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Mixed Variational-like Inequality Problems in Abstract Convex Spaces

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Abstract

P.K. Das and G.C. Nayak [3] introduced the concept of generalized variational like inequalities in H-spaces in the presence of T-η-invex function. There the existence theorems of mixed generalized variational like inequalities are studied in H-spaces and also in Riesz spaces in the presence of T-η-invex function. In the present note, we extend their results to partial KKM spaces, which contain H-spaces as very particular subclass.

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1. Introduction

The KKM theory is originated from the celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem in 1929. In 1961-1984, Ky Fan investigated various results in the theory on Hausdorff topological vector spaces. His results were elaborated and extended by many authors for various types of general spaces. Since 2006, such results have been unified and abstracted by Park's KKM theory on abstract convex spaces. For the history of such research, see [11].

For a long period, H-spaces (or c-spaces) due to Horvath [4] had been an interesting area of research domain for studying variational type inequality and other topics. However, it is well-known that certain results on H-spaces can be extended to more general spaces belonging abstract convex spaces; see Park [10,11,13,14].

Very recently, P.K. Das and G.C. Nayak [3] introduced the concept of generalized variational like inequalities in H-spaces in the presence of T-η-invex function. There the existence theorems of mixed generalized variational like inequalities are studied in H-spaces and also in Riesz spaces in the presence of T-η-invex function.

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In the present note, we extend their results to partial KKM spaces, which contain H-spaces as a very particular subclass. Actually, we can show that their results can be extended to more general and more clear versions.

2. Abstract convex spaces

Recall the following in [6-14] and the references therein.

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \to 2^E$ a multimap with nonempty values $\Gamma_N := \Gamma(N)$ for $N \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_N : N \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z be a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y\in N} G(y) \qquad \text{for all} \quad N \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. A multimap $F : E \multimap Z$ to a set Z is called *a* $\mathfrak{K}$-*map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

Recall that a topological space X is *homotopically trivial* if for any natural number n and any continuous function $f : \partial\Delta_n \to X$, defined on the boundary of the standard n-dimensional simplex Δ_n, there exists its continuous extension $g : \Delta_n \to X$.

Definition. A triple $(X \supset D; \Gamma)$ is called an H-*space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X; \Gamma) := (X, X; \Gamma)$ is called a *c*-space by Horvath [4] or an H-space by Bardaro - Ceppitelli [1, 2].

In case Γ is a family of homotopically trivial sets, then $(X \supset D; \Gamma)$ will be called a *Horvath space* which is more general than H-spaces and becomes clearly the well-known G-convex spaces due to Park [14].

Now the following diagram for triples $(E, D; \Gamma)$ is well-known:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space}$$

$$\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}$$

Note that some subclasses of abstract convex spaces have large numbers of examples; see Park [13,14].
Now we prepare to introduce one of the most general forms of the KKM theorem.
Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Note that Luc et al. showed (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d), and not conversely in each step.

The following is one of the most general KKM type theorems in [7] for abstract convex spaces:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*
(1) *$\overline{G}$ is a KKM map w.r.t. F; and*
(2) *there exists a nonempty compact subset K of Z such that either*
(i) *$K = Z$;*
(ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*
(iii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,
(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*
(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

From now on, we are mainly concerned with the partial KKM spaces of the form $(E; \Gamma)$ for the simplicity.

3. Various coercivity conditions

In this section, we consider some particular cases of the coercivity condition (iii) in Theorem C.
The following is a simplified form of (iii) in case $X = E = D = Z$ and $F = 1_E$:

(A) *Let $(X; \Gamma)$ be a partial KKM space having a compact subset $K \subset X$. For each $N \in \langle X\rangle$, there exists a compact Γ-convex subset $L_N \subset X$ relative to some $D \subset X$ such that $N \subset D$ and*

$$L_N \cap \bigcap_{y\in D} G(y) \subset K.$$

The following is an H-space version of (A):

(B) *Let $(X;\{\Gamma_A\})$ be an H-space having a compact subset $K \subset X$. For each $N \in \langle X\rangle$, there exists a compact Γ-convex subset $L_N \subset X$ relative to some $D \subset X$ such that $N \subset D$, $(L_N, \{\Gamma_A \cap D\})$ is an H-space, and*

$$L_N \cap \bigcap_{y\in D} G(y) \subset K.$$

A subset $Y \subset X$ is said to be *weakly H-convex* if $\Gamma_A \cap Y$ is nonempty and contractible for every finite subset $A \subset Y$. This is equivalent to saying that the pair $(Y, \{\Gamma_A \cap Y\})$ is an H-space.

A subset $L \subset X$ is said to be *H-compact* if there exists a compact and weakly H-convex set $Y \subset X$ such that $L \cup A \subset Y$ for every finite subset $A \subset X$.

In view of such definitions, the following follows from (B):

(C) *Let $(X;\{\Gamma_A\})$ be an H-space having a compact subset $K \subset X$ and an H-compact subset $L \subset X$, such that for each weakly H-convex subset Y with $L \subset Y \subset X$, we have*

$$Y \cap \bigcap_{x\in Y} G(x) \subset K.$$

Proof of (B)$\Longleftarrow$(C): For each $N \in \langle X\rangle$, since L is H-compact, there exists a compact weakly H-convex set $Y \subset X$ such that $L \cup N \subset Y$. If we let $L_N := Y, D := Y$, L_N is a compact Γ-convex subset of X relative to $D \subset X$ and for each $N \subset D$, $(L_N, \{\Gamma_A \cap D\})$ is an H-space. $Y \cap \bigcap_{x\in Y} G(x) \subset K$ clearly implies $L_N \cap \bigcap_{y\in D} G(y) \subset K$.

(D) *Let $(X;\{\Gamma_A\})$ be an H-space having a compact subset $L \subset X$ and an H-compact subset $K \subset X$, such that for each weakly H-convex set D with $K \subset D \subset X$, we have*

$$D \cap \bigcap_{x\in D} G(x) \subset L.$$

Note that (D) is the avatar of (C) by changing the symbols.

The following result of Theorem C has fundamental importance in partial KKM spaces:

Theorem 3.1. *Let $(X;\Gamma)$ be a partial KKM space and $G : X \multimap X$ a KKM multimap such that*
(a) *for each $x \in X$, $G(x)$ is closed,*
(b) *the condition (A) holds.*
Then, $\bigcap_{x\in X} G(x) \neq \emptyset$.

Corollary 3.2. ([1], Theorem 1, p.486) *Let $(X, \{\Gamma_A\})$ be an H-space and $G : X \multimap X$ an H-KKM multimap such that*
(a) *for each $x \in X$, $G(x)$ is compactly closed,*
(b) *the condition (D) holds.*
Then, $\bigcap_{x\in X} G(x) \neq \emptyset$.

Compactly closedness is a closedness in some weak topology, so we may take this topology as the topology of given partial KKM space X; see Park [12].

4. Generalized Variational Like Inequalities in Partial KKM Spaces

In this section we follow Section 3 of [3] and present an application of Theorem 3.1. In fact we establish an inequality associated with the variational inequality or variational type inequality and prove certain results in partial KKM spaces instead of H-spaces.

Let X be a topological vector space, let (Y, P) be an ordered topological vector space equipped with closed convex pointed cone P with $intP \neq \emptyset$. Let K be a convex set in X, let $T : K \to L(X, Y)$ be any map, and let $\eta : X \times X \to Y$ be a vector-valued function as in [5]. For $y \in X$, we set K_y as the smallest convex set containing K and y.

4.1 GVLIP in partial KKM spaces

The *generalized variational like inequality problem* is to find $x_0 \in K$ such that for all $x \in K$,

$$\langle T(x_0), \eta(x_0, x)\rangle \notin -intP.$$

To prove the existence of the above problem, we show the following result.

Theorem 4.1. *Let $(X; \Gamma)$ be a partial KKM space and let $f : X \times X \to Y$ be a mapping satisfying the following conditions:*

(a) *For every $(x, u) \in X \times X$,*

$$f(x, u) + f(u, x) \geq 0.$$

(b) *For every $u \in X$, the set*

$$\{x \in X : f(x, u) \not\leq 0\}$$

is Γ-convex.

(c) *For every $x \in X$, the set*

$$G(x) := \{u \in X : f(x, u) \geq 0\}$$

is closed.

(d) *For every $x \in X$, $f(x, x) = 0$.*

(e) *The coercivity condition* **(A)***: There exists a compact subset $K \subset X$. For each $N \in \langle X\rangle$, there exists a compact Γ-convex subset $L_N \subset X$ relative to some $D \subset X$ such that $N \subset D$ and*

$$L_N \cap \bigcap_{y \in D} G(y) \subset K.$$

Then there is an $x_0 \in X$ such that $f(x, x_0) \geq 0$ for every $x \in X$.

PROOF. Since the conclusion claims that $\bigcap_{x \in X} G(x) \neq \emptyset$, we only have to show that the closed valued map $G : X \multimap X$ is KKM. The coercivity condition to apply Theorem 3.1 is also assumed. If G is not a KKM map, there is an $A \in \langle X\rangle$ such that $\Gamma_A \not\subset \bigcup_{x \in A} G(x)$.This means some $y \in \Gamma_A$ satisfies $f(x, y) \not\geq 0$ for all $x \in A$. Therefore, $A \subset \{x \in X : f(x, y) \not\geq 0\} \subset \{x \in X : f(y, x) \not\leq 0\}$ by the condition (a), The condition (b) implies the finite set A belongs to a Γ-convex set, so $\Gamma_A \subset \{x \in X : f(y, x) \not\leq 0\}$, particularly, $f(y, y) \not\leq 0$ which contradict to the condition (d).

Corollary 4.2. *Let $(X; \{\Gamma\})$ be an H-space and let $f : X \times X \to Y$ be a mapping satisfying the following conditions:*

(a) *For every $(x, u) \in X \times X$,*

$$f(x, u) + f(u, x) \geq 0.$$

(b) *For every $u \in X$, the set*

$$\{x \in X : f(x, u) \not\leq 0\}$$

is H-convex.

(c) *For every $x \in X$, the set*

$$\{u \in X : f(x, u) \geq 0\}$$

is compactly closed.

1. *For every $x \in X$, $f(x, x) = 0$.*

2. *There exists a compact set* $L \subset X$ *and an H-compact set* $K \subset X$ *such that for every* $y \in X \setminus L$, *there is* $x \in K_y$ *with* $f(x,y) \not\geq 0$.

Then there is $x_0 \in X$ *such that* $f(x, x_0) \geq 0$ *for every* $x \in X$.

This is a corrected form of Theorem 3.1 of Das and Nayak [3] for H-spaces, which is a particular case of ([2], Theorem 3, p.53). Since the set K_y is the smallest H-convex set containing K and y, the second condition implies our coercivity condition (D).

The following result gives an application of Theorem 4.1.

Theorem 4.3. *Let* $(X; \Gamma)$ *be a partial KKM space and* Y *be a Riesz space. Let* (Y, P) *be an ordered topological vector space equipped with closed convex pointed cone* P *with* $intP \neq \emptyset$. *Let* K *be a convex set in* X. *Assume that* T *and* η *satisfy the following conditions:*

(a) *For every* $(x, u) \in X \times X$,

$$\langle T(u), \eta(u, x)\rangle + \langle T(x), \eta(x, u)\rangle \notin -intP.$$

(b) *For every* $u \in X$, *the set*

$$\{x \in X : \langle T(u), \eta(u, x)\rangle \notin intP\}$$

is Γ*-convex.*

(c) *For every* $x \in X$, *the set*

$$G(x) := \{u \in X : \langle T(u), \eta(u, x)\rangle \notin -intP\}$$

is closed.

1. *For every* $x \in X$, $\langle T(x, x), \eta(x, x)\rangle \notin -intP \cup intP$.

2. *The coercivity condition* **(A)***: There exists a compact subset* $K \subset X$. *For each* $N \in \langle X\rangle$, *there exists a compact* Γ*-convex subset* $L_N \subset X$ *relative to some* $D \subset X$ *such that* $N \subset D$ *and*

$$L_N \cap \bigcap_{y \in D} G(y) \subset K.$$

Then there is an $x_0 \in X$ *such that* $\langle T(x_0), \eta(x_0, x)\rangle \notin -intP$ *for every* $x \in X$.

Proof. Define a map $f : K \times K \to Y$ by $f(x, u) := \langle T(u), \eta(u, x)\rangle$. We can see that f satisfies all the conditions of Theorem 4.1 in an ordered topological vector space (Y, P). So, there exists an $x_0 \in X$ such that $f(x, x_0) = \langle T(x_0), \eta(x_0, x)\rangle \notin -intP$ for every $x \in X$.

As a Corollary, we obtain the following H-space version of Das and Nayak ([3], Theorem 3.2).

Corollary 4.4. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* Y *be a Riesz space. Let* (Y, P) *be an ordered topological vector space equipped with closed convex pointed cone* P *with* $intP \neq \emptyset$. *Let* K *be a convex set in* X, *with* $0 \in K$. *Assume that* X *is Hausdorff. Assume that* T *and* η *satisfy the following conditions:*

(a) *For every* $(x, u) \in X \times X$,

$$\langle T(u), \eta(u, x)\rangle + \langle T(x), \eta(x, u)\rangle \notin -intP.$$

(b) *For every* $u \in X$, *the set*

$$\{x \in X : \langle T(u), \eta(u, x)\rangle \notin intP\}$$

is H-convex.

(c) *For every* $x \in X$, *the set*

$$\{u \in X : \langle T(u), \eta(u, x)\rangle \notin -intP\}$$

is compactly closed.

1. *For every* $x \in X$, $\langle T(x,x), \eta(x,x)\rangle \notin -intP \cup intP$.

2. *There exists a compact set* $L \subset X$ *and an H-compact set* $W \subset X$ *such that for every* $y \in X \setminus L$, *there is a* $x \in W_y = W \cup \{y\}$ *with* $\langle T(y), \eta(y,x)\rangle \notin intP$.

Then there is $x_0 \in X$ *such that* $\langle T(x_0), \eta(x_0,x)\rangle \notin -intP$ *for every* $x \in X$.

As an extension of the above problem, we consider a variational-like inequality problems as in [3]:
The *generalized variational like inequality problem* is to find $x_0 \in K_h$ such that

$$\langle T(x_0), \frac{x_0 u}{\eta(x_0,u)}\rangle \notin -intP,$$

and

$$\langle T(x_0), \frac{x_0 v}{\eta(x_0,v)}\rangle \notin -intP,$$

or fixed $x \in K_h$ and $u, v \in K_h$.

Theorem 4.5. *Let* $(X;\Gamma)$ *be a partial KKM space. Assume that* X *is Hausdorff. Let* M *be a subset of* $X \times X$ *having the following properties:*

(a) *For each* $x \in X$, $(x,x) \in M$.

(b) *For each* $t \in X$, *the set* $M(t) = \{x \in X : (x,t) \in M\}$ *is closed in* X.

(c) *For each* $x \in X$, *the set* $N(x) = \{t \in X : (x,t) \notin M\}$ *is* Γ*-convex.*

(d) *There exists a compact subset* $K \subset X$. *For each* $N \in \langle X\rangle$, *there exists a compact* Γ*-convex subset* $L_N \subset X$ *relative to some* $D \subset X$ *such that* $N \subset D$ *and*

$$L_N \cap \bigcap_{t\in D} M(t) \subset K.$$

Then there exists $x_0 \in X$ *such that* $\{x_0\} \times X \subset M$.

Proof. By considering M as a set-valued map, it has non-empty closed values by (a) and (b). We only have to show that $M : X \multimap X$is a KKM map. To the contrary, if it is not a KKM map. Then there exists a finite set $A \subset X$ such that $\Gamma_A \not\subset \bigcup_{t\in A} M(t)$. Thus there exists some $u \in \Gamma_A$ such that $(u,t) \notin M$ for all $t \in A$, which means $A \subset N(u)$. Since $N(u)$ is Γ-convex by (c), we have $u \in \Gamma_A \subset N(u)$, that is $(u,u) \notin M$, which contradicts to (a).

In the H-space case, if we change the condition (d) as the coercivity condition (D), we obtain the following Corollary.

Corollary 4.6. *Let* $(X, \{\Gamma_A\})$ *be an H-space. Assume that* X *is Hausdorff. Let* M *be a subset of* $X \times X$ *having the following properties:*

(a) *For each* $x \in X$, $(x,x) \in M$.

(b) *For each* $t \in X$, *the set* $M(t) = \{x \in X : (x,t) \in M\}$ *is closed in* X.

(c) *For each* $x \in X$, *the set* $N(x) = \{t \in X : (x,t) \notin M\}$ *is H-convex.*

(d) *There exists a compact set* $L \subset X$ *and an H-compact set* $W \subset X$ *such that for each weakly H-convex set* D *with* $W \subset D \subset X$,

$$\bigcap_{t\in D} (\{x \in X : x \in M(t)\} \cap D) \subset L.$$

Then there exists $x_0 \in X$ *such that* $\{x_0\} \times X \subset M$.

The following result is slightly different form of Theorem 4.5.

Theorem 4.7. *Let* $(X;\Gamma)$ *be a partial KKM space and* R *be a Riesz space. Let* K *be a convex set in* X. *Assume that* X *is Hausdorff. Let for any fixed* $y \in X$, $f : K_y \times K_y \to R$ *be a continuous map having the following properties:*

(a) *For each* $x \in X$, $f(x,x) \geq 0$.
(b) *For every* $v \in K_y$, *the set* $\{x \in K_y : f(x,v) \geq 0\}$ *is closed in* X.
(c) *For every* $x \in K_y$, *the set* $\{v \in K_y : f(x,v) < 0\}$ *is* Γ*-convex.*
(d) *For any fixed* $y \in X$ *outside a compact subset* $K \subset X$, *and for each* $N \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* $L_N \subset X$ *relative to some* $D \subset X$ *such that* $N \subset D$ *and*

$$L_N \cap \bigcap_{v \in D} \{x \in K_y : f(x,v) \geq 0\} \subset K,$$

Then, there exists $x_0 \in K_y$ *such that* $f(x_0, x) \geq 0$ *for all* $x \in K_y$.

Proof. For any fixed $y \in X$, let M be the set $\{(x,u) : f(x,u) \geq 0\} \subset K_y \times K_y$. By (a), M is nonempty. For each $u \in K_y$, $M(u) := \{x \in K_y : (x,u) \in M\}$ is closed by (b). For each $x \in K_y$, $N(x) := \{u \in K_y : (x,u) \notin M\}$ is Γ-convex by (c). The condition (d) completes all the conditions of Theorem 4.5 and therefore, there exists an $x_0 \in K_y$ such that $\{x_0\} \times K_y \subset M$, that is, for all $x \in K_y$, $f(x_0, x) \geq 0$.

Das and Nayak made an H-space version [3, Theorem 3.4] with the concept of H*-concavity in condition (c). Since this condition used only to show that $N(x)$ in the above proof is H-convex, we have the following;

Corollary 4.8. *Let* $(X, \{\Gamma_A\})$ *be an H-space and* R *be a Riesz space. Let* K *be a convex set in* X. *Assume that* X *is Hausdorff. Let for any fixed* $y \in X$, $f : K_y \times K_y \to R$ *be a continuous map having the following properties:*
(a) *For each* $x \in X$, $f(x,x) \geq 0$.
(b) *For every* $v \in K_y$, *the set* $\{x \in K_y : f(x,v) \geq 0\}$ *is compactly closed in* X.
(c) *For every* $x \in K_y$, *the set* $\{v \in K_y : f(x,v) \leq 0\}$ *is H*-concave on* K.
(d) *For any fixed* $y \in X$, *there exists a compact set* $L \subset X$ *and an H-compact set* $W \subset X$, *such that for each weakly H-convex set* D *with*

$$W \subset D \subset X, \ \bigcap_{v \in D} (\{x \in K_y : f(x,v) \geq 0\} \cap D) \subset L.$$

Then, there exists $x_0 \in K_y$ *such that* $f(x_0, x) \geq 0$ *for all* $x \in K_y$.

4.2 Uniqueness Principle

The following result characterizes the uniqueness of *the solution of the variational-like inequality problem*:

To find $x_0 \in K$ such that

$$f(x_0, x) \notin -intP$$

for all $x \in K$ obtained as in the former Theorems.

Theorem 4.9. *Let* (X, Γ) *be a partial KKM space and let* (Y, P) *be an ordered topological vector space equipped with closed convex pointed cone* P *with* $int\, P \neq \emptyset$. *Let* K *be a* Γ*-convex set in* X. *Let* $f : K \times K \to Y$ *be a continuous map such that*
(a) $f(x,u) + f(u,x) \notin intP$ *for all* $x, u \in K$,
(b) $f(x,u) + f(u,x) =_P 0$ *implies* $x = u$.
Then if the above variational-like inequality problem is solvable, then it has a unique solution.

Proof. Let $x_1, x_2 \in K$ solve the variational-like inequality, then we have $f(x_1, x) \notin intP$ and $f(x_2, x) \notin intP$. By substitution, we see that $f(x_1, x_2)$ and $f(x_2, x_1)$ is not in $intP$, so $f(x_1, x_2) + f(x_2, x_1) \notin intP$. This combined with (a) gives $f(x_1, x_2) + f(x_2, x_1) =_P 0$ which implies $x_1 = x_2$ by (b).

This proof is from Das and Nayak [3]. It does not require any properties of the underlying space X and the convex set K. So it can be applied almost variational problems discussed in this paper including Das and Nayak's result specially. Note that their requirements X is Hausdorff and $0 \in K$ are not necessary and an abstract convex space needs not contain 0 or to be Hausdorff.

Corollary 4.10. *Let $(X, \{\Gamma_A\})$ be an H-space and let (Y, P) be an ordered topological vector space equipped with closed convex pointed cone P with int $P \neq \emptyset$. Let K be a convex set in X, with $0 \in K$. Assume that X is Hausdorff. Let $f : K \times K \to Y$ be a continuous map such that*

(a) $f(x, u) + f(u, x) \notin intP$ *for all* $x, u \in K$,

(b) $f(x, u) + f(u, x) =_P 0$ *implies* $x = u$.

Then if the problem, find x_0 such that $f(x_0, x) \notin -intP$ for all $x \in K$, is solvable, then it has a unique solution.

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