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THE FORMULAE FOR COMPUTING THE COEFFICIENTS OF THE POLYNOMIAL INTERPOLATION PASSING THROUGH $n+1$ DISTINCT POINTS

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ABSTRACT

This paper deals with the polynomial interpolation of degree at most n passing through $n+1$ distinct points. The coefficients of the polynomial interpolation are written as a system of the linear equations. The system consisting of the coefficients is solved by the use of the closed form of the inverse of the Vandermonde matrix. The coefficients of the interpolation are obtained by using the sum and product symbols. The algorithm for the coefficients of the polynomial interpolation is developed by generating formulae. Also, these coefficients for equidistant points are formulated by forward difference. It is seen that the coefficients of the interpolation of degree at most n passing through $n+1$ distinct points can be computed directly by generating special formulae and can be applied easily to the polynomial interpolation. Numerical examples are represented.

***Key words**:* Vandermonde matrix; coefficients of the polynomial interpolation, forward difference.

$n+1$ FARKLI NOKTADAN GEÇEN ARADEĞER POLİNOMUNUN KATSAYILARININ HESAPLAMASI İÇİN FORMÜLLER

ÖZET

Bu çalışma, $n+1$ farklı noktadan geçen en fazla $n.$ dereceden bir aradeğer polinomu ile ilgilidir. Aradeğer polinomunun katsayıları doğrusal denklem sistemi olarak yazılmıştır. Bu katsayılardan oluşan denklem sistemi, Vandermonde matrisinin tersinin kapalı biçiminin kullanımı ile çözülmüştür. Aradeğer polinomunun katsayıları, toplam ve çarpım simgelerini kullanarak elde edilmiştir. Geliştirilen formülleri kullanarak aradeğer polinomunun katsayıları için bir algoritma üretilmiştir. Ayrıca, eşit aralıklı noktalar için katsayılar ileri fark ile de formüle edilmiştir. $n+1$ farklı noktadan geçen, en fazla $n.$ dereceden olan aradeğer polinomunun katsayıları geliştirilen formüller ile hesaplanabileceği ve ara değer hesabına kolaylıkla uygulanabileceği görülmüştür. Bu görüşü destekleyecek örneklere yer verilmiştir.

Anahtar Kelimeler: Vandermonde matrisi; aradeğer polinomunun katsayıları; ileri fark.

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1. INTRODUCTION

The Vandermonde matrix appears in the interpolation problems, especially in Lagrange interpolation, and plays an important role both in mathematics and applied sciences. Also, there are many applications of polynomial interpolations in scientific areas and engineering applications [1, 2]. The polynomial interpolation is investigated by different forms and algorithms and it is easily solvable either numerically or using a computer algebra packages. The same polynomial is produced by using these forms and algorithms [1 – 5].

Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points on the real axis and let $f(x)$ be a real-valued function defined on the interval $\left[a, b\right]$ containing these points. We wish to construct a polynomial $\deg\{ p_n(x) \} \le n$ which interpolates $y = f(x)$ at the distinct points $x_0, x_1, \ldots, x_n$ such that $p_n(x_i) = f(x_i)$ for (x_i, y_i), $0 \le i \le n$ [1 – 5]. Then there exists a unique polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{1}$$

of degree at most n that passes through (x_i, y_i) $0 \le i \le n$ distinct points [1 , 3].

The coefficients a_i, $i = 0,1,2,\ldots,n$ must satisfy the equations

$$\begin{aligned}
a_0 + a_1 x_0 + a_2 {x_0}^2 + \cdots + a_n {x_0}^n &= y_0 \\
a_0 + a_1 x_1 + a_2 {x_1}^2 + \cdots + a_n {x_1}^n &= y_1 \\
a_0 + a_1 x_2 + a_2 {x_2}^2 + \cdots + a_n {x_2}^n &= y_2 \, . \\
&\vdots \\
a_0 + a_1 x_n + a_2 {x_n}^2 + \cdots + a_n {x_n}^n &= y_n
\end{aligned} \tag{2}$$

Writing this $(n+1)\times(n+1)$ system as

$$\begin{bmatrix} 1 & x_0 & {x_0}^2 & \cdots & {x_0}^n \\ 1 & x_1 & {x_1}^2 & \cdots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{3}$$

or

$$V\boldsymbol{a} = \boldsymbol{y} \tag{4}$$

reveals that the coefficient matrix is a square Vandermonde matrix, where

$$\boldsymbol{a} = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_n]^T, \quad \boldsymbol{y} = [y_0 \quad y_1 \quad y_2 \quad \cdots \quad y_n]^T$$

and

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}.$$

If given $n+1$ points are distinct, the Vandermonde matrix is nonsingular and consequently the system (3) has unique solution and thus there is one and only one solution set of coefficients for polynomial (1) [1, 6, 7].

There is a large amount of literature on the computation of the inverse of the Vandermonde matrix and its applications [6, 8-10]. The inverse of the Vandermonde matrix is investigated by using various methods and algorithms. Some of them are referred in this study. LU factorization of the Vandermonde matrix and the inverse of the matrix using symmetric functions are investigated, and the applications are given in [8, 11, 12]. Explicit closed form expression for the inverse matrix and algorithms of $n \times n$ generalized Vandermonde matrix are given by using the elementary symmetric functions [7]. In [13], the inverse matrix of lower and upper triangular factors of Vandermonde matrix using symmetric functions is investigated. The Vandermonde matrix plays great importance of approximation problems such as interpolation and least squares, and the solutions of Vandermonde systems are analytically obtained by using the inverse of the Vandermonde matrix [1, 2, 6, 14] and the solution systems of linear equations with Vandermonde matrices of coefficients are solved by researches using algorithms [15, 16].

The trivariate polynomial interpolation is formulated as a matrix equation and the coefficients of the trivariate polynomial interpolation are computed directly from the matrix equation [17]. In addition, the bivariate polynomial interpolation is the special case of the trivariate polynomial interpolation. Finally, the closed formulae of the coefficients of the bivariate and univariate polynomial interpolations which are the special cases of trivariate polynomial interpolation are obtained using the inverse of the Vandermonde matrix [17].

In this study, the inverse of the matrix V in equation (4) is written by using the sum and product symbols. All the formulae for the entries of the matrices L^{-1} and U^{-1}, being L and U the triangular matrices in the LU factorization of V can be found by replacing values of the elementary symmetric functions in [8]. The coefficients of the polynomial interpolation of degree n passing through $n+1$ distinct points in the equation (1) are obtained from the inverse of V. The formulations of coefficients of the polynomial (1) are defined as the closed form and the algorithm for the coefficients of the polynomial interpolation is developed by generating formulae. Numerical examples are solved with these formulae and the algorithm. Also, these coefficients of the polynomial interpolation are formulated with making use of forward difference, when the points x_i's are equally spaced.

2. THE FORMULAE FOR COMPUTING THE COEFFICIENTS OF THE POLYNOMIAL INTERPOLATION

In this section, the system of the linear equations defined in (3) is considered and the inverse of the matrix V in equation (4) can be computed by using LU decomposition. The inverses of matrices L and U are formulated in an closed form as

$$U^{-1}=\begin{bmatrix} 1 & -x_0 & x_0x_1 & -\prod_{i_0=0}^{2} x_{i_0} & \cdots & (-1)^n\prod_{i_0=0}^{n-1} x_{i_0} \\ 0 & 1 & -(x_0+x_1) & \sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0}x_{i_1} & \cdots & (-1)^{n-1}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-2}}\right) \\ 0 & 0 & 1 & -\sum_{i_0=1}^{2} x_{i_0} & \cdots & (-1)^{n-2}\left(\sum_{i_0=0}^{2}\sum_{i_1=1}^{3}\cdots\sum_{i_{n-3}=n-3}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-3}}\right) \\ \bullet & \bullet & \bullet & \bullet & \vdots & \bullet \\ 0 & 0 & 0 & 0 & \cdots & -\sum_{i_0=0}^{n-1} x_{i_0} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$L^{-1}=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{x_1-x_0} & -\frac{1}{x_0-x_1} & 0 & \cdots & 0 \\ \frac{1}{(x_1-x_0)(x_2-x_0)} & \frac{1}{(x_0-x_1)(x_2-x_1)} & \frac{1}{(x_0-x_2)(x_1-x_2)} & \cdots & 0 \\ \bullet & \bullet & \bullet & \vdots & \bullet \\ \frac{(-1)^{n-1}}{\prod_{i=1}^{n-1}(x_i-x_0)} & \frac{(-1)^{n-1}}{\prod_{i=0,i\neq 1}^{n-1}(x_i-x_1)} & \frac{(-1)^{n-1}}{\prod_{i=0,i\neq 2}^{n-1}(x_i-x_2)} & \cdots & 0 \\ \frac{(-1)^{n}}{\prod_{i=1}^{n}(x_i-x_0)} & \frac{(-1)^{n}}{\prod_{i=0,i\neq 1}^{n}(x_i-x_1)} & \frac{(-1)^{n}}{\prod_{i=0,i\neq 2}^{n}(x_i-x_2)} & \cdots & \frac{(-1)^{n-1}}{\prod_{i=0}^{n-1}(x_i-x_n)} \end{bmatrix}$$

where $0 \le i_0 < i_1 < ... < i_n \le n$ and $i_r = i_{r-1} + 1, \quad r = 1, 2, ..., n$.
Using L^{-1} and U^{-1}, the following result is obtained.

Corollary 1 Let $V^{-1} = U^{-1}L^{-1}$. Then the closed form of the inverse of V is stated as

$$V^{-1} = [\boldsymbol{v}_0 \quad \boldsymbol{v}_1 \quad \cdots \quad \boldsymbol{v}_n] \tag{5}$$

where the columns vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \cdots \boldsymbol{v}_n$ are

$$\boldsymbol{v}_0 = \frac{1}{\prod_{i=1}^{n}(x_i - x_0)} \begin{bmatrix} \sum_{i_1=1}^{1}\sum_{i_2=2}^{2}\cdots\sum_{i_n=n}^{n} x_{i_1}x_{i_2}\cdots x_{i_n} \\ -\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n} x_{i_1}x_{i_2}\cdots x_{i_{n-1}} \\ \vdots \\ (-1)^{n-2}\sum_{i_1=1}^{n-1}\sum_{i_2=2}^{n} x_{i_1}x_{i_2} \\ (-1)^{n-1}\sum_{i_1=1}^{n} x_{i_1} \\ (-1)^n \end{bmatrix},$$

$$\boldsymbol{v}_1 = \frac{1}{\prod_{\substack{i=0 \\ i\neq 1}}^{n}(x_i - x_1)} \begin{bmatrix} \sum_{i_0\neq 1, i_0=0}^{0}\sum_{i_2=2}^{2}\cdots\sum_{i_n=n}^{n} x_{i_0}x_{i_2}\cdots x_{i_n} \\ -\sum_{i_0\neq 1, i_0=0}^{2}\sum_{i_3=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n} x_{i_0}x_{i_2}\cdots x_{i_{n-1}} \\ \vdots \\ (-1)^{n-2}\sum_{i_0\neq 1, i_0=0}^{n-1}\sum_{i_2=2}^{n} x_{i_0}x_{i_2} \\ (-1)^{n-1}\sum_{i_0=0, i_0\neq 1}^{n} x_{i_0} \\ (-1)^n \end{bmatrix}, \ldots,$$

$$\boldsymbol{v}_n = \frac{1}{\prod_{i=0}^{n-1}(x_i - x_n)} \begin{bmatrix} \sum_{i_0=0}^{0}\sum_{i_1=1}^{1}\cdots\sum_{i_{n-1}=n-1}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-1}} \\ -\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-2}} \\ \vdots \\ (-1)^{n-2}\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1} x_{i_0}x_{i_1} \\ (-1)^{n-1}\sum_{i_0=0}^{n-1} x_{i_0} \\ (-1)^n \end{bmatrix}. \tag{6}$$

It is seen that the columns of V^{-1} can be expressed as a closed form of the inverse of the $(n+1)\times(n+1)$ Vandermonde matrix in (3). Note that the first elements of $\boldsymbol{v}_0, \boldsymbol{v}_1, \cdots \boldsymbol{v}_n$ can be written in terms of product symbol, respectively

$$\sum_{i_1=1}^{1}\sum_{i_2=2}^{2}\cdots\sum_{i_n=n}^{n} x_{i_1}x_{i_2}\cdots x_{i_n} = \prod_{i=1}^{n} x_i\,,\quad \sum_{i_0\neq 1, i_0=0}^{0}\sum_{i_2=2}^{2}\cdots\sum_{i_n=n}^{n} x_{i_0}x_{i_1}\cdots x_{i_n} = \prod_{i=0, i\neq 1}^{n} x_i\,,\ \ldots,$$
$$\sum_{i_0=0}^{0}\sum_{i_1=1}^{1}\cdots\sum_{i_{n-1}=n-1}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-1}} = \prod_{i=0}^{n-1} x_i\,.$$

Using the equation (6) for $n=3$, the inverse of the 4×4 matrix V for x_0, x_1, x_2, x_3 four distinct points is computed as

$$V^{-1} = \begin{bmatrix} \frac{x_1x_2x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)} & \frac{x_0x_2x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)} & \frac{x_0x_1x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)} & \frac{x_0x_1x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)} \\ -\frac{x_1x_2+x_1x_3+x_2x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)} & -\frac{x_0x_2+x_0x_3+x_2x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)} & -\frac{x_0x_1+x_0x_3+x_1x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)} & -\frac{x_0x_1+x_0x_2+x_1x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)} \\ \frac{x_1+x_2+x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)} & \frac{x_0+x_2+x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)} & \frac{x_0+x_1+x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)} & \frac{x_0+x_1+x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)} \\ -\frac{1}{(x_1-x_0)(x_2-x_0)(x_3-x_0)} & -\frac{1}{(x_0-x_1)(x_2-x_1)(x_3-x_1)} & -\frac{1}{(x_0-x_2)(x_1-x_2)(x_3-x_2)} & -\frac{1}{(x_0-x_3)(x_1-x_3)(x_2-x_3)} \end{bmatrix}$$

It is seen that the inverse of the Vandermonde matrix for any distinct points can be easily applied to the equation (6).

The unique solution of the system (3) is obtained by using $\boldsymbol{a}=V^{-1}\boldsymbol{y}$. If V^{-1} in (5) are applied to the equation (4), the following results are obtained.

Corollary 2 Let a_i for $i=0,1,2,..,n$ be the coefficients of the polynomial (1) passing through $n+1$ distinct points. Then the coefficients of the polynomial interpolation defined in (1) of degree n are

$$a_0=\frac{\prod_{i=1}^{n}x_i}{\prod_{i=1}^{n}(x_i-x_0)}y_0+\frac{\prod_{i=0,i\neq 1}^{n}x_i}{\prod_{i=0,i\neq 1}^{n}(x_i-x_1)}y_1+\cdots+\frac{\prod_{i=0}^{n-1}x_i}{\prod_{i=0}^{n-1}(x_i-x_n)}y_n,$$

$$a_1=-\left(\frac{\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n}x_{i_1}x_{i_2}\cdots x_{i_{n-1}}}{\prod_{i=1}^{n}(x_i-x_0)}y_0+\frac{\sum_{i_0\neq 1,i_0=0}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n}x_{i_0}x_{i_2}\cdots x_{i_{n-1}}}{\prod_{i=0,i\neq 1}^{n}(x_i-x_1)}y_1+\cdots+\frac{\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1}x_{i_0}x_{i_1}\cdots x_{i_{n-2}}}{\prod_{i=0}^{n-1}(x_i-x_n)}y_n\right),$$

$$\vdots \tag{7}$$

$$a_{n-2}=(-1)^{n-2}\left(\frac{\sum_{i_1=1}^{n-1}\sum_{i_2=2}^{n}x_{i_1}x_{i_2}}{\prod_{i=1}^{n}(x_i-x_0)}y_0+\frac{\sum_{i_0\neq 1,\,i_0=0}^{n-1}\sum_{i_2=2}^{n}x_{i_0}x_{i_2}}{\prod_{i=0,\,i\neq 1}^{n}(x_i-x_1)}y_1+\cdots+\frac{\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1}x_{i_0}x_{i_1}}{\prod_{i=0}^{n-1}(x_i-x_n)}y_n\right),$$

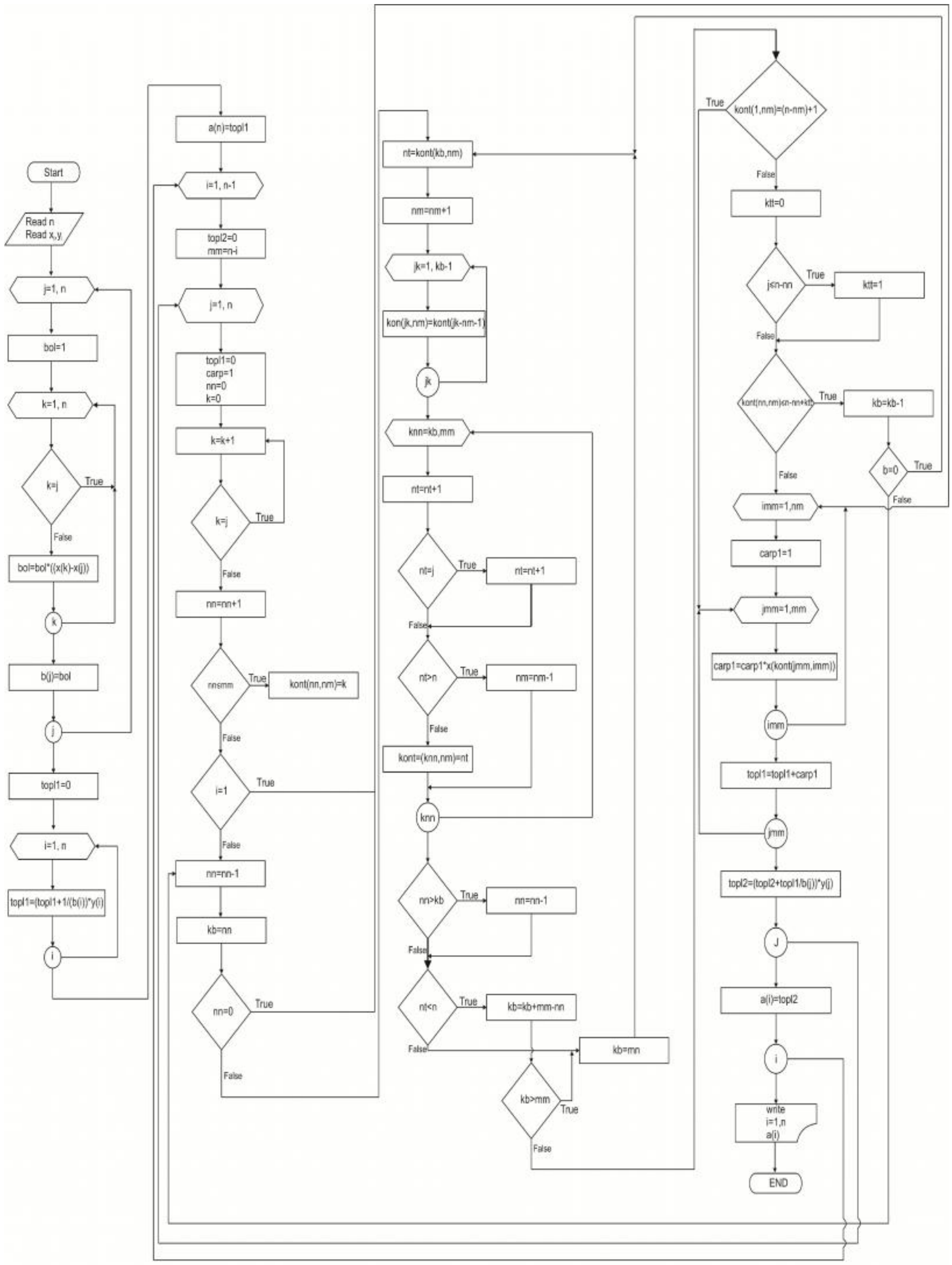
$$a_{n-1}=(-1)^{n-1}\left(\frac{\sum_{i_1=1}^{n}x_{i_1}}{\prod_{i=1}^{n}(x_i-x_0)}y_0+\frac{\sum_{i_0=0,i_0\neq 1}^{n}x_{i_0}}{\prod_{i=0,i\neq 1}^{n}(x_i-x_1)}y_1+\cdots+\frac{\sum_{i_0=0}^{n-1}x_{i_0}}{\prod_{i=0}^{n-1}(x_i-x_n)}y_n\right)$$

and

$$a_n=(-1)^{n}\left(\frac{1}{\prod_{i=1}^{n}(x_i-x_0)}y_0+\frac{1}{\prod_{i=0,i\neq 1}^{n}(x_i-x_1)}y_1+\cdots+\frac{1}{\prod_{i=0}^{n-1}(x_i-x_n)}y_n\right)$$

Proof Using $\boldsymbol{a}=V^{-1}\boldsymbol{y}$, it is easily proved from the product the matrix V^{-1} and the vector $\boldsymbol{y}$.

Using these formulae, the following algorithm is developed to compute the coefficients of the polynomial interpolation by Fortran. This algorithm flow chart is given in Figure 1.



(xi, yi) : distinct points
a(i) : the coffiecients of polynomial interpolation
n : polynomial of degree

Figure 1 Flow chart of the computer algorithm

Example 1 Given four distinct points $(-1,4),(0,2),\ (1,2),\ (2,10)$ satisfying the polynomial $p_3(x)=x^3+x^2-2x+2$. The coefficients of third degree polynomial interpolation passing through four distinct points are obtained by writing $n=3$ in (7) as

$$a_0=\frac{x_1x_2x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)}y_0+\frac{x_0x_2x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)}y_1+\frac{x_0x_1x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)}y_2+\frac{x_0x_1x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)}y_3$$

$$a_1=-\left(\frac{x_1x_2+x_1x_3+x_2x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)}y_0+\frac{x_0x_2+x_0x_3+x_2x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)}y_1+\frac{x_0x_1+x_0x_3+x_1x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)}y_2+\frac{x_0x_1+x_0x_2+x_1x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)}y_3\right)$$

$$a_2=\frac{x_1+x_2+x_3}{(x_1-x_0)(x_2-x_0)(x_3-x_0)}y_1+\frac{x_0+x_2+x_3}{(x_0-x_1)(x_2-x_1)(x_3-x_1)}y_2+\frac{x_0+x_1+x_3}{(x_0-x_2)(x_1-x_2)(x_3-x_2)}y_3+\frac{x_0+x_1+x_2}{(x_0-x_3)(x_1-x_3)(x_2-x_3)}y_3$$

$$a_3=-\left(\frac{1}{(x_1-x_0)(x_2-x_0)(x_3-x_0)}y_0+\frac{1}{(x_0-x_1)(x_2-x_1)(x_3-x_1)}y_1+\frac{1}{(x_0-x_2)(x_1-x_2)(x_3-x_2)}y_2+\frac{1}{(x_0-x_3)(x_1-x_3)(x_2-x_3)}y_3\right)$$

and $a_0=2,\ a_1=-2,\ a_2=1$ and $a_3=1$ are calculated by using these formulae.

Using the algorithm, the following example given on page 211in [1] is computed and the same coefficients of the polynomial are found.

Example 2 Consider $(-2,-39),\ (-1,1),\ (0,1),\ (1,3),\ (2,25)$, $(3,181)$ and $(4,801)$ seven distinct points for the polynomial $p_5(x)$ in Example 5.3 [1]. The coefficients defined in (1) of fifth degree polynomial interpolation passing through seven distinct points are obtained by using the algorithm.
The coefficients of $p_5(x)$ are calculated easily 1,0,2,0,-1,1,0 and this polynomial is given by

$$p_5(x)=1+2x^2-x^4+x^5.$$

The coefficients of the interpolation obtained above can be defined by forward difference, when the points x_i's are equally spaced. The following results are concerned with these formulations of the coefficients.

Corollary 3 If the (x_i,y_i) for $0\le i\le n$ distinct points are equally spaced, then the coefficients of the interpolation of degree n are formulated as

$$a_0=y_0-\frac{1}{h}x_0\Delta y_0+\frac{1}{2!h^2}x_0x_1\Delta^2y_0-\frac{1}{3!h^3}x_0x_1x_2\Delta^3y_0+\cdots+(-1)^n\frac{1}{n!h^n}x_0x_1\cdots x_{n-1}\Delta^ny_0$$

$$a_1 = \frac{1}{h}\Delta y_0 - \frac{1}{2!h^2}\left(\sum_{i_0=0}^{1} x_{i_0}\right)\Delta^2 y_0 + \frac{1}{3!h^3}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0}x_{i_1}\right)\Delta^3 y_0 + \cdots + (-1)^{n-1}\frac{1}{n!h^n}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-2}}\right)\Delta^n y_0$$

$$\vdots$$

$$a_{n-2} = \frac{1}{(n-2)!h^{n-2}}\Delta^{n-2}y_0 - \frac{1}{(n-1)!h^{n-1}}\left(\sum_{i_0=0}^{n-2} x_{i_0}\right)\Delta^{n-1}y_0 + \frac{1}{n!h^n}\left(\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1} x_{i_0}x_{i_1}\right)\Delta^n y_0 \qquad (8)$$

$$a_{n-1} = \frac{1}{(n-1)!h^{n-1}}\Delta^{n-1}y_0 - \frac{1}{n!h^n}\left(\sum_{i_0=0}^{n-1} x_{i_0}\right)\Delta^n y_0$$

$$a_n = \frac{1}{n!h^n}\Delta^n y_0$$

where $x_{j+1} - x_j = h$ for $j = 0,1,2,...,n-1$, $x_n = x_0 + nh$ and Δ forward difference operator.

Proof The result can be proved by using induction on k equidistant points .The result is true for $k=1$ and $k=2$ from (7). For $k=1$ and $k=2$, the polynomial interpolations are $p_0(x) = a_0$ and $p_1(x) = a_0 + a_1 x$, respectively. From (7),

$$a_0 = \frac{x_1}{x_1 - x_0}y_0 + \frac{x_0}{x_0 - x_1}y_1 = \frac{x_0 + h}{h}y_0 - \frac{x_0}{h}y_1 = y_0 - x_0\frac{\Delta y_0}{h},$$

$$a_1 = -\left(\frac{1}{x_1 - x_0}y_0 + \frac{1}{x_0 - x_1}y_1\right) = \frac{\Delta y_0}{h}$$

are obtained. Assume that the formulations of the coefficients (8) are true for k, that is, the polynomial interpolation of degree $k-1$ passing through k points is found as

$$p_{k-1}(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{k-1}x^{k-1}$$

where the coefficients $b_0, b_1, b_2, \cdots, b_{k-1}$ are defined in (8).

Consider the $k+1$ equidistant points, and then the interpolation of degree k from Newton's forward difference formula is written as

$$p_k(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{k-1}x^{k-1} + \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})}{k!h^k}\Delta^k y_0$$

where $x_n = x_0 + nh$ for $n = 0,1,2,\cdots,k$.

Using the well known equality

$$(x-x_0)(x-x_1)\cdots(x-x_{k-1}) = x^k - \left(\sum_{i_0=0}^{k} x_{i_0}\right)x^{k-1} + \left(\sum_{i_0=0}^{k-1}\sum_{i_1=1}^{k} x_{i_0}x_{i_1}\right)x^{k-2} + \cdots + (-1)^k \prod_{i=0}^{k} x_i$$

where $0 \le i_0 < i_1 < \ldots < i_k \le k$ and $i_r = i_{r-1}+1, \quad r = 1,2,\ldots,k$, and rearranging the terms in the polynomial $p_k(x)$, the coefficients $a_0, a_1, \cdots, a_k$ are obtained as defined in (8). Therefore, by mathematical induction, the proof of the result is complete.
Note that the polynomial interpolation for equidistant points can be found by the use of the equation (8). The following example is given by using the forward difference table and the equation (8). Using the connection between forward differences and divided differences,

$$\Delta^k y_i = k!h^k f\left[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}\right].$$

It is seen that the relationship between the coefficients of the polynomial interpolation and divided differences as

$$a_n = \frac{1}{n!h^n}\Delta^n y_0 = f\left[x_0, x_1, x_2, \cdots, x_n\right].$$

Example 3 Consider $(-1,4), (0,2), (1,2), (2,10)$ four distinct equidistant points for the polynomial $p_3(x)$ in Example 1. The coefficients defined in (8) of third degree polynomial interpolation passing through four distinct points are obtained by using the Table 1. The coefficients of $p_3(x)$ are calculated as

$$a_0 = y_0 - \frac{1}{h}x_0\Delta y_0 + \frac{1}{2!h^2}x_0x_1\Delta^2 y_0 - \frac{1}{3!h^3}x_0x_1x_2\Delta^3 y_0 = 2$$

$$a_1 = \frac{1}{h}\Delta y_0 - \frac{1}{2!h^2}(x_0+x_1)\Delta^2 y_0 + \frac{1}{3!h^3}(x_0x_1 + x_0x_2 + x_1x_2)\Delta^3 y_0 = -2$$

$$a_2 = \frac{1}{2!h^2}\Delta^2 y_0 - \frac{1}{3!h^3}(x_0+x_1+x_2)\Delta^3 y_0 = 1$$

$$a_3 = \frac{1}{3!h^3}\Delta^3 y_0 = 1$$

and $p_3(x) = 2 - 2x + x^2 + x^3$ is obtained.

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
-1	**4**			
		−2		
0	2		**2**	
		0		6
1	2		8	
		8		
2	10			

Table 1 The forward difference table

3. CONCLUSIONS

There are many ways to represent this interpolating polynomial (1). Some representations are more useful for computation than others to find $p_n(x)$. The most commonly used polynomial interpolations are the Lagrange and Newton's forms. The most convenient form of these operations is usually $p_n(x)$ defined in (1) when given $n+1$ distinct points are large, but to find the values a_i's we must go through the effort of solving the system of coefficients of the polynomial interpolation satisfying $n+1$ distinct points or collecting coefficients of like powers of x from another form of $p_n(x)$ such as Lagrange and Newton's form. In this study, the system of the linear equations consisting of coefficients of the polynomial interpolation is solved by use of the formulation of the inverse of Vandermonde matrix and the solution set of coefficients is obtained in the closed form. These coefficients for equidistant points are written by forward difference. It is seen that the coefficients of the polynomial interpolation of degree at most n can be found directly by generating special formulae. Also, the values of coefficients can be easily computed by and the algorithm for the coefficients of the polynomial interpolation. The closed formulations of them can be applied to the polynomial interpolations passing through $n+1$ distinct points and engineering problems.

REFERENCES

[1] Johnson L.W., Riess R.D., *Numerical Analysis* (1982), Addison Wesley Publishing, Canada.

[2] Kreyszig E., *Advanced Engineering Mathematics* (1993), 7th Ed., John Wiley &Sons, New York.

[3] Conte S.D and de Boor C., *Elementary Numerical Analysis* (1972), McGraw Hill, Japan.

[4] Gerald C. F. and Wheatley G., *Applied Numerical Analysis* (2004), Seventh International Edition, Pearson Education Inc., USA.

[5] Fausett L., *Numerical Methods Algorithms and Applications* (2003), Pearson Education, Inc., New Jersey.

[6] Turner L.R., Inverse of the Vandermonde matrix with applications, *NASA Technical Note*, 1966; *NASA-TN-D*-3547:1-14.

[7] El-Mikkawy M.E.A., Explicit inverse of a generalized Vandermonde matrix, *Appl. Math. Comput*. (2003), 146 p. 643-651.

[8] Oruç H., *LU* factorization of the Vandermonde matrix and its applications, *Appl. Math. Letters*, (2007), 20 p. 982-987.

[9] Jog C.S., The accurate inversion of Vandermonde matrices, *Comput. Math. Appl.* (2004), 47 p.921-929.
[10] Eisinberg A. and Fedele G., On the inversion of the Vandermonde matrix, *Appl. Math. Comput.* (2006), 174 p.1384-1397.
[11] Martinez J.J. and Pena J.M., Factorizations of Cauchy-Vandermonde matrices, *Linear Algebra Appl.* (1998), 284 p.229-237.
[12] Oruç H. and Phillips G.M., Explicit factorization of the Vandermonde matrix, *Linear Algebra Appl.* (2000), 315 p.113-123.
[13] Oruç H. and Akmaz H.K., Symmetric functions and Vandermonde matrix, *J. Comput. Appl. Math.* (2004), 172 p. 49-64.
[14] Meyer C.D., *Matrix Analysis and Applied Linear Algebra,* (2000), Siam.
[15] Björck A. and Pereyra V., Solution of Vandermonde systems of equations, *Math. Comput.* (1970), Vol.24, 112 p. 893-903.
[16] Mühlbach G., Interpolation by Cauchy-Vandermonde systems and applications, *J. Comput. Appl. Math.* (2000), 122 p. 203-222.
[17] Safak S., On the Trivariate Polynomial Interpolation, *Wseas Transactions on Mathematics*,(2012), Is. 8, Vol 11, p. 722-730.