a given differential equation. When considering the numerical solution of ordinary differential equations (ODEs), a predictor-corrector method typically uses an explicit method for the predictor step and an implicit method for the corrector step. Bratu-Type equation is widely used in science and engineering to describe complicated physical and chemical models [1]. As author [2] stated, recently much attention has been given to develop several iterative methods for solving nonlinear equations of Bratu-type of equations. The nonlinear models of real-life problems are still difficult to solve analytically. Authors [3], [4] said that there has been recently much attention devoted to the search for better and more efficient numerical methods for determining a solution to nonlinear models. Recently, authors [5-9] solves Bratu type equation using different numerical method but still there is a room for accuracy of the governing problem under consideration. Therefore, it is important to develop more accurate and convergent numerical method for solving second order Bratu-type equation. Thus, the purpose of this study is to develop stable, convergent and more accurate numerical method for solving initial value problems of Bratu-Type equations. We first linearize the given equation using quasi-linearization formula and then used fourth order Adams-Bash forth method as a predictor and Adams-Moulton fourth order method as a corrector. The starting values (u_1, u_2, u_3) were calculated using Runge-Kutta fourth order method.

2. Formulation of the method

Bratu-type of Eq. (1) can be transformed to a linear differential problem using the quasi linearization method and we get the iterative scheme as

$$u''_{k+1}(x) + \lambda e^{u_k(x)} + u'_{k+1}(x) = \lambda e^{u_k(x)}(u_k(x) - 1) \tag{3}$$

with initial condition

$$u_{k+1}(0) = \beta \text{ and } u'_{k+1}(0) = \gamma \tag{4}$$

where $k = 1, 2, 3, ...$
Eq. (3) can be used to compute $u_{k+1}(x)$ provided $u_k(x)$ is known. In particular, the initial approximation $u_0(x)$ must be specified so that we compute $u_1(x)$. Once $u_1(x)$ is known, we compute $u_2(x)$ using Eq. (3) and so on.
Eqs. (3) and (4) can be reduced to the equations

$$Lu \equiv u''(x) + a(x)u(x) = b(x), \quad 0 \le x \le l, \tag{5}$$

where, $a(x) = \lambda e^{u(x)}$ *and* $b(x) = \lambda e^{u(x)}(u(x) - 1)$

with initial condition

$$u(0) = \alpha \text{ and } u'(0) = \gamma \tag{6}$$

Therefore, the given second order IVP of Bratu equation is linearized to Eq. (5) with initial condition (6) can be solved by explicit-implicit Adams-Bashforth-Moulton predictor-corrector method. Eq. (5) can be reduced to first order system of equations using the substitutions $v(x) = u'(x)$ and $v'(x) = u''(x)$. Then Eq. (5) and Eq. (6) can be re-written as:

$$\begin{cases} u'(x) = v(x) = F(x,u,v), & u(0) = \alpha \\ v'(x) = b(x) - a(x)u(x) = G(x,u,v), & v(0) = \gamma \end{cases} \tag{7}$$

Dividing the interval $[0, l]$ into N equal subinterval of mesh length h and the mesh point is given by $x_n = x_0 + nh,$ for $n = 1,2,...,N-1$. For the sake of simplicity let use the notation: $u(x_n) = u_n$, $v(x_n) = v_n$, etc. Thus, at the nodal point x_n Eq. (7), written as:

$$\begin{cases} u_n' = F(x_n,u_n,v_n), & u(0) = \alpha \\ v_n' = G(x_n,u_n,v_n), & v(0) = \gamma \end{cases} \tag{8}$$

where $G(x_n,u_n,v_n) = -a(x_n)u(x_n) + b(x_n)$

To solve the system of equations given in Eq. (8), we use explicit-implicit multi step methods that require information about the solution at x_n to calculate at x_{n+1} from the solution at a number of previous solutions using Runge-Kutta method as self-starter.

For the general case let's consider the first order nonlinear equal spaced initial value problem (IVP) of the form

$$u'(x) = f(x,u(x)), \quad u(x_0) = \alpha \tag{9}$$

The IVP of the form of Eq. (9) can be solved by using fourth order Runge-Kutta method. The general fourth order Runge-Kutta method of Eq. (9) is given by [10].

$$u_{n+1} = u_n + h\sum_{n=1}^{4} w_n k_n \tag{10}$$

where $k_n = f(x_n + c_n h,\ u_n + \sum_{j=1}^{4} a_{n,j} k_j)$

For particular fourth order classical Runge-Kutta method we have:

$$u_{n+1} = u_n + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \tag{11}$$

where
$$k_1 = f(x_n, u_n),\ \ k_2 = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1),\ \ k_3 = f(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2)$$
$$k_4 = f(x_n + h, u_n + k_3)$$

For the fourth order Runge-Kutta method of the system of equations of the form of Eq. (8) can also be expressed as:

$$\begin{cases} u_{n+1} = u_n + \sum_{n=1}^{4} w_n k_n \\ v_{n+1} = v_n + \sum_{n=1}^{4} w_n k_n \end{cases} \tag{12}$$

where
$$\begin{cases} k_n = hF(x_n + c_n h,\ u_n + \sum_{j=1}^{4} a_{nj} k_j,\ v_n + \sum_{j=1}^{4} a_{nj} m_j) \\ m_n = hG(x_n + c_n h,\ u_n + \sum_{j=1}^{4} a_{nj} k_j,\ v_n + \sum_{j=1}^{4} a_{nj} m_j) \end{cases}$$

Eq. (12) can also be simplified to the fourth order of classical Runge-Kutta method as:

$$\begin{cases} u_{n+1} = u_n + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \\ v_{n+1} = v_n + \frac{1}{6} h(m_1 + 2m_2 + 2m_3 + m_4) \end{cases} \tag{13}$$

where

$$k_1 = F(x_n, u_n, v_n) \qquad m_1 = G(x_n, u_n, v_n)$$

$$k_2 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1) \qquad m_2 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1)$$

$$k_3 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2) \qquad m_3 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$$

$$k_4 = F(x_n + h, u_n + k_3, v_n + m_3) \qquad m_4 = G(x_n + h, u_n + k_3, v_n + m_3)$$

Using Eq. (13) we can derive the general formula of the linearized Bratu equation of Eq. (8). Let calculate the values of k_i and m_i for $i = 1, 2, 3$ and 4 as follow:

$$k_1 = F(x_n, u_n, v_n) = u_n'$$

$$m_1 = G(x_n, u_n, v_n) = -a_n u_n + b_n$$

$$k_2 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1) = u_n' + \frac{1}{2}u_n''$$

$$m_2 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1) = -a_n(u_n + \frac{1}{2}u_n') + b_n$$

$$k_3 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2) = u_n' + \frac{1}{2}u_n'' + \frac{1}{4}u'''$$

$$m_3 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2) = -a_n(u_n + \frac{1}{2}u_n' + \frac{1}{4}u_n'') + b_n$$

$$k_4 = F(x_n + h, u_n + k_3, v_n + m_3) = u_n' + u_n'' + \frac{1}{2}u''' + \frac{1}{4}u_n^{(4)}$$

$$m_4 = G(x_n + h, u_n + k_3, v_n + m_3) = -a_n(u_n + u_n' + \frac{1}{2}u_n'' + \frac{1}{4}u''') + b_n$$

Using Eq. (13) we can derive the general formula of the linearized Bratu equation of Eq. (8). Let calculate the values of k_i and m_i for $i = 1, 2, 3$ and 4 as follow:

$$k_1 = F(x_n, u_n, v_n) = u_n'$$

$$m_1 = G(x_n, u_n, v_n) = -a_n u_n + b_n$$

$$k_2 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1) = u_n' + \frac{1}{2}u_n''$$

$$m_2 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_1, v_n + \frac{1}{2}m_1) = -a_n(u_n + \frac{1}{2}u_n') + b_n$$

$$k_3 = F(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2) \qquad m_3 = G(x_n + \frac{1}{2}h, u_n + \frac{1}{2}k_2, v_n + \frac{1}{2}m_2)$$
$$= u_n' + \frac{1}{2}u_n'' + \frac{1}{4}u''' \qquad = -a_n(u_n + \frac{1}{2}u_n' + \frac{1}{4}u_n'') + b_n$$

$$k_4 = F(x_n + h, u_n + k_3, v_n + m_3) \qquad m_4 = G(x_n + h, u_n + k_3, v_n + m_3)$$
$$= u_n' + u_n'' + \frac{1}{2}u''' + \frac{1}{4}u_n^{(4)} \qquad = -a_n(u_n + u_n' + \frac{1}{2}u_n'' + \frac{1}{4}u''') + b_n$$

Then substituting these values of k_i's and m_i's $(\mathrm{i} = 1,\ 2,\ 3,\ 4)$ in Eq. (13) and simplifying the equations separately for u_{n+1} and v_{n+1} we get:

$$\begin{aligned} u_{n+1} &= u_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \\ &= u_n + h(u_n' + \frac{1}{2}u_n'' + \frac{1}{6}u''' + \frac{1}{24}u_n^{(4)}) \end{aligned} \tag{14}$$

and the values of v_{n+1} can also be calculated as follows:

$$\begin{aligned} v_{n+1} &= v_n + \frac{1}{6}h(m_1 + 2m_2 + 2m_3 + m_4) \\ &= v_n - h(a_n u_n + \frac{1}{2}a_n u_n' + \frac{1}{6}a_n u_n'' + \frac{1}{24}a_n u_n''' + b_n) \end{aligned} \tag{15}$$

Therefore the system of equation (13) simplified to:

$$\begin{cases} u_{n+1} = u_n + h(u_n' + \frac{1}{2}u_n'' + \frac{1}{6}u_n''' + \frac{1}{24}u_n^{(4)}) \\ v_{n+1} = v_n - h(a_n u_n + \frac{1}{2}a_n u_n' + \frac{1}{6}a_n u_n'' + \frac{1}{24}a_n u_n''' + b_n) \end{cases} \tag{16}$$

This equation is Runge-Kutta fourth order formula used to approximate the values of u_n and v_n for $\mathrm{n} = 1,\ 2,\ 3$ since the Adams-Bashforth-Moulton predictor-corrector method requires these values.

To solve Eq. (9), we can apply the explicit-implicit multistep method that requires information about the solution at x_{n+1} from the solution at a number of previous solutions.

To begin the derivation of the multi-step methods, if we integrate the initial-value problem over the interval $[x_n, x_{n+1}]$, then the following property exists:

$$u(x_{n+1}) = u(x_n) + \int_{x_n}^{x_{n+1}} f(x, u(x))dx \tag{17}$$

where $f(x, u(x))$ is the first derivative of $u(x)$. To derive an Adams-Bashforth method, Newton backward difference formula with a set of equal spacing points, x_{n+1-k}, ..., x_{n-1}, x_n, is used to approximate the integral and the fourth order Adams-Bashforth method is given by [2].

$$u_{n+1} = u_n + \frac{h}{24}\left[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right] + T_k \tag{18}$$

where, T_k is the truncation error of the fourth order Adams-Bashforth method and is given by:

$$T_k = \frac{251}{720}h^5u^{(5)}(\xi) = O(h^5) \tag{19}$$

To use Eq. (18), we require the starting values u_n, u_{n-1}, u_{n-2} and u_{n-3} which are calculated by self-starting single step method, Runge-Kutta fourth order method for our case. The fourth order Adams-Bashforth method for the system of Eq. (8), can be solved using Eq. (18) and it becomes

$$\begin{cases} u_{n+1} = u_n + \frac{h}{24}\left[55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3}\right] \\ v_{n+1} = v_n + \frac{h}{24}\left[55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3}\right] \end{cases} \tag{20}$$

Using Eq. (20) we can formulate the general form of the systems of Eq. (8) for $n \geq 4$. Therefore, Eq. (20) can be derived as follow:

$$u_{n+1} = u_n + \frac{h}{24}(55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3}) \tag{21}$$

But, since the values of F_n, F_{n-1}, F_{n-2} and F_{n-3}, for $n \geq 4$, can be calculated using the linearized system of Eq. (8), we have

$$F_n = u'_n,\ F_{n-1} = u'_{n-1},\ F_{n-2} = u'_{n-2},\ F_{n-3} = u'_{n-3}, \tag{22}$$

Then

$$u_{n+1} = u_n + \frac{h}{24}(55u'_n - 59u'_{n-1} + 37u'_{n-2} - 9u'_{n-3}) \tag{23}$$

For

$$v_{n+1} = v_n + \frac{h}{24}(55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3}) \tag{24}$$

where the values of G_n, G_{n-1}, G_{n-2}, G_{n-3} are given by:

$$\begin{cases} G_n = -a_n u_n + b_n, \quad G_{n-1} = -a_{n-1}u_{n-1} + b_{n-1}, \\ G_{n-2} = -a_{n-2}u_{n-2} + b_{n-2}, \quad G_{n-3} = -a_{n-3}u_{n-3} + b_{n-3} \end{cases} \tag{25}$$

So, Eq. (24) becomes

$$\begin{aligned} v_{n+1} = v_n + \frac{h}{24}(&55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ &- 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{aligned} \tag{26}$$

Then summarizing Eq. (23) and (26), we have

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}(55u'_n - 59u'_{n-1} + 37u'_{n-2} - 9u'_{n-3}) \\ v_{n+1} = v_n + \dfrac{h}{24}(55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ \qquad\qquad - 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{cases} \tag{27}$$

Therefore, Eq. (27) is the fourth order Adams-Bashforth predictor method for the given system of Eq. (8).

Similarly, to solve the given nonlinear differential equation using fourth order Adams-Moulton method, first let's consider the first order nonlinear IVP of the form Eq. (9) and the method is derived by using the set of equal spacing points, $x_{n+2-k}, \ldots, x_n, x_{n+1}$.
Integrating both sides of Eq. (9) with respect to x from x_n to x_{n+1} we have,

$$u(x_{n+1}) = u(x)_n + \int_{x_n}^{x_{n+1}} f(x, u(x))dx \tag{28}$$

Replace $f(x,u)$ of Eq. (27) by the polynomial $p_k(x)$ of degree k, which interpolates $f(x,u)$ at $k+1$ points and Newton backward interpolation formula, gives polynomial of degree k and the fourth order Adams-Moulton method is given by:

$$u_{n+1} = u_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] + T_l \tag{29}$$

where, the truncation error T_l is given by:

$$T(x) = \frac{-19}{720}h^5 u^{(5)}(\xi) = O(h^5) \tag{30}$$

The system of Eq. (8), is then given by

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}\left[9F_{n+1} + 19F_n - 5F_{n-1} + F_{n-2}\right] \\ v_{n+1} = v_n + \dfrac{h}{24}\left[9G_{n+1} + 19G_n - 5G_{n-1} + G_{n-2}\right] \end{cases} \tag{31}$$

To apply Eq. (31) on Bratu equation, we simplify this equation using the same procedures as we have done for the predictor (Adams-Bashforth method) above.

That is, the values of $F_{n+1}, F_n, F_{n-1}, F_{n-2}$ and $G_{n+1}, G_n, G_{n-1}, G_{n-2}$ are as calculated for the predictor method. Therefore, the system of Eq. (31) can be written as:

$$\begin{cases} u_{n+1} = u_n + \dfrac{h}{24}(9u'_{n+1} + 19u'_n - 5u'_{n-1} + u'_{n-2}) \\ v_{n+1} = v_n + \dfrac{h}{24}(9(-a_{n+1}u_{n+1} + b_{n+1}) + 19(-a_n u_n + b_n) - 5(-a_{n-1}u_{n-1} + b_{n-1}) \\ \qquad\qquad + (-a_{n-2}u_{n-2} + b_{n-2})) \end{cases} \tag{32}$$

This is the Adams-Moulton corrector formula. We use the fourth order Adams-Bashforth method as a predictor and Adams-Moulton method as a corrector and we have the following equations.

$$\begin{cases} u^p_{n+1} = u_n + \dfrac{h}{24}\left[55F_n - 59F_{n-1} + 37F_{n-2} - 9F_{n-3}\right] \\ v^p_{n+1} = v_n + \dfrac{h}{24}\left[55G_n - 59G_{n-1} + 37G_{n-2} - 9G_{n-3}\right] \end{cases} \tag{33}$$

where $F^*_{n+1} = F(x_{n+1}, u^p_{n+1}, v^p_{n+1})$
$G^*_{n+1} = G(x_{n+1}, u^p_{n+1}, v^p_{n+1})$

$$\begin{cases} y^c_{n+1} = y_n + \dfrac{h}{24}\left[9F^*_{n+1} + 19F_n - 5F_{n-1} + F_{n-2}\right] \\ z^c_{n+1} = z_n + \dfrac{h}{24}\left[9G^*_{n+1} + 19G_n - 5G_{n-1} + G_{n-2}\right] \end{cases} \tag{34}$$

u_{n+1}^{p} and v_{n+1}^{p} are calculated from Eq. (34) and applying these equations on the linearized Bratu equations is the same as combining Eq. (27) and Eq. (32), using Eq. (27) as a predictor and Eq.(32) as a corrector and it becomes:

Predictor Formula

$$\begin{cases} u_{n+1}^{p} = u_n + \dfrac{h}{24}(55u_n' - 59u_{n-1}' + 37u_{n-2}' - 9u_{n-3}') \\ v_{n+1}^{p} = v_n + \dfrac{h}{24}(55(-a_n u_n + b_n) - 59(-a_{n-1}u_{n-1} + b_{n-1}) + 37(-a_{n-2}u_{n-2} + b_{n-2}) \\ \qquad\qquad - 9(-a_{n-3}u_{n-3} + b_{n-3})) \end{cases} \tag{35}$$

and corrector formula

$$\begin{cases} u_{n+1}^{c} = u_n + \dfrac{h}{24}(9(u_{n+1}^{p})' + 19u_n' - 5u_{n-1}' + u_{n-2}') \\ v_{n+1}^{c} = v_n + \dfrac{h}{24}(9(-a_{n+1}u_{n+1}^{p} + b_{n+1}) + 19(-a_n u_n + b_n) - 5(-a_{n-1}u_{n-1} + b_{n-1}) \\ \qquad\qquad + (-a_{n-2}u_{n-2} + b_{n-2})) \end{cases} \tag{36}$$

3. Truncation Error, Convergence and Stability Analysis

Let's consider the more general multistep method of the following

$$\begin{aligned} &\frac{\left[U(t_{k+1}) + \alpha_1 U(t_k) + \alpha_2 U(t_{k-1}) + \ldots + \alpha_m U(t_{k+1-m})\right]}{h} \\ &= \beta_0 f(t_{k+1}, U(t_{k+1})) + \beta_1 f(t_k, U(t_k)) + \ldots + \beta_m f(t_{k+1-m}, U(t_{k+1-m})) \end{aligned} \tag{37}$$

where α_i and β_j, (for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., m$) are constants.

Theorem 1: If a sequence of numbers e_k satisfies

$$e_{k+1} + \rho_1 e_k + \rho_2 e_{k-1} + \ldots + \rho_m e_{k+1-m} = hT_k \tag{38}$$

for $k \geq m-1 (m \geq 1)$ and if all the roots of the corresponding characteristic polynomial

$$\lambda^m + \rho_1 \lambda^{m-1} + \rho_2 \lambda^{m-2} + \ldots + \rho_m \tag{39}$$

are less than or equal to one in absolute value, and all multiple roots are strictly less than one in absolute value, then

$|e_k| \leq M_\rho \left[\max\{|e_0|, \ldots, |e_{m-1}|\} + t_k T\right]$, where $t_k = kh$, $T = \max|T_j|$, and M_ρ is a constant depending only on the ρ_i.

Definition: The *region of absolute stability* of a multistep method consists of those values of *ah* in the complex plane for which all roots of polynomial

$$(1-\beta_0 ah)\lambda^m + (\alpha_1 - \beta_1 ah)\lambda^{m-1} + (\alpha_2 - \beta_2 ah)\lambda^{m-2} + \ldots + (\alpha_m - \beta_m ah) \tag{40}$$

are less than or equal to one in absolute value, and all multiple roots are strictly less than one in absolute value.

Theorem 2: The multistep method (29) is stable provided all roots of

$$\lambda^m + \alpha_1 \lambda^{m-1} + \alpha_2 \lambda^{m-2} + \ldots. + \alpha_m \tag{41}$$

are less than or equal to one in absolute value, and all multiple roots are strictly less than one in absolute value.

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order $O(h^5)$. Therefore, the local truncation errors of predictor and corrector are respectively

$$\begin{cases} u(t_{n+1}) - p_{n+1} = \dfrac{251}{720} h^5 u^{(5)}(\xi) \\ u(t_{n+1}) - u_{n+1} = \dfrac{-19}{720} h^5 u^{(5)}(\zeta) \end{cases} \tag{42}$$

where $u(t_{n+1})$ is given by Eq. (15) for the predictor and Eq. (20) for corrector and p_{n+1} and u_{n+1} are calculated values for Adams-Bash forth predictor and Adams-Moulton corrector given by Eqs. (16) and (29) respectively

3. Stability Analysis

Some of the most popular higher-order, stable, multistep methods are the Adams methods, which ensure stability by choosing $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$. The characteristic polynomial corresponding to theorem 1 is $\lambda^m - \lambda^{m-1}$ which has 1 as a simple root and 0 as a multiple root. Thus these methods are stable regardless of the values chosen for the β_i's.

The values of β_i's are determined in order to maximize the order of the truncation error. For Adams-Bashforth method we can calculate the value of β_i as [2]:

$$\begin{cases} \beta_0 = \int_0^1 (-1)^0 \binom{-s}{0} ds = 1,\ \beta_1 = \int_0^1 (-1)^1 \binom{-s}{1} ds = \dfrac{1}{2},\ \beta_2 = \int_0^1 (-1)^2 \binom{-s}{2} ds = \dfrac{5}{12}, \\ \beta_3 = \int_0^1 (-1)^3 \binom{-s}{3} ds = \dfrac{3}{8},\ \beta_4 = \int_0^1 (-1)^4 \binom{-s}{4} ds = \dfrac{251}{720} \end{cases} \tag{43}$$

And also for Adams-Moulton method we have

$$\beta_0 = 1,\ \beta_1 = -\frac{1}{2},\ \beta_2 = -\frac{1}{12},\ \beta_3 = -\frac{1}{24},\ \beta_4 = -\frac{19}{720} \tag{44}$$

Since for all Adams methods the values of $\alpha_1 = -1$ and $\alpha_2 = \alpha_3 = ... = \alpha_m = 0$, the fourth order Adams-Bashforth method (Eq. 18) and fourth order Adams-Moulton method Eq. (31) have the characteristic equation of

$$\rho(\lambda) = \lambda^4 - \lambda^3 = 0 \Rightarrow \lambda^3(\lambda - 1) = 0 \tag{45}$$

$\Rightarrow \lambda = 1$ is a simple root and 0 is a multiple root with multiplicity 3.

Therefore, since the simple root is 1, and multiple roots are 0 which is strictly less than 1, by Theorem 1, Adams-Bash forth and Adams-Moulton methods are stable.

4. **Numerical Examples and Results**

To demonstrate the applicability of the method, we implemented the method on four numerical examples To show the applicability and efficiency of the method, we have taken two examples of Bratu-type model and compared the numerical solutions with different other numerical methods and exact solution as follow.

Example 1: Consider the Bratu-type initial value problem

$$\begin{cases} y'' - 2e^y = 0, \ 0 < x < 1 \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \tag{46}$$

whose exact solution is $y(x) = -2\ln(\cos(x))$

Table 1. The comparison of absolute errors for Example 1 at different values of the mesh size *h* with different numerical methods

	Absolute errors at $h = 0.1$			
x	Method in[7]	Method in[8]	Method in [10]	Present Method
0.1	6.7100e-6	4.3876e-13	6.4102e-7	2.8436e-9
0.2	9.5500e-6	4.5402e-10	9.7469e-6	1.2788e-7
0.3	3.3100e-6	2.6638e-8	4.5299e-5	3.9593e-7
0.4	8.0400e-6	4.8488e-7	1.2711e-4	3.4141e-6
0.5	8.4800e-6	4.6664e-6	2.6867e-4	4.4904e-6
0.6	2.0300e-5	3.0124e-5	4.8365e-4	6.8988e-6
0.7	7.1500e-5	1.4821e-4	8.3679e-4	1.1741e-5
0.8	2.9100e-4	6.0039e-4	1.6005e-3	2.1580e-5
0.9	1.0500e-3	2.1074e-3	3.6497e-3	4.2756e-5
1.0	3.5300e-3	6.6498e-3	9.3915e-3	9.2517e-5

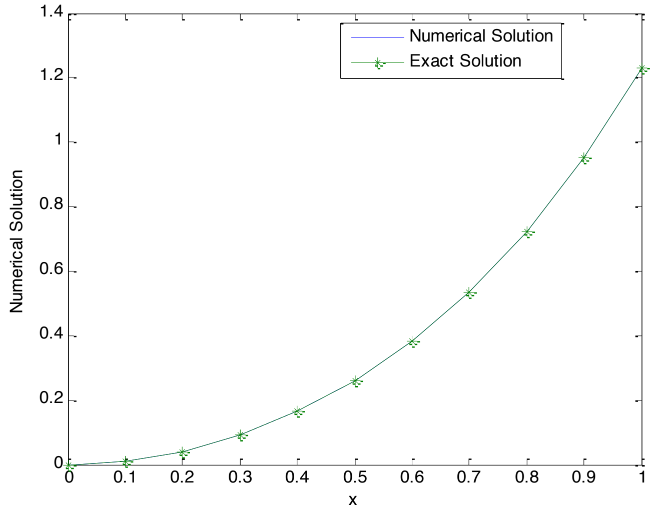

Fig. 1. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for Example 1 with mesh length $h = 0.1$.

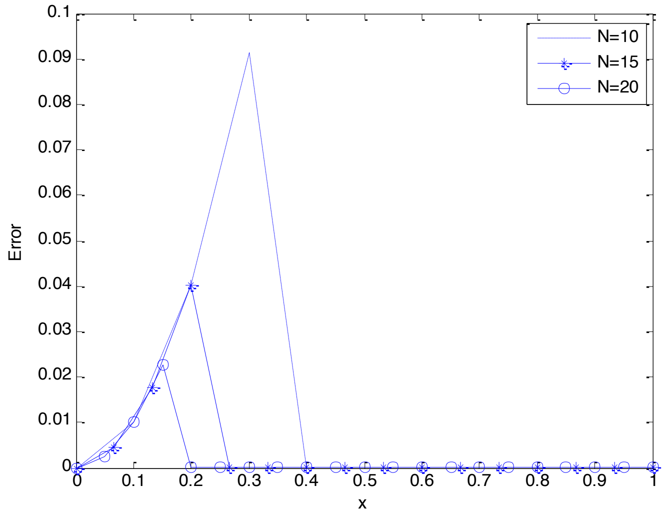

Fig. 2. Point-wise absolute error of Example 1 for different values of number of meshes points.

Example 2: Consider the Bratu-type initial value problem

$$\begin{cases} \dfrac{d^2y}{dx^2} = -\pi^2 e^{-y}, \\ y(0) = 0, \quad y'(0) = \pi \end{cases} \tag{47}$$

whose exact solution is $y(x) = \ln(1 + \sin(\pi x))$

Table 2. The comparison of absolute errors for Example 2 at different values of the mesh size h

x	Absolute errors at $h = 0.1$		
	Exact value	Method in [9]	Present Method
0.1	0.26928	3.20777e-4	3.4129e-5
0.2	0.46234	2.37600e-5	5.7752e-5
0.3	0.59278	3.58700e-5	7.9099e-5
0.4	0.66837	8.01000e-5	2.7368e-4
0.5	0.69315	1.19500e-4	4.2841e-5
0.6	0.66837	1.66200e-4	6.8607e-5
0.7	0.59278	2.20200e-4	1.3754e-4
0.8	0.46234	2.85100e-4	1.8845e-4
0.9	0.26928	4.03400e-4	2.2350e-4
1.0	2.2204e-16	5.37400e-4	2.1737e-4

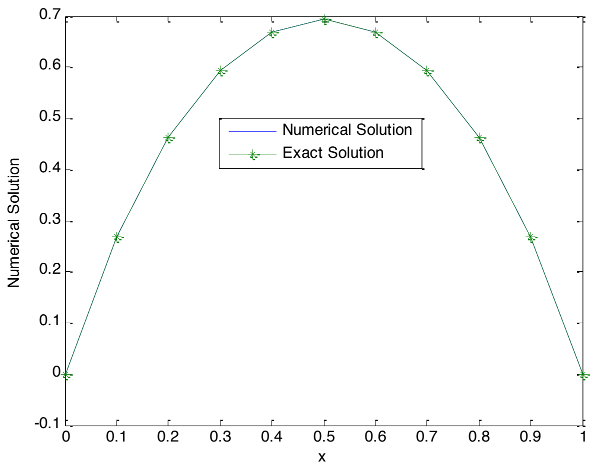

Fig. 3. Plot of exact and approximated solution of Bratu-type equation using predictor-corrector method for Example 2 with mesh size $h = 0.1$.

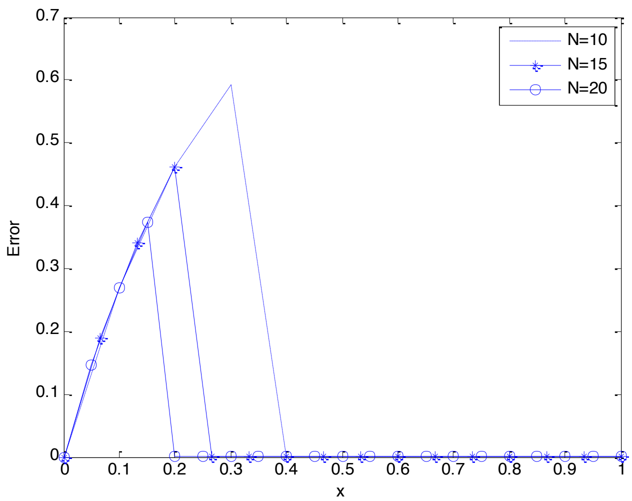

Fig. 4. Point-wise absolute errors of Example 2 for different values of number of mesh points.

5. Discussion and Conclusion

This study introduces numerical solutions of second order initial value problems of Bratu-type equations using predictor-corrector method. The stability and convergence of the scheme are investigated and established well. The numerical solutions are tabulated in terms of point wise absolute errors and observed that the present method improves the findings of some existing numerical methods reported in the literature (Table 1 and 2). Moreover, behaviors of the numerical solution (Figure 1 and 3) and point-wise absolute errors (Figure 2 and 4) were shown in figures. According to the plotted figures one can clearly observe that the numerical and exact solutions agree very well and as number of mesh point increases or as step size decreases, the point-wise absolute error decreases which clearly indicates the convergence of the present scheme. Concisely, the present method gives more accurate solution for solving second order initial value problems of Bratu-type equations.

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