SOLUTION OF EIGENVALUE PROBLEM OF TIMOSHENKO BEAM ON ELASTIC FOUNDATION BY DIFFERENTIAL TRANSFORM METHOD AND TRANSFER MATRIX METHOD

Seval CATAL[1*] Oktay DEMİRDAĞ[2]

[1]*Dokuz Eylul University, Department of Civil Engineering (Applied Mathematics), Faculty of Engineering, İzmir - TURKEY*

[2]*Pamukkale University, Department of Civil Engineering, Faculty of Engineering, Denizli – TURKEY*

Abstract

In this study, equation of motion for free vibration of both ends simply supported Timoshenko beam resting on two different elastic foundation are obtained considering P-Δ effects. The eigenvalues being the fundamental frequencies of Timoshenko beam on elastic foundation and the frequency factors related to these eigenvalues are calculated using two different methods. Free vibration equation of Timoshenko beam is solved by using respectively differential transform method (DTM) and transfer matrix method (TMM), respectively. Frequency factor values obtained by both methods depending on the beam length ratios at two different foundation regions, the modulus of subgrade reactions and the axial load ratios are presented in the tables.

Key words: Differential transformation method, transfer matrix method, partial differential equation, equation of motion, Timoshenko beam, elastic foundation.

Özet

Bu çalışmada, iki farklı elastik zemine oturan, iki ucu basit mesnetli, Timoshenko kirişinin serbest titreşimine ait hareket denklemi P-Δ etkileri altında elde edilmiştir. Elastik zemine oturan Timoshenko kirişinin temel

* Corresponding Author: *seval.catal@deu.edu.tr*

frekansları olan özdeğerler ve bu özdeğerlere bağlı frekans faktörleri iki farklı yöntem kullanılarak hesaplanmıştır. Timoshenko kirişinin serbest titreşim denklemleri, sırasıyla diferansiyel dönüşüm yöntemi (DTM) ve taşıma matrisi yöntemi (TMM) kullanılarak çözülmüştür. İki farklı zemin bölgesindeki kirişin uzunluk oranları, zemin yatak katsayısı ve eksenel kuvvet oranları bağlı olarak her iki yöntemle elde edilen frekans faktör değerleri tablolarda sunulmuştur.

Anahtar kelimeler: Diferensiyel dönüşüm yöntemi, taşıma matrisi yöntemi, kısmi diferansiyel denklem, hareket denklemi, Timoshenko kirişi, elastik zemin.

1. INTRODUCTION

Static and dynamic analysis problems of beams resting on elastic foundation is encountered at many engineering applications related to soil-structure interactions in structure and geotechnical engineering like strip foundations, railroads tracks, pipelines embedded in soil.

It is assumed in many studies related to beam on elastic foundation problem that the soil behavior is modeled by linear-elastic spring according to Winkler soil. Yokoyama studied the vibration of Timoshenko beam on two-parameter elastic foundation considering both bending moment and shear force effects [1]. Doyle et all, solved the equation of motion of the beam on partial elastic foundation including only bending moment effect by using separation of variables [2]. Chen examined the static analysis using differential quadrature element method of Bernoulli-Euler beam on elastic foundation considering only the bending moment effect by discretizing differential equation of the beam [3]. Chen and Huang obtained the dynamic stiffness matrix of Timoshenko beam on viscoelastic foundation [4]. Karami studied free vibration analysis of non-uniform Timoshenko beam resting on elastic supports by differential quadrature element method [5]. Catal obtained the free vibration circular frequencies of the piles partially embedded in the soil due to supporting conditions of top and bottom ends of the pile using separation of variables [6]. Hsu investigated vibration analysis of axially loaded clamped-free and hinged-hinged Bernoulli-Euler beams on elastic foundation with single edge crack using differential quadrature method [7]. Kim obtained dynamic stiffnes matrix of non-symmetric thin-walled beams on elastic foundation by power series method [8]. The differential equation for bending of Timoshenko beam resting on Kerr-type three-parameter elastic foundation is obtained an analytically solved by Avramidis and Morfidis [9].

The differential transform method (DTM) which was introduced by Zhou in 1986 for the solution of initial value problems in electric circuit analysis is based on Taylor series expansions [10]. In recent works, DTM is applied to vibration analysis of continuous systems as beams and plates. Jang and Chen, the differential transformation method is applied to solve a second order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitations [11]. According to types of conditions at both end of a prismatic Bernoulli-Euler beam, frequency equations and fundamental frequencies of the beam have obtained using DTM by Malik and Dang [12]. Chen and Ho, using differential transform technique proposed a method to solve eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timeshenko beam under axial loading [13]. Özdemir and Kaya, flapwise bending vibration of a rotating tapered cantilever Bernoulli-Euler Beam is considered by using differential transform technique [14]. Kaya and Özgümüş, flexural-torsional-coupled vibration analysis of axially loaded closed-section composite Timoshenko beam is considered by using DTM [15]. Ruotolo and Surace calculated natural frequencies of a bar with many cracks using transfer matrix approach and finite element method [16]. Hosking studied natural flexural vibrations of Bernoulli-Euler beam mounted on discrete elastic supports using transfer matrices [17]. Coupling lateral and torsional vibrations of symmetric rotating shaft modeled by the Timoshenko beam is examined using modified TMM by Hsieh [18]. Free vibration of semi-rigidly connected piles embedded in soils with different subgrades problem are taken by Yesilce and Catal [19]. Differential transform method is used for free vibration analysis of a moving beam [20]. Demirdag and Yesilce, the problem of free vibration equation of elastically supported Timoshenko columns with a tip mass are solved by using differential transform method [21].

In this study, forth-order partial differential equations of motion for free vibration of Timoshenko beam on two different elastic foundations are developed considering P-Δ effect. These governing equations are solved using two different methods, the first being differential transform method (DTM) the other being transfer matrix method (TMM) approach, and frequency factors for the first three modes of the beam are obtained and presented in tables.

2. THE MATHEMATICAL MODEL

A Timoshenko beam with total length of L and with lengths of L_1 and L_2 on elastic foundations respectively called as the first and the second regions and having modulus of subgrade reactions of C_{r1} and C_{r2} is presented in Figure 1a; whereas internal forces and deformations of differential beam segment of the first and the second regions in Figures 1b and c.

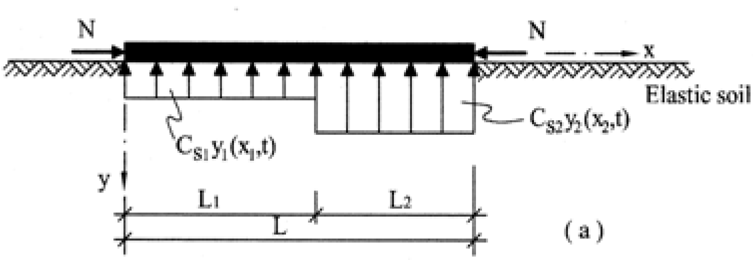

Figure 1a. Beam on elastic foundation

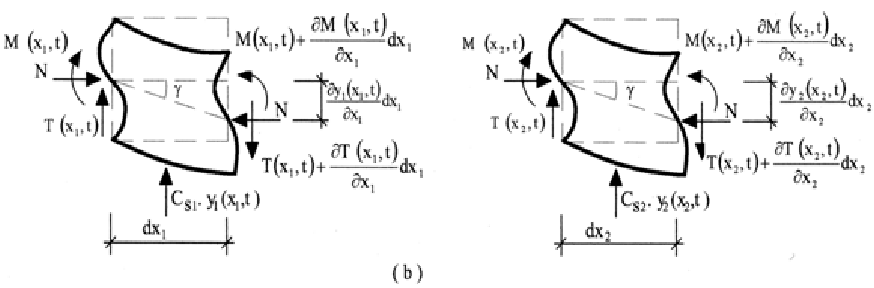

Figure 1b. Internal forces and deformations of segments of the beam in first and second regions.

The relation between the distributed forces acting on differential beam segment of the first and the second regions and the elastic curve functions of the beam are written as $q_1(x_1,t) = C_{s1}*y(x_1,t)$ and $q_2(x_2,t) = C_{s2}*y(x_2,t)$ according to Winkler hypothesis where $C_{s1} = C_{r1}*b$, $C_{s2} = C_{r2}*b$, $y_1(x_1,t)$ and $y_2(x_2,t)$ are elastic curve functions respectively at the first and the second regions, b is beam width. Equations of motion for the first and the second regions of Timoshenko beam on elastic foundation are obtained by using the equilibrium equations of forces and moments acting to differential beam segments of the first and the second regions, and by considering also P-Δ effect under the assumptions that cross-section and density of the beam is constant and the beam is made of linear elastic material, respectively as in the following [6].

$$\frac{\partial^4 y_1(x_1,t)}{\partial x_1^4} + \left[\frac{N}{EI_x} - \frac{kC_{s1}}{AG}\right]\frac{\partial^2 y_1(x_1,t)}{\partial x_1^2} - \frac{mk}{AG}\frac{\partial^4 y_1(x_1,t)}{\partial x_1^2 \partial t^2} + \frac{m}{EI_x}\frac{\partial^2 y_1(x_1,t)}{\partial t^2} + \frac{C_{s1}}{EI_x} y_1(x_1,t) = 0 \quad (0 \le x_1 \le L_1) \tag{1}$$

$$\frac{\partial^4 y_2(x_2,t)}{\partial x_2^4} + \left[\frac{N}{EI_x} - \frac{kC_{s2}}{AG}\right]\frac{\partial^2 y_2(x_2,t)}{\partial x_2^2} - \frac{mk}{AG}\frac{\partial^4 y_2(x_2,t)}{\partial x_2^2 \partial t^2} + \frac{m}{EI_x}\frac{\partial^2 y_2(x_2,t)}{\partial t^2} + \frac{C_{s2}}{EI_x} y_2(x_2,t) = 0 \quad (0 \le x_2 \le L_2) \tag{2}$$

Writing the dimensionless parameters z_1, z_2 instead of the position parameters x_1, x_2 and $y_1(z_1,t) = \phi(z_1).\sin(\omega t + \theta)$, $y_2(z_2,t) = \phi(z_2).\sin(\omega t + \theta)$ instead of the elastic curve functions in equations (1) and (2) gives the equation of motion for the beam at the first and the second regions as

$$\left\{\phi_1^{IV}(z_1) + \left[\pi^2 N_r + \frac{(m\omega^2 - Cs_1)\bar{k}L^2}{AG}\right]\phi_1^{II}(z_1) + \frac{(Cs_1 - m\omega^2)L^4}{EI}\phi_1(z_1)\right\}\mathrm{Sin}(\omega t + \theta) = 0 \tag{3}$$

$$0 \le z_1 \le L_1 / L$$

$$\left\{\phi_2^{IV}(z_2) + \left[\pi^2 N_r + \frac{(m\omega^2 - Cs_2)\bar{k}L^2}{AG}\right]\phi_2^{II}(z_2) + \frac{(Cs_2 - m\omega^2)L^4}{EI}\phi_2(z_2)\right\}\mathrm{Sin}(\omega t + \theta) = 0 \tag{4}$$

$$0 \le z_2 \le L_2 / L$$

where $\phi_1(z_1)$ and $\phi_2(z_2)$ are dimensionless displacement functions of the beam in the first and the second region, respectively; t is time variable; θ is phase angle; $Nr = N L^2 / (\pi^2 EI)$ is the ratio of axial load N acting to the beam to Euler buckling load; m is distributed mass of the beam; ω is beam circular frequency; $\bar{k}$ is shape factor due to cross-section area of the beam. N is constant axial compressive force, L_1 and L_2 are length of the beam in the first and the second region, respectively; L is total length of the beam, A, G, E, I are respectively cross-section area, shear modulus, elastic modulus and moment of inertia of the beam respectively.

3. DIFFERENTIAL TRANSFORMATION

The differential transformation technique, which was first proposed by Zhou in 1986 [10], is one of the numerical methods for ordinary and partial differential equations that use the form of polynomials as the approximation to the exact solutions that are sufficiently differentiable. The function that will be solved and the calculation of following derivatives necessary in the solution become more difficult when the order increases. This is in contrast with the traditional high-order Taylor series method. Instead, the differential transform technique provides an iterative procedure to obtain higher-order series; therefore, it can be applied to the case high order.

The differential transformation of the function $\phi(z)$ is defined as follows:

$$\Phi(k) = \frac{1}{k!}\left[\frac{d^k\phi(z)}{dz^k}\right]_{z=z_0} \tag{5}$$

Where $\phi(z)$ is the original function and $\Phi(k)$ is transformed function which is called the T-function (it is also called the spectrum of the $\phi(z)$ at $z = z_0$, in the K domain). The differential inverse transformation of $\Phi(k)$ is defined as:

$$\phi(z) = \sum_{k=0}^{\infty}(z - z_0)^k\,\Phi(k) \tag{6}$$

from Eq. (3) and Eq. (4) we get

$$\phi(k) = \sum_{k=0}^{\infty}\frac{(z - z_0)^k}{k!}\left[\frac{d^k\phi(z)}{dz^k}\right]_{z=z_0} \tag{7}$$

Equation (6) implies that the concept of the differential transformation is derived from Taylor’s series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions.

The basic operations of transformed functions which are given Table-1 can be easily proofed using equations (5) and (6).

The function is expressed by finite series and equation (6) can be written as $\phi(z) = \sum_{k=0}^{n} (z - z_0)^k \Phi(k)$. Equation (4) implies that $\phi(z) = \sum_{k=n+1}^{\infty} (z - z_0)^k \Phi(k)$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this paper.

Table 1. Some basic mathematical operations of DTM

Original function $\phi(z)$	Transformed function $\Phi(k)$
$A\phi(z)$	$a\Phi(k)$
$\phi_1(z) \pm \phi_2(z)$	$\Phi_1(k) \pm \Phi_2(k)$
$d\phi(z)/dz$	$(k+1)\ \Phi(k+1)$
$d^2\phi(z)/dz^2$	$(k+1)(k+2)\ \Phi(k+2)$
$d^3\phi(z)/dz^3$	$(k+1)(k+2)(k+3)\ \Phi(k+3)$
$d^4\phi(z)/dz^4$	$(k+1)(k+2)(k+3)(k+4)\ \Phi(k+4)$

4. SOLUTION OF EQUATIONS OF MOTION BY DIFFERENTIAL TRANSFORMATION METHOD

The boundary conditions of the Timeshenko beam resting on two different elastic foundation and both ends simply supported shown in Figure 2 are given in equations (8) - (15).

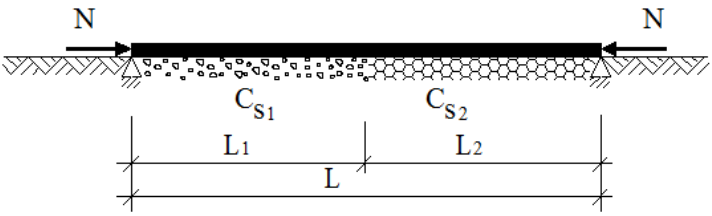

Figure 2. Both ends simply supported beam on elastic foundation.

$$\phi_1(z_1 = 0) = 0 \tag{8}$$

$$\phi_2(z_2 = L_2 / L) = 0 \tag{9}$$

$$\left.\frac{d^2\phi_1(z_1)}{dz_1{}^2}\right|_{z_1=0} = -C_1\phi_1(z_1 = 0) \tag{10}$$

$$\left.\frac{d^2\phi_2(z_2)}{dz_2{}^2}\right|_{z_2=\frac{L_2}{L}} = -C_2\phi_2(z_2 = L_2 / L) \tag{11}$$

$$\phi_1(z_1 = L_1 / L) = \phi_2(z_2 = 0) \tag{12}$$

$$\left.\frac{d\phi_1(z_1)}{dz_1}\right|_{z_1=\frac{L_1}{L}} = \left.\frac{d\phi_2(z_2)}{dz_2}\right|_{z_2=0} \tag{13}$$

$$\left.\frac{d^3\phi_1(z_1)}{dz_1{}^3}\right|_{z_1=\frac{L_1}{L}} + C_1\left.\frac{d\phi_1(z_1)}{dz_1}\right|_{z_1=\frac{L_1}{L}} = \left.\frac{d^3\phi_2(z_2)}{dz_2{}^3}\right|_{z_2=0} + \left.\frac{d\phi_2(z_2)}{dz_2}\right|_{z_2=0} \tag{14}$$

$$\left.\frac{d^2\phi_1(z_1)}{dz_1{}^2}\right|_{z_1=\frac{L_1}{L}} + C_1\phi_1(z_1 = L_1 / L) = \left.\frac{d^2\phi_2(z_2)}{dz_2{}^2}\right|_{z_2=0} + C_2\phi_2(z_2 = 0) \tag{15}$$

By applying the DTM to equations (3),(4),(8),(10) and using the relationship in Table-1 following equations are obtained.

$$\Phi_1(k+4) = -C_1\frac{\Phi_1(k+2)}{(k+3)(k+4)} - D_1\frac{\Phi_1(k)}{(k+1)(k+2)(k+3)(k+4)} \tag{16}$$

$$\Phi_2(k+4) = -C_2\frac{\Phi_2(k+2)}{(k+3)(k+4)} - D_2\frac{\Phi_2(k)}{(k+1)(k+2)(k+3)(k+4)} \tag{17}$$

$$\Phi_1(0) = 0 \tag{18}$$

$$\Phi_1(2) = 0 \tag{19}$$

$\gamma = \sqrt[4]{\dfrac{m\omega^2 L^4}{EI}}$, γ being the frequency factor. Where

$$\lambda_1 = \frac{Cs_1 L^4}{EI}, \quad \lambda_2 = \frac{Cs_2 L^4}{EI}$$

$$C_1 = \pi^2 N_r + \frac{(m\omega^2 - Cs_1)\bar{k}L^2}{AG}, C_2 = \pi^2 N_r + \frac{(m\omega^2 - Cs_2)\bar{k}L^2}{AG}, D_1 = \lambda_1 - \gamma^4, D_2 = \lambda_2 - \gamma^4$$

The recurrence relations of the first region for k = 0(1)n are obtained from equation (16) using equations (18) and (19) as follows:

$$\left.\begin{aligned}
&\Phi_1(2k) = 0\\
&\Phi_1(5) = \frac{1}{5!}\left\{-C_1 3!\Phi_1(3) - D_1\Phi_1(1)\right\}\\
&\Phi_1(7) = \frac{1}{7!}\left\{\left(C_1^2 - D_1\right)3!\Phi_1(3) - C_1 D_1\Phi_1(1)\right\}\\
&\Phi_1(9) = \frac{1}{9!}\left\{\left(-C_1^3 + 2C_1 D_1\right)3!\Phi_1(3) + \left(-C_1^2 D_1 + D_1^2\right)\Phi_1(1)\right\}\\
&\Phi_1(11) = \frac{1}{11!}\left\{\left(C_1^4 - 3C_1^2 D_1 + D_1^2\right)3!\Phi_1(3) + \left(C_1^3 - 2C_1 D_1^2\right)\Phi_1(1)\right\}\\
&\Phi_1(13) = \frac{1}{13!}\left\{\left(-C_1^5 + 4C_1^3 D_1 - 3C_1 D_1^2\right)3!\Phi_1(3) + \left(-C_1^4 D_1 + 3C_1^2 D_1^2 - D_1^3\right)\Phi_1(1)\right\}\\
&\quad\vdots
\end{aligned}\right\} \tag{20}$$

The recurrence relations of the second region for k = 0(1) n are obtained from Eq. (17) as:

$$\left.\begin{aligned}
\Phi_2(4) &= \frac{1}{4!}\left\{-C_2 2!\Phi_2(2) - D_2\Phi_2(0)\right\}\\
\Phi_2(5) &= \frac{1}{5!}\left\{-C_2 3!\Phi_2(3) - D_2\Phi_2(1)\right\}\\
\Phi_2(6) &= \frac{1}{6!}\left\{\left(C_2^2 - D_2\right)2!\Phi_2(2) + \left(C_2D_2\right)\Phi_2(0)\right\}\\
\Phi_2(7) &= \frac{1}{7!}\left\{\left(C_2^2 - D_2\right)3!\Phi_2(3) + \left(C_2D_2\right)\Phi_2(1)\right\}\\
\Phi_2(8) &= \frac{1}{8!}\left\{\left(-C_2^3 + 2C_2D_2\right)2!\Phi_2(2) + \left(-C_2^2D_2 + D_2^2\right)\Phi_2(0)\right\}\\
\Phi_2(9) &= \frac{1}{9!}\left\{\left(-C_2^3 + 2C_2D_2\right)3!\Phi_2(3) + \left(-C_2^2D_2 + D_2^2\right)\Phi_2(1)\right\}\\
\Phi_2(10) &= \frac{1}{10!}\left\{\left(C_2^4 - 3C_2^2D_2 + D_2^2\right)2!\Phi_2(2) + \left(C_2^3D_2 - 2C_2D_2^2\right)\Phi_2(0)\right\}\\
\Phi_2(11) &= \frac{1}{11!}\left\{\left(C_2^4 - 3C_2^2D_2 + D_2^2\right)3!\Phi_2(3) + \left(C_2^3D_2 - 2C_2D_2^2\right)\Phi_2(1)\right\}\\
\Phi_2(12) &= \frac{1}{12!}\left\{\left(-C_2^5 + 4C_2^3D_2 - 3C_2D_2^2\right)2!\Phi_2(2) + \left(-C_2^4D_2 + 3C_2^2D_2^2 - D_2^3\right)\Phi_2(0)\right\}\\
\Phi_2(13) &= \frac{1}{13!}\left\{\left(-C_2^5 + 4C_2^3D_2 - 3C_2D_2^2\right)3!\Phi_2(3) + \left(-C_2^4D_2 + 3C_2^2D_2^2 - D_2^3\right)\Phi_2(1)\right\}\\
&\vdots
\end{aligned}\right\} \tag{21}$$

By applying the DTM to equations (9), (11), (12), (13), (14), (15) and using the recurrence relations (20), (21) following equations are obtained

$$b_{11}\Phi_2(0) + b_{12}\Phi_2(1) + b_{13}2!\Phi_2(2) + b_{14}3!\Phi_2(3) = 0 \tag{22}$$

$$b_{21}\Phi_2(0) + b_{22}\Phi_2(1) + b_{23}2!\Phi_2(2) + b_{24}3!\Phi_2(3) = 0 \tag{23}$$

$$b_{35}\Phi_1(1) + b_{36}3!\Phi_1(3) = \Phi_2(0) \tag{24}$$

$$b_{45}\Phi_1(1) + b_{46}3!\Phi_1(3) = \Phi_2(1) \tag{25}$$

$$b_{55}\Phi_1(1) + b_{56}3!\Phi_1(3) = 3!\Phi_2(3) + C_2\Phi_2(1) \tag{26}$$

$$b_{65}\Phi_1(1) + b_{66}3!\Phi_1(3) = 2!\Phi_2(2) + C_2\Phi_2(0) \tag{27}$$

where

$$b_{11} = 1 + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_2^{k-2m}D_2^m(-1)^m\right\}$$

$$b_{12} = \frac{L_2}{L} + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_2^{k-2m}D_2^m(-1)^m\right\}$$

$$b_{13} = \sum_{k=1}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}C_2^{k-2m+1}D_2^{m-1}(-1)^m\right\}$$

$$b_{14} = \sum_{k=1}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}C_2^{k-2m+1}D_2^{m-1}(-1)^m\right\}$$

$$b_{21} = C_2 + \left(\frac{L_2}{L}\right)^{2}\frac{-D_2}{2!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}C_2^{k-2m-1}D_2^{m+1}(-1)^m\right\}$$

$$b_{22} = \left(\frac{L_2}{L}\right)C_2 + \left(\frac{L_2}{L}\right)^{3}\frac{-D_2}{3!} + \sum_{k=3}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}C_2^{k-2m-1}D_2^{m+1}(-1)^m\right\}$$

$$b_{23} = 1 + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_2^{k-2m}D_2^{m}(-1)^m\right\}$$

$$b_{24} = \frac{L_2}{L} + \sum_{k=2}^{n}\left(\frac{L_2}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_2^{k-2m}D_2^{m}(-1)^m\right\}$$

$$b_{35} = \frac{L_1}{L} + \sum_{k=2}^{n}\left(\frac{L_1}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_1^{k-2m}D_1^{m}(-1)^m\right\}$$

$$b_{36} = \sum_{k=1}^{n}\left(\frac{L_1}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}C_1^{k-2m+1}D_1^{m-1}(-1)^m\right\}$$

$$b_{45} = 1 + \sum_{k=2}^{n}\left(\frac{L_1}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_1^{k-2m}D_1^{m}(-1)^m\right\}$$

$$b_{46} = \sum_{k=1}^{n}\left(\frac{L_1}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m-1}\binom{k-m}{m-1}C_1^{k-2m+1}D_1^{m-1}(-1)^m\right\}$$

$$b_{55} = C_1 + \left(\frac{L_1}{L}\right)^{2}\frac{-D_1}{2!} + \sum_{k=3}^{n}\left(\frac{L_1}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}C_1^{k-2m-1}D_1^{m+1}(-1)^m\right\}$$

$$b_{56} = 1 + \sum_{k=2}^{n}\left(\frac{L_1}{L}\right)^{2k}\frac{(-1)^k}{(2k)!}\left\{\sum_{m=1}^{k\geq 2m}\binom{k-m-1}{m-1}C_1^{k-2m}D_1^{m}(-1)^m\right\}$$

$$b_{65} = \left(\frac{L_1}{L}\right)C_1 + \left(\frac{L_1}{L}\right)^{3}\frac{-D_1}{3!} + \sum_{k=3}^{n}\left(\frac{L_1}{L}\right)^{2k+1}\frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m+1}\binom{k-m-2}{m-1}C_1^{k-2m-1}D_1^{m+1}(-1)^m\right\}$$

$$b_{66} = \left(\frac{L_1}{L}\right) + \sum_{k=2}^{n}\left(\frac{L_1}{L}\right)^{2k+1} \frac{(-1)^k}{(2k+1)!}\left\{\sum_{m=1}^{k\geq 2m}\begin{pmatrix} k-m-1 \\ m-1 \end{pmatrix} C_1^{k-2m} D_1^m (-1)^m\right\}$$

Substituting equations (24) and (25) into equations (26) and (27), respectively, gives:

$$3!\Phi_2(3) = (b_{55} - C_2 b_{45})\Phi_1(1) + (b_{56} - C_2 b_{46})3!\Phi_1(3) \qquad (28)$$

$$2!\Phi_2(2) = (b_{65} - C_2 b_{35})\Phi_1(1) + (b_{66} - C_2 b_{36})3!\Phi_1(3) \qquad (29)$$

Substituting equations (24),(25),(28) and (29) into equations (22) and (23), respectively, gives:

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{Bmatrix} \Phi_1(1) \\ 3!\Phi_1(3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (30)$$

where

$B_{11} = b_{11}b_{35} + b_{12}b_{45} + b_{13}(b_{65} - C_2 b_{35}) + b_{14}(b_{55} - C_2 b_{45})$

$B_{12} = b_{11}b_{36} + b_{12}b_{46} + b_{13}(b_{66} - C_2 b_{36}) + b_{14}(b_{56} - C_2 b_{46})$

$B_{21} = b_{21}b_{35} + b_{22}b_{45} + b_{23}(b_{65} - C_2 b_{35}) + b_{24}(b_{55} - C_2 b_{45})$

$B_{22} = b_{21}b_{36} + b_{22}b_{46} + b_{23}(b_{66} - C_2 b_{36}) + b_{24}(b_{56} - C_2 b_{46})$

Thus, the frequency equation of the beam resting on elastic foundation is obtained using Eq. (30) as:

$$f^{(n)} = B_{11} B_{22} - B_{12} B_{21} = 0 \qquad (31)$$

Solving (31) we get $\omega = \omega_i^{(n)}$, i = 1, 2, 3,… where $\omega_i^{(n)}$ is the nth estimated ω circular frequency corresponding to n, and n is indicated by

$$\left|\omega_i^{(n)} - \omega_i^{(n-1)}\right| \leq \varepsilon$$

where $\omega_i^{(n-1)}$ is the ith estimated circular frequency corresponding to n-1 and ε is a positive and small value.

5. TRANSFER MATRIX METHOD

The relations of displacement and internal force vector between the simply supported right end and the left end of the beam in the first region, and between the simply supported left end and the right end of the beam in the second region are as in the following [22].

$$\{U(\frac{L_1}{L})\} = [H_1]\{U(0)\} \tag{32}$$

$$\{U(\frac{L_2}{L})\} = [H_2]\{U(\frac{L_1}{L})\} \tag{33}$$

where $\{U(0)\}$, $\{U(\frac{L_1}{L})\}$, $\{U(\frac{L_2}{L})\}$ are displacement and force vectors of the beam respectively at the positions of z_1=0 ,z_1=L_1/L , z_2=L_2/L and are as in the following.

$$\{U(z_1 = 0)\}^T = \{0 \quad \frac{d\phi_1(0)}{dz_1} \quad 0 \quad T(z_1 = 0)\} \tag{34}$$

$$\{U(z_1 = \frac{L_1}{L})\}^T = \{\phi(z_1 = \frac{L_1}{L}) \quad \frac{d\phi_1(\frac{L_1}{L})}{dz_1} \quad M(z_1 = \frac{L_1}{L}) \quad T(z_1 = \frac{L_1}{L})\} \tag{35}$$

$$\{U(z_2 = \frac{L_2}{L})\}^T = \{0 \quad \frac{d\phi_2(\frac{L_2}{L})}{dz_2} \quad 0 \quad T(z_2 = \frac{L_2}{L})\} \tag{36}$$

$[H_1]$ and $[H_2]$ are transfer matrices respectively for the first and the second regions.

$\phi_1(z_1)$ function in the relations (34), (35) and $\phi_2(z_2)$ function in the relation (36) are obtained according to the signs of parameters S_1, S_2, S_3 ,S_4 ,S_5 and S_6 from the solutions of differential equations (3) and (4). Following five conditions exist according to the signs of parameters S_1, S_2, S_3 ,S_4 ,S_5 and S_6 [6].

I. Condition: in the first region $S_1>0$, $S_2>0$ ve $S_3>0$

$$\phi_1(z_1) = \left[C_3 \cosh(D_3 z_1) + C_4 \sinh(D_3 z_1) + C_5 \cosh(D_4 z_1) + C_6 \sinh(D_4 z_1)\right] \tag{37}$$

$(0 \leq z_1 \leq L_1/L)$

in the second region $S_4>0$, $S_5>0$ ve $S_6>0$

$$\phi_2(z_2) = \left[C_7 \cosh(D_5 z_2) + C_8 \sinh(D_5 z_2) + C_9 \cosh(D_6 z_2) + C_{10} \sinh(D_6 z_2)\right] \quad (38)$$

$(0 \leq z_2 \leq L_2/L)$

II. Condition: in the first region $S_1>0$, $S_2>0$ ve $S_3<0$

$$\phi_1(z_1) = \left[C_3 \cosh(D_3 z_1) + C_4 \sinh(D_3 z_1) + C_5 \cos(D_4 z_1) + C_6 \sin(D_4 z_1)\right] \quad (39)$$

$(0 \leq z_1 \leq L_1/L)$

in the second region; $S_4>0$, $S_5>0$ ve $S_6<0$

$$\phi_2(z_2) = \left[C_7 osh(D_5 z_2) + C_8 \sinh(D_5 z_2) + C_9 \cos(D_6 z_2) + C_{10} \sin(D_6 z_2)\right] \quad (40)$$

$(0 \leq z_2 \leq L_2/L)$

III. Condition: in the first region $S_1>0$, $S_2<0$ ve $S_3>0$

$$\phi_1(z_1) = \left[C_3 \cos(D_3 z_1) + C_4 \sin(D_3 z_1) + C_5 \cosh(D_4 z_1) + C_6 \sinh(D_4 z_1)\right] \quad (41)$$

$(0 \leq z_1 \leq L_1/L)$

in the second region; $S_4>0$, $S_5<0$ ve $S_6>0$

$$\phi_2(z_2) = \left[C_7 \cos(D_5 z_2) + C_8 \sin(D_5 z_2) + C_9 \cosh(D_6 z_2) + C_{10} \sinh(D_6 z_2)\right] \quad (42)$$

$(0 \leq z_2 \leq L_2/L)$

IV. Condition: in the first region; $S_1>0$, $S_2<0$ ve $S_3<0$

$$\phi_1(z_1) = \left[C_3 \cos(D_3 z_1) + C_4 \sin(D_3 z_1) + C_5 \cos(D_4 z_1) + C_6 \sin(D_4 z_1)\right] \quad (43)$$

$(0 \leq z_1 \leq L_1/L)$

in the second region; $S_4>0$, $S_5<0$ ve $S_6<0$

$$\phi_2(z_2) = \left[C_7 \cos(D_5 z_2) + C_8 \sin(D_5 z_2) + C_9 \cos(D_6 z_2) + C_{10} \sin(D_6 z_2)\right] \quad (44)$$

V. Condition: in the first region; $S_1<0$

$$\phi_1(z_1)=\{C_3[\cosh(r_1\alpha_1 z_1)\cos(r_1\alpha_2 z_1)]+C_4[\sinh(r_1\alpha_1 z_1)\cos(r_1\alpha_2 z_1)]$$

$$+C_5[\cosh(r_1\alpha_1 z_1)\sin(r_1\alpha_2 z_1)]+C_6[\sinh(r_1\alpha_1 z_1)\sin(r_1\alpha_2 z_1)]\}; \quad (45)$$

$$(0\le z_1\le L_1/L)$$

in the second region; $S_2<0$

$$\phi_2(z_2)=\{C_7[\cosh(r_2\alpha_3 z_2)\cos(r_2\alpha_4 z_2)]+C_8[\sinh(r_2\alpha_3 z_1)\cos(r_2\alpha_4 z_2)]$$

$$+C_9[\cosh(r_2\alpha_3 z_2)\sin(r\alpha_4 z_2)]+C_{10}[\sinh(r_2\alpha_3 z_2)\sin(r_2\alpha_4 z_2)]\} \quad (46)$$

$$(0\le z_2\le L_2/L)$$

where, $D_3 = S_2^{0.5}$; $D_4 = S_3^{0.5}$; $D_5 = S_5^{0.5}$; $D_6 = S_6^{0.5}$; $\beta_1 = \gamma^4 - \lambda_1$; $\beta_2 = \gamma^4 - \lambda_2$

$\alpha_1=\sin(\theta_1/2)$; $\alpha_2=\cos(\theta_1/2)$; $r_1=(-\beta_1)^{0.25}$;

$r_2=(-\beta_2)^{0.25}$; $\alpha_3=\sin(\theta_2/2)$

$$\theta_1=\mathrm{Arctg}\left\{\frac{-2\left[-\left(\frac{C_1}{2}\right)^2-\beta_1)\right]^{0.5}}{C_1}\right\};\quad \theta_2=\mathrm{Arctg}\left\{\frac{-2\left[-\left(\frac{C_2}{2}\right)^2-\beta_2)\right]^{0.5}}{C_2}\right\};$$

$\alpha_4=\cos(\theta_2/2)$; $S_1=(\frac{C_1}{2})^2+\gamma^4-\lambda_1$; $S_2=-\frac{C_1}{2}-(S_1)^{0.5}$;

$S_3=-\frac{C_1}{2}+(S_1)^{0.5}$; $S_4=(\frac{C_2}{2})^2+\gamma^4-\lambda_2$; $S_5=-\frac{C_2}{2}-(S_4)^{0.5}$;

$S_6=-\frac{C_2}{2}+(S_4)^{0.5}$; $_3,C_4,\ldots,C_{10}$ are integration constants.

Shear force and bending moment functions $T(z_1)$, $M(z_1)$ in the relations (34), (35) and shear force function $T(z_2)$ in the relation (36) are obtained using the relation between the derivatives of elastic curve and the internal forces of the beam as in the following [6].

$$T_1(z_1)=\left\{-\frac{EI_x}{L^3}\frac{\partial^3\phi_1(z_1)}{\partial z_1^3}+\left[\frac{kEI_x}{AG}(C_{s1}-mw^2)-N\right]\frac{1}{L}\frac{\partial\phi_1(z_1)}{\partial z_1}\right\}\ (0\leq\ z_1\leq L_1/L\ ) \quad (47)$$

$$T_2(z_2)=\left\{-\frac{EI_x}{L^3}\frac{\partial^3\phi_2(z_2)}{\partial z_2^3}+\left[\frac{kEI_x}{AG}(C_s2-mw^2)-N\right]\frac{1}{L}\frac{\partial\phi_2(z_2)}{\partial z_2}\right\}(0\leq\ z_2\leq L_2/L\ ) \quad (48)$$

$$M_1(z_1)=\left\{-\frac{EI_x}{L^2}\frac{\partial^2\phi_1(z_1)}{\partial z_1^2}+\left[\frac{kEI_x}{AG}(C_{s1}-mw^2)-N\right]\phi_1(z_1)\right\}(0\leq\ z_1\leq L_1/L\ ) \quad (49)$$

Writing the value of $\{U(\frac{L_1}{L})\}$ form the equation (32) in equation (33) gives

$$\{U(\frac{L_2}{L})\}=[H]\{U(0)\} \quad (50)$$

where $[H]=[H_1][H_2]=\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$ and is 4x4 matrix that transforms displacements and forces in the left end of the beam in the first region to displacements and forces in the right end of the beam in the second region [22]. The 2x2 homogeneous matrix equations are obtained from equation (50) according to the boundary conditions of the beam ends as in the following.

$$\begin{bmatrix} h_{12} & h_{14} \\ h_{32} & h_{34} \end{bmatrix}\begin{pmatrix} \dfrac{d\phi_2(z_2=\frac{L_2}{L})}{dz_2} \\ T(z_2=\frac{L_2}{L}) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (51)$$

Frequency equation of the both ends simply supported beam on elastic foundation is obtained equating the determinant of the coefficient matrix in the matrix equation (51) to zero as in the following.

$$h_{12} * h_{34} - h_{32} * h_{14} = 0 \qquad (52)$$

Thus, circular frequency values obtained using equation (52) is the eigenvalues of the beam.

6. NUMERICAL ANALYSIS

Frequency factor values (γ) for the first three modes of the both ends simply supported IPB 900 steel beam resting on two different elastic foundation are calculated considering three groups of modulus of subgrade reactions and using DTM and transfer matrix method for parametric studies in this paper. I., II. and III. groups of modulus of subgrade reactions are considered as respectively C_{s1}=70000 kN/m^2, C_{s2}= 0 kN/m^2; C_{s1}=70000 kN/m^2, C_{s2} = 20000 kN/m^2 and C_{s1}=70000 kN/m^2, C_{s2}=50000 kN/m^2. 0.25, 0.5, 0.75 values are taken for both N_r and L_1/L in the study. Characteristics of the steel IPB 900 profile used in the numerical analysis are presented in the following.

$L = 6$ m; $I = 444.1*10^{-5}$ m^4; $A = 3.71*10^{-2}$m^2; $m = 0.296$ Nsec^2/m^2; $\bar{k}$ = 2.55; $E = 2.1*10^8$ kN/m^2; $G = 8.1*10^7$ kN/m^2

Frequency factor values are calculated according to N_r, L_1/L and series size (n) values using DTM and according to N_r and L_1/L values using transfer matrix method; and the values obtained for respectively C_{s1}=70000 kN/m^2, C_{s2} = 0 kN/m^2; C_{s1}=70000 kN/m^2, C_{s2} = 20000 kN/m^2 and C_{s1}= 70000 kN/m^2, C_{s2} = 50000 kN/m^2 are presented in Tables 2, 3 and 4.

Frequency factor values for the third mode cannot be calculated for each modulus of subgrade reaction considered in numerical analysis and n = 2 using DTM.

Frequency factor values obtained for the first mode using DTM for series size n = 2 and n>2 are same. DTM results indicate that frequency factor values of the first mode are very fast converging for each N_r, L_1/L,

C_{s1}, C_{s2} value, and that converging speed decreases as the number of modes increase.

It is seen from Tables-2, 3 and 4 that all frequency factors obtained using TMM and obtained using DTM for n = 16 overlap.

7. CONCLUSION

Eigenvalues for the first three modes of the both ends simply supported Timoshenko beam resting on two different foundations are calculated using DTM and TMM according to the axial compressive force, modulus of subgrade reactions and variation of L_1/L values.

Frequency factor values of all modes increase for each N_r and L_1/L values as the modulus of subgrade reaction C_{S2} increases.

Frequency factor values of all modes decrease for each C_{s1} and C_{s2} value as L_1/L value remains constant and axial compressive force increases. This variation in frequency factors is clearer in the first and the second modes.

Frequency factor values of all modes decrease for each C_{s1} and C_{s2} value as N_r value remains constant and L_1/L value increases.

Nr	METHOD	n	$L_1/L = 0.25$			$L_1/L = 0.50$			$L_1/L = 0.75$		
			γ_1	γ_2	γ_3	γ_1	γ_2	γ_3	γ_1	γ_2	γ_3
0.25	**D T M**	**2**	7.64419889	7.70511430	-	3.20497274	5.20944023	-	3.61614609	3.80194879	-
		4	2.85668182	4.86087608	7.37156820	3.16665697	5.30849934	6.56168461	3.42784381	5.00870800	7.31811142
		8	2.85533524	5.25938606	7.11325979	3.16665697	5.29980135	7.04365253	3.42756343	5.33288670	7.11461258
		6	2.85533524	5.23997831	7.58061171	3.16665697	5.29980135	7.01819420	3.42756343	5.31899071	7.52173376
		10	2.85533524	5.25938606	7.03285074	3.16665697	5.29980135	7.04392576	3.42756343	5.33308894	7.05716515
		12	2.85533524	5.25938606	7.03175592	3.16665697	5.29980135	7.04392576	3.42756343	5.33308894	7.05634689
		14	2.85533524	5.25938606	7.03175592	3.16665697	5.29980135	7.04392576	3.42756343	5.33308894	7.05634689
		16	2.85533524	5.25938606	7.03175592	3.16665697	5.29980135	7.04392576	3.42756343	5.33308894	7.05634689
	TMM		2.85533524	5.25938606	7.03175592	3.16665697	5.29980135	7.04392576	3.42756343	5.33308894	7.05634689
0.50	**D T M**	**2**	7.60216188	7.66657495	-	3.04089999	5.12161446	-	3.47380137	3.68799472	-
		4	2.61820149	4.76037741	7.32639122	2.99887466	5.22362804	6.50408792	3.29896164	4.91710281	7.27194071
		6	2.61636472	5.15211344	7.53809214	2.99855399	5.21478844	6.96796560	3.29867029	5.23520708	7.47823620
		8	2.61636472	5.17203665	7.06438875	2.99855399	5.21478844	6.99374437	3.29867029	5.24932480	7.06575060
		10	2.61636472	5.17222214	6.98272753	2.99855399	5.21478844	6.99401951	3.29867029	5.24950790	7.00762796
		12	2.61636472	5.17222214	6.98162508	2.99855399	5.21478844	6.99401951	3.29867029	5.24950790	7.00680399
		14	2.61636472	5.17222214	6.98162508	2.99855399	5.21478844	6.99401951	3.29867029	5.24950790	7.00680399
		16	2.61636472	5.17222214	6.98162508	2.99855399	5.21478844	6.99401951	3.29867029	5.24950790	7.00680399
	TMM		2.61636472	5.17222214	6.98162508	2.99855399	5.21478844	6.99401951	3.29867029	5.24950790	7.00680399
0.75	**DTM**	**2**	7.55950928	7.62503958	-	2.84487772	5.02881718	-	3.31349707	3.56070113	-
		4	2.28694654	4.65296173	7.28040504	2.79647589	5.13454723	6.44478226	3.15296865	4.81997108	7.22480869
		6	2.28484344	5.05949545	7.49481821	2.79647589	5.12536669	6.91653776	3.15266371	5.14707422	7.43383694
		8	2.28484344	5.08016014	7.01449108	2.79647589	5.12536669	6.94278479	3.15266371	5.16161919	7.01586294
		10	2.28484344	5.08034945	6.93140936	2.79647589	5.12536669	6.94292307	3.15266371	5.16180515	6.95690823
		12	2.28484344	5.08034945	6.93029881	2.79647589	5.12536669	6.94292307	3.15266371	5.16180515	6.95607853
		14	2.28484344	5.08034945	6.93029881	2.79647589	5.12536669	6.94292307	3.15266371	5.16180515	6.95607853
		16	2.28484344	5.08034945	6.93029881	2.79647589	5.12536669	6.94292307	3.15266371	5.16180515	6.95607853
	TMM		2.28484344	5.08034945	6.93029881	2.79647589	5.12536669	6.94292307	3.15266371	5.16180515	6.95607853

Table-2: Frequency factors for the first, second and third modes of the beam resting on foundation having modulus of subgrade reaction of C_{s1}=70000 kN/m^2, C_{s2}=0 kN/m^2

Nr	METHOD	n	$L_1/L = 0.25$			$L_1/L = 0.50$			$L_1/L = 0.75$		
			γ_1	γ_2	γ_3	γ_1	γ_2	γ_3	γ_1	γ_2	γ_3
0.25	**D T M**	**2**	7.65702915	5.23392200	-	3.29808736	5.23392200	-	3.68569844	3.74771404	-
		4	3.07579494	5.32783747	7.37182903	3.26793242	5.32783747	6.57311392	3.44295192	7.33374310	7.33374310
		6	3.07454443	5.31917143	7.57959604	3.26793242	5.31917143	7.02723837	3.44267273	7.53758144	7.53758144
		8	3.07454443	5.31917143	7.12150764	3.26793242	5.31917143	7.05266380	3.44267273	7.12245369	7.12245369
		10	3.07454443	5.31917143	7.04515505	3.26793242	5.31917143	7.05280018	3.44267273	7.06248140	7.06248140
		12	3.07454443	5.31917143	7.04419899	3.26793242	5.31917143	7.05280018	3.44267273	7.06166363	7.06166363
		14	3.07454443	5.31917143	7.04419899	3.26793242	5.31917143	7.05280018	3.44267273	7.06166363	7.06166363
		16	3.07454443	5.31917143	7.04419899	3.26793242	5.31917143	7.05280018	3.44267273	7.06166363	7.06166363
	TMM		3.07454443	5.31917143	7.04419899	3.26793242	5.31917143	7.05280018	3.44267273	7.06166363	7.06166363
0.50	**D T M**	**2**	7.61518908	7.65489244	-	3.14869809	5.14726067	-	3.53659534	3.64236474	-
		4	2.89113998	4.81218767	7.32665396	3.11709166	5.24401236	6.51591396	3.31610680	4.92218256	7.28053713
		6	2.89014220	5.18688202	7.53694296	3.11709166	5.23502350	6.97721291	3.31552696	5.24547863	7.49366236
		8	2.89014220	5.20574903	7.07282925	3.11709166	5.23502350	7.00295734	3.31552696	5.26029968	7.02299166
		10	2.89014220	5.20593357	6.99525785	3.11709166	5.23502350	7.00309467	3.31552696	5.26029968	6.94430923
		12	2.89014220	5.20593357	6.99429464	3.11709166	5.23502350	7.00309467	3.31552696	5.26029968	6.94333887
		14	2.89014220	5.20593357	6.99429464	3.11709166	5.23502350	7.00309467	3.31552696	5.26029968	6.94333887
		16	2.89014220	5.20593357	6.99429464	3.11709166	5.23502350	7.00309467	3.31552696	5.26029968	6.94333887
	TMM		2.89014220	5.20593357	6.99429464	3.11709166	5.23502350	7.00309467	3.31552696	5.26029968	6.94333887
0.75	**D T M**	**2**	7.57260990	7.61316681	-	2.97440886	5.05607462	-	3.37500215	3.51815860	-
		4	2.66226411	4.70820093	7.28053713	2.94027257	5.15584326	6.45701504	3.17241859	4.82535219	7.24104071
		6	2.66118050	5.09621668	7.49366236	2.94027257	5.14670038	6.92599249	3.17211556	5.15789366	7.45025921
		8	2.66118050	5.11560583	7.02299166	2.94027257	5.14670038	6.95206547	3.17211556	5.17315149	7.02408791
		10	2.66118050	5.11579370	6.94430923	2.94027257	5.14670038	6.95234203	3.17211556	5.17315149	6.96160984
		12	2.66118050	5.11579370	6.94333887	2.94027257	5.14670038	6.95234203	3.17211556	5.17315149	6.96160984
		14	2.66118050	5.11579370	6.94333887	2.94027257	5.14670038	6.95234203	3.17211556	5.17315149	6.96160984
		16	2.66118050	5.11579370	6.94333887	2.94027257	5.14670038	6.95234203	3.17211556	5.17315149	6.96160984
	TMM		2.66118050	5.11579370	6.94333887	2.94027257	5.14670038	6.95234203	3.17211556	5.17315149	6.96160984

Table-3: Frequency factors for the first, second and third modes of the beam resting on foundation having modulus of subgrade reaction of C_{s1}=70000 kN/m^2, C_{s2}=20000 kN/m^2

Nr	METHOD	N	$L_1/L = 0.25$			$L_1/L = 0.50$			$L_1/L = 0.75$		
			γ_1	γ_2	γ_3	γ_1	γ_2	γ_3	γ_1	γ_2	γ_3
0.25	**D T M**	**2**	3.45286410	5.01472513	-	3.42307377	5.26851559	-	4.14953709	5.55713272	-
		4	3.33489347	4.98003340	7.37209034	3.40025377	5.35752153	6.59007502	3.46465850	5.02059174	7.35693216
		6	3.33431697	5.32151985	7.57781839	3.40025377	5.34890318	7.04078245	3.46438098	5.34350967	7.56116390
		8	3.33431697	5.33829069	7.13419867	3.40025377	5.34890318	7.06602335	3.46438098	5.35877705	7.13473797
		10	3.33431697	5.33847094	7.06370735	3.40025377	5.34890318	7.06615925	3.46438098	5.35877705	7.07051611
		12	3.33431697	5.33847094	7.06275368	3.40025377	5.34890318	7.06615925	3.46438098	5.35877705	7.06969929
		14	3.33431697	5.33847094	7.06275368	3.40025377	5.34890318	7.06615925	3.46438098	5.35877705	7.06969929
		16	3.33431697	5.33847094	7.06275368	3.40025377	5.34890318	7.06615925	3.46438098	5.35877705	7.06969929
	TMM		3.33431697	5.33847094	7.06275368	3.40025377	5.34890318	7.06615925	3.46438098	5.35877705	7.06969929
0.50	**D T M**	**2**	4.13642195	4.99713092	-	3.29079366	5.18354559	-	3.38104217	4.96401763	-
		4	3.19415832	4.88671064	7.32691669	3.26793242	5.27489710	6.53317070	3.34035617	4.92959738	7.31153345
		6	3.19325542	5.23777676	7.53528309	3.26793242	5.26614332	6.99099207	3.33978963	5.26084757	7.51827860
		8	3.19325542	5.25499821	7.08574343	3.26793242	5.26614332	7.01654863	3.33978963	5.27635431	7.08628654
		10	3.19325542	5.25518084	7.01421642	3.26793242	5.26614332	7.01668596	3.33978963	5.27653646	7.02121019
		12	3.19325542	5.25518084	7.01325607	3.26793242	5.26614332	7.01668596	3.33978963	5.27653646	7.02038765
		14	3.19325542	5.25518084	7.01325607	3.26793242	5.26614332	7.01668596	3.33978963	5.27653646	7.02038765
		16	3.19325542	5.25518084	7.01325607	3.26793242	5.26614332	7.01668596	3.33978963	5.27653646	7.02038765
	TMM		3.19325542	5.25518084	7.01325607	3.26793242	5.26614332	7.01668596	3.33978963	5.27653646	7.02038765
0.75	**D T M**	**2**	3.16403715	4.97130219	-	3.14044523	5.09433031	-	3.30571425	4.92170324	-
		4	3.03171706	4.78775787	7.28093386	3.11740017	5.18836451	6.47487450	3.20017076	4.83331299	7.26518822
		6	3.03076553	5.14987421	7.49199247	3.11740017	5.17927933	6.94015074	3.19987035	5.17408037	7.47463226
		8	3.03076553	5.16757441	7.03627062	3.11740017	5.17927933	6.96603203	3.19987035	5.19003153	7.03681803
		10	3.03076553	5.16757441	6.96368313	3.11740017	5.17927933	6.96630812	3.19987035	5.19021654	6.97086573
		12	3.03076553	5.16757441	6.96271563	3.11740017	5.17927933	6.96630812	3.19987035	5.19021654	6.96989918
		14	3.03076553	5.16757441	6.96257734	3.11740017	5.17927933	6.96630812	3.19987035	5.19021654	6.96989918
		16	3.03076553	5.16757441	6.96257734	3.11740017	5.17927933	6.96630812	3.19987035	5.19021654	6.96989918
	TMM		3.03076553	5.16757441	6.96257734	3.11740017	5.17927933	6.96630812	3.19987035	5.19021654	6.96989918

Table-4: Frequency factors for the first, second, third modes of the beam resting on foundation having modulus of subgrade reaction of C_{s1}=70000 kN/m^2, C_{s2}=50000 kN/

69

REFERENCES

[1] Yokoyoma Y., *Earthquake Engineering and Structure Dynamics*, **(1991)**, 20, p.355-370.
[2] Doyl, P.F., Pavlovic M.N., *Engineering and Structural Dynamics*, **(1982)**, 10, p.663-674.
[3] Chen C.N., *Journal of Engineering Mechanics*,**(1998)**, 124, p.1381-1384.
[4] Chen Y.H., Huang Y.H., *International Journal for Numerical Methods in Engineering* **(2000)**, 48, p.1-18.
[5] Karami G., Malekzadeh P., Shahpari S.A., *Engineering Structures*,**(2003)**, 25,p.1169-1178.
[6] Catal H.H., *Engineering Structures* **(2002)**, 24, p.1615-1622.
[7] Hsu M.H., *Computuer Methods in Applied Mechanics Engineering*, **(2005)**, 194, p.1-17.
[8] Kim N.I., Lee J.H., Kim M.Y., *Advances in Engineering Software* **(2005)**, 36, p.518-532.
[9] Avramidis I.E., Morfidis K., *International Journal of Solid and Structures*, **(2005),**43, p.357-375.
[10] Zhou J.K., **(1986)**, Differential transformation and its applications for electrical circuits, Wuhan China, Huazhong University Press.
[11] Jang M.J., Chen C.L., *Applied Mathematics and Computation*, **(1997)**, 88, p.137-151.
[12] Malik M., Dang, H.H., *Applied Mathematics and Computation* **(1998)**,96, p.17-26.
[13] Chen C.K., Ho S.H., **(1996)**, *Journal of Applied Mathematics and Computation*, 79, p.173-188.
[14] Özdemir Ö., Kaya M.O., *Journal of Sound on Vibration*,**(2006)**, 289, p.413-420.
[15] Kaya M.O, Ozgumus O.O., *Journal of Sound and Vibration*, **(2007)**, 306, p.495-506.
[16] Ruotolo R., Surace C., *Journal of Sound and Vibration*, **(2004)**, 272, p.301-316.
[17] Hosking R.J., Husain S.A., Milinazzo F., *Journal of sound and Vibration* **(2004)**, 272, p.169-185.
[18] Hsieh S.C., Chen J.H., Lee A.C., *Journal of Sound and Vibration*, **(2006)**, 209, p.294-333.
[19] Yesilce Y., Catal HH., *International Journal of Structural Stability and Dynamics* **(2008)**, 8(2), p.299-320.
[20] Yesilce Y., *Structural Engineering and Mechanic*, **(2010)**,35(5), p.645-658.
[21] Demirdag O., Yesilce Y., *Advances in Engineering Software* **(2011)**, 42(10), p.860-867.
[22] Zhou D., Ji T., *International Journal of Solid and Structures*, **(2006)**, 43, p.5555-5569.