RESULTS ON ADOMIAN DECOMPOSITION METHOD

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ABSTRACT

In this work, the Adomian decomposition method has been considered for solving linear and non-linear parabolic and hyperbolic type partial differential equation with initial conditions. The both of two type's partial differential equations have been applied for the considered method according to the functions of the initial conditions. Then, it has been obtained that the solution is finite term series. It has been taken into some examples to emphasize their results.

Keywords**:** Parabolic partial equations, hyperbolic partial differential equations, initial value problems, Adomian decomposition method.

ADOMIAN AYRIŞTIRMA YÖNTEMİNİN UYGULANMASI ÜZERİNE SONUÇLAR

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ÖZET

Bu çalışmada, Adomian ayrıştırma yöntemi, başlangıç koşulları ile tanımlı, doğrusal ve doğrusal olmayan, parabolik ve hiperbolik kısmi diferansiyel denklemleri çözmek için dikkate alınmıştır. Ele alınan bu iki tip kısmi diferansiyel denklemlere başlangıç koşullarının bağlı olduğu fonksiyonlarına göre dikkate alınan yöntem uygulanmıştır. Daha sonra, bu yöntem uygulanarak oluşan sonlu terimli serilere ait sonuçlar elde edilmiştir. Bu sonuçları vurgulamak için bazı örneklere yer verilmiştir.

Anahtar Kelimeler: Parabolik tip kısmi diferansiyel denklemler, hiperbolik tip kısmi diferansiyel denklemler, başlangıç değer problemleri, Adomian ayrıştırma yöntemi.

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1. INTRODUCTION

When we want to obtain mathematical models of physical or engineering science, generally we have linear or non-linear differential equations. To obtain analytical solution of such differential equation is not so easy. They're some methods to find approximate solution of these kinds of equations. Recently, some of these are differential transform [1-10], spectral [11-13], and Adomian methods [14-30].
In the 1980's G. Adomian developed a new powerful method for solving linear and non-linear functional equations of any kind. As a result, many authors were interested in this method for solving problems whose mathematical models involve algebraic, integral, integro-differential, difference, ordinary and partial differential equation and systems.
This paper is concerned with Adomian decomposition method for solving parabolic and hyperbolic type partial differential equation, linear and non-linear initial value problem.

Now, we give some basic idea of the Adomian decomposition method:

1.THE ADOMIAN DECOMPOSITION METHOD (ADM)

The main idea of the ADM when applied to a general non-linear partial differential equation is in the form

$$L\,u(x,t) + R\,u(x,t) + N\,u(x,t) = h(x,t) \tag{1}$$

For simplicity, after this point, drop the notation and just call $u = u(x,t)$.
The linear terms are decomposed into L + R, while the non-linear terms are represented by Nu. L is taken as the highest order derivative and R is the remainder term of the linear operator, L^{-1} is inverse operator of L, and is defined by a definite integration from 0 to x or from 0 to t, i.e.,

$$L^{-1}(*) = \int_0^x\int_0^x (*)dxdx \qquad \text{or} \qquad L^{-1}(*) = \int_0^t\int_0^t (*)dtdt \tag{2.a-b}$$

If L is second-order operator, inverse operator L^{-1} is a two-fold indefinite integral, then we have

$$L^{-1}Lu = u(x,t) - u(0,t) - x\frac{\partial u(0,t)}{\partial x} \quad \text{or} \quad L^{-1}Lu = u(x,t) - u(x,0) - t\frac{\partial u(x,0)}{\partial t} \tag{3.a-b}$$

Operating on both sides of Eq.(1) with two fold inverse operator L^{-1} comes into

$$L^{-1}\,L\,u = L^{-1}\,h(x,t) - L^{-1}\,R\,u - L^{-1}\,N\,u \tag{4}$$

And, then we get the following equations;

$$u(x,t) = u(0,t) - x\frac{\partial u(0,t)}{\partial x} + L^{-1}h \quad - L^{-1}Ru - L^{-1}Nu \tag{5}$$

$$u(x,t) = u(x,0) - t\frac{\partial u(x,0)}{\partial t} + L^{-1}h \quad - L^{-1}Ru - L^{-1}Nu \tag{6}$$

The decomposition method consists in searching the solution of Eqs. (5) or (6) as a infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (7)$$

The non-linear operator, N u, is also decomposed in a infinite series form:

$$Nu = \sum_{n=0}^{\infty} A_n \qquad (8)$$

where A_n are polynomials depending on u_0, u_1, …, u_n and named Adomian polynomials. They are obtained by the following formula [15]:

$$A_n = \frac{1}{n!}\frac{d^n}{da^n}\left[F\left(\sum_{k=0}^{\infty} a^k u_k\right)\right]_{a=0} \quad , \quad n = 0,1,2,... \qquad (9)$$

where a is a parameter introduced for convenience. Substituting Eqs. (7) and (8) into Eq.(6), then we get following equations.

$$\sum_{n=0}^{\infty} u_n(x,t) = u(0,t) - x\frac{\partial u(0,t)}{\partial x} + L^{-1}h(x,t) \quad - L^{-1}R\sum_{n=0}^{\infty} u_n(x,t) - L^{-1}\sum_{n=0}^{\infty} A_n$$

or

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) - t\frac{\partial u(x,0)}{\partial t} + L^{-1}h(x,t) \quad - L^{-1}R\sum_{n=0}^{\infty} u_n(x,t) - L^{-1}\sum_{n=0}^{\infty} A_n$$

(10.a-b)

From Eq.(10), we obtain the following recurrent scheme:

$$u_0 = u(0,t) - x\frac{\partial u(0,t)}{\partial x} + L^{-1}h(x,t) = f_0(x,t) + f_1(x,t)$$

or (11.a-b)

$$u_0 = u(x,0) - t\frac{\partial u(x,0)}{\partial t} + L^{-1}h(x,t) = f_0(x,t) + f_1(x,t)$$

and consequently,

$$u_1 = - L^{-1} R\, u_0 - L^{-1} A_0$$
$$u_2 = - L^{-1} R\, u_1 - L^{-1} A_1 \qquad (12)$$
$$\vdots$$
$$u_{n+1} = - L^{-1} R\, u_n - L^{-1} A_n \;,\; n \geq 0$$

where A_n are Adomian polynomials that gives are obtained from the Eq.(9)

$$A_0 = F(u_0)$$
$$A_1 = u_1 F'(u_0)$$
$$A_2 = u_2 F'(u_0) + (u_1)^2 F''(u_0) / 2 \qquad (13)$$
$$A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + (u_1)^3 F'''(u_0) / 6$$
$$\vdots$$

Application of Adomian method to the differential equation allows us to find an approximated solution as in Eq. (7). By approximated solution we mean a solution based on a truncated series involving finite number of terms. The convergency of this method is proposed by Cherruault and his friends [31, 32].

Now, we give the proposed Adomian method (PADM):

The basic principles of the ADM depend on selecting the zeroth component $u_0(x,t)$. Generally $u_0(x,t)$ is defined by a function which is obtained initial conditions and non-homogeneous term of differential equation as in Eq.(11). The components u_0,

u_1, u_2 ,… are obtained recursively. So the series solution u(x,t) is defined by Eq. (7) is determined.

The main idea of the proposed Adomian decomposition method (PADM) puts forward that in Eq.(11) the function $u_0(x,t)$ is decomposed into two parts (where $f_0(x,t)$ is obtained from initial condition and $f_{1.2}(x,t)$ is obtained from inhomogeneous parts), such that

$$u_0(x,t)= f_0(x,t) + f_{1,2}(x,t) \qquad (14)$$

In this proposed method, only the initial condition part $f_0(x,t)$ be assigned to the zeroth components of $u_0(x,t)$ whereas the remaining part $f_{1,2}(x,t)$ be combined with the other terms given as in Eq. (10) to determine $u_1(x,t)$. In this assumption, we rearranged Eqs. (11)-(12), and we get the following recurrent scheme for the PADM,

$$\begin{aligned} u_0 &= f_0(x,t) \\ u_1 &= f_{1,2}(x,t) - L^{-1} R u_0 - L^{-1} A_0 \\ u_2 &= - L^{-1} R u_1 - L^{-1} A_1 \qquad (15) \\ &\vdots \\ u_{n+1} &= - L^{-1} R u_n - L^{-1} A_n , \; n \geq 0 \end{aligned}$$

When the Eq.(12) and Eq.(15) are compared, it is shown that, there are some differences to choose u_0. In this meaning, the zeroth component u_0 is defined only by $f_0(x,t)$ a part of $u_0(x,t)$. The remaining part $f_{1,2}(x,t)$ of $u_0(x,t)$ is added to the definition of the component u_1 in Eq.(12). Applying this proposed method to the problem, generally; it is shown that only two iterations are sufficient to determine the exact solution for most of the examined cases, which is given below. If more than two iterations are needed, then non-linear term Nu(x,t) should be calculated for all types of non-linear form. With the choosing of $f_0(x,t)$ for u_0 to include on making easy the recurrent relation, and speeds up the convergence of the solution.

In this work, the achievement of the PADM relates to special choice of the parts $f_0(x,t)$ and $f_{1,2}(x,t)$. Here, we have been able to establish some criterion to judge what forms of $f_0(x,t)$ and $f_{1,2}(x,t)$ can be used to advantage of required speed. It appears that these rules have some specific bases, which are given as in below.

In the following section, after a brief introduction to partial differential equations, and then application of the proposed decomposition method is shown to parabolic and hyperbolic type partial differential equations.

3. PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

The mathematical formulation of the most problem in science involving rates of change with respect to two or more independent variables, usually representing time length or angle, leads either to a partial differential equation or to a set of such equations. Special cases of two-dimensional second-order equation

$$A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU + G = 0 \qquad (16)$$

where A, B, C, D, E, F and G may be functions of the independent variables x and y and of the dependent variable U, occur more frequently than any other because they are often the mathematical form of one of the conservation principles of physics.

This equation is said to be elliptic when B^2-4AC<0, parabolic when B^2-4AC=0, and hyperbolic when B^2-4AC>0.

Parabolic and hyperbolic equations: If the problems involving time t as one independent variable leads usually to parabolic or hyperbolic equations.
The simplest parabolic equation derives from the theory of heat conduction. Heat may be transferred by conduction, convection, and radiation. In conduction the heat (molecular motion or vibration) is transferred locally by impacts of molecules with adjacent molecules. With convection, heat is carried from one region to another by a current flow, and heat radiation occurs via infrared electromagnetic waves. In homogeneous, solid, heat-conducting material, the temperature U(x, t), at the point x at time t, very nearly obeys the heat equation $U_t = k\ U_{xx}$ where k is a positive constant which measures the heat conductivity of the material. The function U can also have the interpretation of being the concentration of a chemical or dye in a liquid without currents, and hence the heat equation is often called the diffusion equation, and its solution gives, for example, the temperature U at a distance x units of length from one end of a thermally insulated bar after t seconds of heat conduction. In such problem the temperatures at the ends of a bar of length -l- are often known for all time. In other words, the boundary conditions are known. It is also usual for the temperature distribution along the bar to be known at some particular instant. This instant is usually taken as zero time and the temperature distribution is called initial condition. The solution gives U for values of x between 0 and -l- and values of t from zero to infinity. Hence the area of integration S in the x-t plane, is the infinite area bounded by the x-axis and the parallel lines x = 0, x = l. this is described as an open area because the boundary curves marked C do not constitute a closed boundary in any finite region of the x-t plane.
The simplest hyperbolic equations generally originate from vibration problems, or from problems where discontinuities can persist in time, such as with shock waves, across which there are discontinuities in speed, pressure and density. The simplest equation is the one-dimensional wave equation $U_{tt} = c^2\ U_{xx}$ giving, for example, the transverse displacement U at a distance x from one end of a vibrating string of the length l after a time t. as the values of U at the ends of the string are usually known for all time (the boundary conditions) and the shape and velocity of the string are prescribed at zero time (the initial conditions), that the solution is similar to that of parabolic equation in that the calculation of U for given x and t $(0 \leq x \leq l)$, entails integration of the equation over the open area S bounded by the open curve C. Although hyperbolic equations can be solved numerically by finite difference methods, those involving only two independent variables, x and t say, are often deal with by the method of characteristics, especially if the initial conditions and/or boundary conditions involve discontinuities. This method finds special curves in the x-t plane, called characteristic curves, along which the solution of the partial differential equation is reduced to the integration of an ordinary differential equation. This ordinary differential equation is generally integrated by numerical methods.
After this section, PADM is considered to solve some typical problems according to the initial conditions.

3.1. The Proposed Adomian Method For Parabolic Partial Differential Equations

The ADM, when applied to the first-order parabolic differential equation is

$u_t = \alpha\, u_x + N\, u + h(x, t)$ (17.a)

with initial condition

$u(x,0) = f(x)$ (17.b)

where α is a scalar, h(x,t) non-homogeneous part of differential equation while the non-linear terms are presented by N u, L^{-1} is the inverse operator form of L which is defined $L^{-1}(*) = \int_0^t (*)dt$, we get

$u(x,t) = u(x,0) + L^{-1}\{ \alpha\, u_x + N\, u + h(x,t)\}$ (18)

if f(x) is a polynomial function, then

$u_0 = f(x)$ + { the parts of coming from $L^{-1}\{h(x,t)\}$ that are considering the degree of non-linear terms} (19)

is choosing and finite term series solutions are obtained. The ADM, when applied to a general second-order non-linear parabolic equation with initial conditions are in the form as

D.E. $u_t = \alpha\, u_{xx} + N\, u + h(x, t)$ (20a)

I.C. $u(0,t) = f(t)$ $u_x(0,t) = g(t)$ (20b)

where α, h(x, t), and N u are the same as stated above. L is taken as the highest order derivative, L^{-1} as regard as the inverse operator of L and is defined as two-fold integration operator from 0 to x, i.e., $L^{-1}(*) = \int_0^x\int_0^x (*)dxdx$, we get

$L^{-1}L\, u(x,t) = u(x,t)- u(0,t)-xu_x(0,t) + L^{-1}\{ h(x,t) + N\, u\}$ (21)

In Eq. (21) , if only Nu is equal to zero, means linear parabolic equation, where f(t) and g(t) are;

- a polynomial functions or sine and cosine from trigonometric functions, then
  $u_0 = f(t) + x\, g(t)$ + { the parts of coming from $L^{-1}\{h(x,t)\}$ which is including time t dependent variable part}
- exponential functions, then $u_0 = f(t) + x\, g(t)$ is choosing and the finite term series solution is obtained.

And also, if Nu is different from zero, means non-linear, where f(t) and g(t) are;

- polynomial, logarithmic, exponential sine and cosine trigonometric functions then $u_0 = f(t) + x\, g(t)$ is choosing,
- if both f(t) and g(t) are zero and h(x, t) is trigonometric function sine or cosine then u_0 = { the parts of coming from $L^{-1}\{h(x, t)\}$ that are considering the degree of non-linear terms} is choosing

and finite term series solutions are obtained.

3.2 The Proposed Adomian Method For Hyperbolic Partial Differential Equations

The first-order hyperbolic type initial boundary value problem is interested in two parts: First part, if the problem for the continuous conditions is defined as follows

D.E. $.u_t + \alpha\, u_x = 0$ (22)

I.C. $u(x,0) = f(x)$ (23)

where α is a scalar. By applying the proposed ADM to Eq.(22), where $L^{-1}(*) = \int_0^t (*)dt$, we get

$u(x,t) = u(x,0) + L^{-1}\{-\alpha\, u_x\} = f(x) + L^{-1}\{-\alpha\, u_x\}$ (24)

in the selection of $u_0(x,t)$, if f(x) is a polynomial function, then we take $u_0 = f(x)$ and $u_1 = + L^{-1}\{-\alpha\, u_x\}$ as a result of this choosing we obtain finite term series because of $u_1(x,t) = 0$. So, the solution of the problem is obtained.

The second part, if the problem (22) for the discontinuous initial condition which is defined as

$$u(x,0) = \begin{cases} f_1(x), & x \ge t \\ f_2(x), & x < t \end{cases} \quad (25)$$

and by applying the PADM to the problem, it is obtained from Eq. (24) under the condition (25) is

$$u(x,t) = \begin{cases} f_1(x), & x \ge t \\ f_2(x), & x < t \end{cases} - \alpha L^{-1}\{u_x\} \quad (26)$$

Choosing $u_0(x, t)$, if $f_1(x)$ and $f_2(x)$ are polynomial function, we take $u_0 = f_i(x)$ (i = 1, 2), hence we obtain finite term series which is named the solution function.

If the second-order partial differential equation is defined as in below

D.E. $u_{xx} + P(x)\, u_{xy} + Q(x)\, u_{yy} = 0$ (27.a)

I.C. $u(x,0) = f(x)$ and $u_y(x,0) = g(x)$ (27.b)

by applying PADM, where $L^{-1}(*) = \int_0^y\int_0^y (*)dydy$, are obtained

$$u(x,y) = f(x) + y\, g(x) + L^{-1}\left\{-\frac{1}{Q(x)}\left[u_{xx} + P(x)u_{xy}\right]\right\} \quad (28)$$

if f(x) and g(x) are polynomial function, we select $u_0 = f(x) + y\, g(x)$, then we get infinite term solution like $u(x, y) = u_0(x, y)$.

Similarly Eq. (27.a); if the problem second-order inhomogeneous equation as

D.E. $u_{xy} = h(x, y)$ (29.a)

I.C. $u(x,0) = f(y)$ and $u(0,y) = g(x)$ (29.b)

applying the PADM,

$$u(x,y) = u(0,y) + \int_0^x \frac{\partial u(x,0)}{\partial x}dx + L^{-1}\{h(x,y)\} \quad (30)$$

when f(x) and g(x) are polynomials, then we have finite term series solution where taking $u_0 = f(y) + g(x)$.

Second order general hyperbolic type initial value problem is determined as

D.E. $u_{tt} = \alpha^2 u_{xx} + N\, u + h(x,t)$ (31.a)

I.C. $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ (31.b)

by using Adomian method, we have

$u(x,t) = u(x,0) + tu_t(x,0) + L^{-1}\{\alpha^2 u_{xx} + h(x,t) + N\, u\}$ (32)

where the inverse operator $L^{-1}(*) = \int_0^t\int_0^t (*)dtdt$. In Eq. (31), first, if both h(x, t) and Nu is equal to zero, it means that linear homogeneous initial value problem,

where f(x) and g(x) are the polynomial functions, then u_0 = f(x) + t g(x) is choosing and the finite term series solution is obtained. Second, in Eq. (31), if only Nu is equal zero, means linear and non-homogeneous initial value problem, where f(x) and g(x) are polynomial functions, then u_0 = f(x) + tg(x) + {the parts of coming from L^{-1}{h(x, t)} which is including x independent variables parts} is choosing and the solution is obtained as finite term series. And, finally, if both h(x, t) and Nu are different from zero, where f(x) and g(x) are polynomial, then u_0 = f(x) + t g(x) + {the parts of coming from L^{-1}{h(x, t)}that are considering the degree of non-linear terms} is choosing and finite term series solutions are obtained.
The first and the second-order parabolic and hyperbolic type partial differential equations are applied to the considering method. For both types, examples are taken into emphasize to obtain results and these examples are presented in Table-1 and Table-2, respectively.

4. CONCLUSION

In this work, different from the other authors, Adomian decomposition method is investigated according to choose of u_0 for the parabolic and hyperbolic type initial value problems, with the considering method is applied some problems it is shown that the convergency speed of the proposed Adomian Decomposition series solution is quite rapid. Because of this rapid significantly depends on the proper choice of f_0 and $f_{1,2}$. For this situation, we gave some clues to judge which parts are selecting u_0. So, in many cases, the exact solutions were obtained by determining two components u_0 and u_1 only without using Adomian polynomials for the non-linear terms. Applying proposed method is obtained finite terms series by the solutions are shown that how rapid and simple integral used in this method.

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