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FLAT MANİFOLDLAR ÜZERİNDE WEYL-HAMİLTON DENKLEMLERİ

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ÖZET

1918 yılında H. Weyl uzay-zaman üzerindeki eğrilik ölçüsü için Riemann geometrisini genelleyerek bir birleşik alan teorisi formüle edip sundu. Klasik alan teorisi geleneksel Hamilton dinamiklerinin dilini kullanır. Bir flat manifold yerel mesafeler açısından Öklid uzay gibi görünür. Bir tam Flat manifoldu evrensel örtüsü Öklid alandır. Bu makale, flat manifoldu üzerindeki Weyl-Hamilton denklemleri ile ilgilenir. Bu çalışmanın sonucunda, uzayda nesnelerin hareketi için elde edilecek kısmi diferansiyel denklemler ve bu denklemlerin kapalı çözümleri sembolik hesaplama programı kullanılarak yapıldı. Ayrıca, matematiksel ve fiziksel sonuçlar ortaya çıkan denklem için sunulmaktadır.

***Anahtar Kelimeler:* :** Weyl Manifold, Flat Manifold, Mekanik Sistem, Dinamik Proğramlama, Hamilton Formalizmi.

WEYL-HAMILTON EQUATIONS ON FLAT MANIFOLD

ABSTRACT

In 1918 H. Weyl introduced a generalization of Riemannian geometry in his attempt to formulate a unified field theory for a measure of the curvature on space-time. Classical field theory utilizes traditionally the language of Hamiltonian dynamics. A flat manifold is locally looks like Euclidean space in terms of distances. The universal cover of a complete flat manifold is Euclidean space. This paper deals with Weyl-Hamilton equations on flat manifold. As a result of this study, partial differential equations have be obtained for movement of objects in space and closed solutions of these equations have be made using symbolic computational program. Also, the mathematical and physical results are presented for the resulting equations.

Keywords: Weyl Manifold, Flat Manifold, Mechanical System, Dynamic Equation, Dynamic Programming, Hamiltonian Formalism

1. INTRODUCTION

Geodesics always have attracted the attention of researchers. It is well known a geodesic is the shortest route between two points. Geodesics can be found with the help of the Hamiltonian equations. It is showed that Hamiltonian mechanics are very important tools for analytical mechanics. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows. In this article, Weyl structures on flat manifolds will be transferred to the mechanical system.

Thus the time-dependent Weyl-Hamilton partial equations of motion of the dynamic systems have be found and an example will be given on the solution of the equations. Now follows, we let some work on this subject.

Olszak investigated paraquaternionic analogy of these ideas applied to conformally flat almost pseudo-Kählerian as well as almost para-Kählerian manifolds [1]. *Schwartz* considered asymptotically flat Riemannian manifolds with non-negative scalar curvature that are conformal to $R^n \backslash \Omega$, $n \geq 3$, and such that their boundary is a minimal hypersurface [2]. *Ge* et al. submitted that the mass of an asymptotically flat n-manifold is a geometric invariant [3]. *Gonzalez* explored complete, locally conformally flat metrics defined on a domain $\Omega \subset S^n$ [4]. *Upadhyay* indicated bounding question for almost at manifolds by looking at the equivalent description of them as infranilmanifolds $\Gamma \backslash L \rtimes G/G$. He showed that infranilmanifolds $\Gamma \backslash L \rtimes G/G$ bound if L is a 2-step nilpotent group [5]. *Kapovich* obtained an existence theorem for flat conformal structures on finite-sheeted coverings over a wide class of Haken manifolds [6]. *Zhu* obtained a classification of complete locally conformally flat manifolds of nonnegative Ricci curvature [7]. *Kulkarni* revealed some new examples of conformally flat manifolds, as a step toward a classification of such manifolds up to conformal equivalence [8]. *Dotti and Miatello* purposed the real cohomology ring of low dimensional compact flat manifolds endowed with one of these special structures [9]. *Szczepanski* revealed that a list of six dimensional at Kähler manifolds and he submitted an example of eight dimensional at Kähler manifold with finite group [10]. *Akbulut and Kalafat* studied infinite families of non-simply connected locally conformally flat (LCF) 4-manifolds realizing rich topological types [11]. *Abood* proved that if M is flat manifold with flat Bochner tensor, then M is an Einstein manifold with a cosmological constant [12]. *Lutowski* showed an example of a *Bieberbach* group Γ for which Out (Γ) is a cyclic group of order 3. He also calculated the outer automorphism group of a direct product of n copies of a *Bieberbach* group with trivial center [13]. *Kasap and Tekkoyun* obtained Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research [14]. *Kasap* found that the Weyl-Euler-Lagrange and Weyl-Hamilton equations on R_n^{2n} which is a model of tangent manifolds of constant W-sectional curvature [15].

2. PRELIMINARIES

Throughout this study, all the manifolds and geometric objects are C^∞ and the Einstein summation convention ($\sum a_j x^j = a_j x^j$) is in use. Also, A, F(TM), χ(TM) and Λ^1(TM) denote the set of para-complex numbers, the set of complex functions on TM, the set of complex vector fields on TM and the set of complex 1-forms on TM, respectively.

3. RIEMANN MANIFOLD

Definition 1: Tangent space given any point $p \in M$, it has a tangent space T_pM isometric to R^n. If we have a metric (inner-product) in this space $<,>_p : T_pM \times T_pM \mapsto R$ defined on every point $p \in M$, we thus call M Riemann Manifold.

4. CONFORMAL GEOMETRY

A conformal map is a function which preserves angles. Conformal maps can be defined between domains in higher dimensional Euclidean spaces and more generally on a (semi) Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on a space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces. In more than two dimensions, conformal geometry may refer either to the study of conformal transformations of flat spaces, such as Euclidean spaces or spheres, to the study of conformal manifolds which are Riemann or pseudo-Riemann manifolds with a class of metrics defined up to scale. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics g′ and g are equivalent if and only if $g' = \Psi^2 g$ where $\Psi > 0$ is a smooth positive function and an equivalence class of such metrics is known as a conformal metric or conformal class [16].

5. CONFORMALLY FLAT MANIFOLD

Definition 2: A manifold with a Riemannian metric, it has zero curvature, is a flat manifold.

A flat manifold is one such that locally looks like Euclidean space in terms of distances and angles, e.g. the interior angles of a triangle add up to 180°. The basic example is Euclidean space with the usual metric $\mathrm{ds}^2 = \sum_{i=1}^{3} dx_i^2$. Indeed any point on a flat manifold has a neighborhood isometric to a neighborhood in Euclidean space. A flat manifold is locally Euclidean in terms of distances and angles, as well as merely topologically locally Euclidean, as all manifolds are. The simplest nontrivial examples occur as surfaces in four dimensional spaces. For example, the flat torus is a flat manifold. It is the image of f(x,y)=(cosx,sinx,cosy,siny). A theorem due to *Bieberbach* says that all compact flat manifolds are tori. More generally, the universal cover of a complete flat manifold is Euclidean space. The integrability of the almost complex structure implies a relation in the curvature. Let $\{x_i, y_i,\ i=1,2,3\}$ be coordinates on R^6 with the standard flat metric $\mathrm{ds}^2 = \sum_{i=1}^{3} (dx_i^2 + dy_i^2)$ [17].

Definition 3: A (pseudo) Riemannian manifold is conformally flat manifold if each point has a neighborhood that can be mapped to flat space by a conformal transformation.

Let (M,g) be a pseudo-Riemannian manifold. Then (M,g) is conformally flat if for each point x in M, there exists a neighborhood U of x and a smooth function f defined on U such that $(U, e^{2f}g)$ is flat (i.e. the curvature of $e^{2f}g$ vanishes on U). The function f need not be defined on all of M. Some authors use locally conformally flat to describe the above notion and reserve conformally flat for the case in which the function f is defined on all of M [18].

6. THE THEORY OF J-HOLOMORPHIC CURVES

A pseudo *J*-holomorphic curve is a smooth map from a Riemannian surface into an almost complex manifold such that satisfies the Cauchy--Riemann equation. Pseudo-holomorphic curves have since revolutionized the study of symplectic manifolds. The theory of *J*-holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry and making it possible to study the global structure of symplectic manifolds. Aforementioned the methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic. This effect of this is to ensure that one is looking at persistent rather than accidental features of these curves. But, the perturbed structure may no longer be integrable, and so again one is led to the study of curves such that it's are holomorphic with respect to some non-integrable almost complex structure *J* [19].

7. WEYL GEOMETRY

A conformal transformation for use in curved lengths has been revealed. The linear distance between two points can be found easily by Riemann metric. Many scientists have used the Riemann metric. Einstein was one of the first studies in this field. Einstein discovered such that the Riemannian geometry and successfully used it to describe General Relativity in the 1910 that is actually a classical theory for gravitation. But, the universe is really completely not like Riemannian geometry. Each path between two points is not always linear. Also, orbits of move objects may change during movement. So, each two points in space may not be linear geodesic. Therefore, new metric is needed for non-linear distances like spherical surface. Then, a method is required for converting nonlinear distance to linear distance. Weyl introduced a metric with a conformal transformation in 1918. The basic concepts related to the topic are listed below [20,21].

Definition 4: Two Riemann metrics g_1 and g_2 on M are said to be conformally equivalent iff there exists a smooth function f: $M \rightarrow R$ with

$$e^f g_1 = g_2 \tag{1}$$

In this case: $g_1 \sim g_2$.

Definition 5: Let M an n-dimensional smooth manifold. A pair (M,G), a conformal structure on M is an equivalence class G of Riemann metrics on M, is called a conformal structure.

Theorem 1: Let D be a connection on M and $g \in G$ a fixed metric. D is compatible with $(M,G) \Leftrightarrow$ there exists a 1-form ω with $D_X g + \omega(X) g = 0$.

Definition 6: A compatible torsion-free connection is called a Weyl connection. The triple (M,G,D) is a Weyl structure.

Theorem 2: To each metric $g\in G$ and 1-form ω, there corresponds a unique Weyl connection D satisfying $D_Xg+\omega(X)g=0$.

Definition 7: Define a function F:{1-forms on M}×G→{Weyl connections} by $F(g,\omega)=D$, where D is the connection guaranteed by **Theorem 2**. We say that D corresponds to (g,ω).

Proposition 1: 1. F is surjective. 2. $F(g,\omega)=F(e^fg,\eta)$ iff $\eta=\omega-df$. So

$$F(e^fg)=F(g)-df. \tag{2}$$

Where, G is a conformal structure. Note that a Riemann metric g and a one-form ω determine a Weyl structure, namely $F: G\rightarrow \Lambda^1M$ where G is the equivalence class of g and $F(e^fg)=\omega-df$.

Proof: Suppose $F(g,\omega)=F(e^fg,\eta)=D$. We have

$$D_X(e^fg)+\eta(X)e^fg=X(e^f)g+e^fD_Xg+\eta(X)e^fg=df(X)e^fg+e^fD_Xg+\eta(X)e^fg=0. \tag{3}$$

Therefore $D_Xg=-(df(X)+\eta(X))$. On the other hand $D_Xg+\omega(X)g=0$. Therefore $\omega=\eta+df$. Set $D=F(g,\omega)$. To show $D=F(e^fg,\eta)$ and $D_X(e^fg)+\eta(X)\, e^fg=0$. To calculate

$$D_X(e^fg)+\eta(X)e^fg=e^fdf(X)g+e^fD_Xg+(\omega(X)-df(X))e^fg=e^f(D_Xg+\omega(X)g)=0. \tag{4}$$

Theorem 3: A connection on the metric bundle ω of a conformal manifold M naturally induces a map $F: G\rightarrow \Lambda^1M$ and (2), and conversely. Parallel translation of points in ω by the connection is the same as their translation by F.

Theorem 4: Let $m\geq 6$. If (M,g,∇,W) is a Kähler--Weyl structure, then the associated Weyl structure is trivial, i.e. there is a conformally equivalent metric $\breve{g}=e^{2f}g$ so that $(M,\breve{g},W)$ is Kähler and so that $\nabla=\nabla^{\breve{g}}$ [22,23].

8. WEYL CURVATURE TENSOR

Definition 8: Weyl curvature tensor is a measure of the curvature of space-time or a pseudo-Riemannian manifold. Like the Riemannian curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic.

The Weyl tensor differs from the Riemannian curvature tensor in that it does not convey information on how the volume of the body changes, but rather only how the shape of the body is distorted by the tidal force. The Ricci curvature, or trace component of the Riemannian tensor contains precisely the information about how volumes change in the presence of tidal forces, so the Weyl tensor is the traceless component of the Riemannian tensor. It is a tensor that has the same symmetries as the Riemannian tensor with the extra condition that it be trace-free: metric contraction on any pair of indices yields zero. In general relativity, the Weyl curvature is the only part of the curvature that exists in free space a solution of the suction Einstein equation-and it governs the propagation of gravitational radiation through regions of space devoid of matter. The Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold. In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions≥4, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension≥4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor. This fact was a key component of Nordstrom's theory of gravitation, which was a precursor of general relativity [25].

9. ALMOST COMPLEX MANIFOLD

Definition 9: Let M be a smooth manifold of real dimension 2n. We say that a smooth atlas A of M is holomorphic if for any two coordinate charts $z:U \to U' \subset C^m$ and $w:V \to V' \subset C^m$ in A, the coordinate transition map $z \circ w^{-1}$ is holomorphic. Any holomorphic atlas uniquely determines a maximal holomorphic atlas, and a maximal holomorphic atlas is called a complex structure for M. We say that M is a complex manifold of complex dimension n if M comes equipped with a holomorphic atlas. Any coordinate chart of the corresponding complex structure will be called a holomorphic coordinate chart of M. A Riemann surface or complex curve is a complex manifold of complex dimension 1.

Definition 10: Let M be a differentiable manifold of dimension 2n, and suppose J is a differentiable vector bundle isomorphism J: TM→TM such that J_x: $T_xM \to T_xM$ is a (almost) complex structure for T_xM, i.e. $J^2=-I$ where I is the identity (unit) operator on V. Then J is called an almost complex structure for the differentiable manifold M. A manifold with a fixed (almost) complex structure is called an (almost) complex manifold. Where $J^2=J\circ J$, and I is the identity operator on V.

A celebrated theorem of *Newlander and Nirenberg* says that an almost (para) complex structure is a (para) complex structure if and only if its Nijenhuis tensor or torsion vanishes [26].

Theorem 5: The almost (para)complex structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J[X,Y]=[JX,JY]-J[X,JY]-J[JX,Y]-[X,Y]. \qquad (5)$$

The tensor (2,1) is called the *Nijenhuis* tensor (5). We say that J is torsion free if $N_J=0$. Complex *Nijenhuis* tensor of an almost complex manifold (M,J) is given by (5).

10. WEYL HOLOMORPHIC STRUCTURES

It vanishes if and only if *J* is an integrable almost complex structure, i.e. given any point P∈M, there exist local coordinates (x_i,yi), i=1,2,3 centered at P and $f=f(x_i,y_i)$, following structures taken from;

$$J\partial x_1=\cos(x_3)\partial y_1+\sin(x_3)\partial y_2,\ J\partial x_2=-\sin(x_3)\partial y_1+\cos(x_3)\partial y_2,\ J\partial x_3=\partial y_3,$$

$$J\partial y_1=-\cos(x_3)\partial x_1+\sin(x_3)\partial x_2,\ J\partial y_2=-\sin(x_3)\partial x_1-\cos(x_3)\partial x_2,\ J\partial y_3=-\partial x_3. \qquad (6)$$

The above structures (6) have been taken from [27]. Here, instead of *J* conformal structure representing the structure of W will be used and $\partial x_i=dx_i$, $\partial y_i=dy_i$.

Proposition 2: W* is the dual of the W. If we extend the equation (6) by means of conformal structure [18,28] and Theorem 4, we can give equations as follows:

1. $W^*(dx_1)=e^{2f}\cos(x_3)dy_1+e^{2f}\sin(x_3)dy_2,$
2. $W^*(dx_2)=-e^{2f}\sin(x_3)dy_1+e^{2f}\cos(x_3)dy_2,$
3. $W^*(dx_3)=e^{2f}dy_3,$ (7)
4. $W^*(dy_1)=-e^{-2f}\cos(x_3)dx_1+e^{-2f}\sin(x_3)dx_2,$
5. $W^*(dy_2)=-e^{-2f}\sin(x_3)dx_1-e^{-2f}\cos(x_3)dx_2,$
6. $W^*(dy_3)=-e^{-2f}dx_3,$

such that are base structures for Hamilton equations. Where, W is a conformal complex structure to be similar to an integrable almost complex J given in (6). From now on, we continue our studies thinking of the (TM,g,∇,W^*). Now, W* denote the structure of the holomorphic property:

$$\begin{aligned}W^{*2}(dx_1)&=W^*\circ W^*(dx_1)=e^{2f}\cos(x_3)W^*(dx_1)+e^{2f}\sin(x_3)\,W^*(dy_2)\\&=e^{2f}\cos(x_3)[-e^{-2f}\cos(x_3)dx_1+e^{-2f}\sin(x_3)dx_2]+e^{2f}\sin(x_3)[-e^{-2f}\sin(x_3)dx_1\\&\quad-e^{-2f}\cos(x_3)dx_2]\\&=-\cos^2(x_3)dx_1+\cos(x_3)\sin(x_3)dx_2-\sin^2(x_3)dx_1-\sin(x_3)\cos(x_3)dx_2\\&=-[\cos^2(x_3)+\sin^2(x_3)]dx_1=-dx_1,\end{aligned} \qquad (8)$$

and similar manner it is shown that

$$W^{*2}(dx_i)=-dx_i,\ W^{*2}(dy_i)=-dy_i,\quad i=1,2,3. \qquad (9)$$

As can be seen from (8) and (9) $W^{*2}=-I$ are the complex structures.

11. HAMILTONIAN SYSTEM

Definition 11: Let M is the base manifold of dimension n and its cotangent manifold T^*M. By a symplectic form we mean a 2-form Φ on T^*M. Let (T^*M,Φ) be a symplectic manifold, there is a unique vector field X_H on T^*M and H: $T^*M \to \mathbb{R}$ is called as Hamiltonian Function (H=T+V, T is the kinetic energy and V is the potential energy) such that Hamiltonian Dynamical Equation is determined by

$$i_{X_H}\Phi = dH. \tag{10}$$

We can say that X_H is locally Hamiltonian vector field. Φ is closed and also shows the canonical symplectic form so that $\Phi=-d\Omega$, $\Omega=J^*(\omega)$, J^* a dual of J, ω a 1-form on T^*M. The triple (T^*M,Φ,X_H) is named Hamiltonian System which is defined on the cotangent bundle T^*M. From the local expression for X_H we see that $(q^i(t),p_i(t))$ is an integral curve of X_H if Hamilton's Equations is expressed as follows [29,30].

$$q^i=(\partial H)/(\partial p_i)\ ,\ p_i=-(\partial H)/(\partial q^i). \tag{11}$$

12. WEYL-HAMILTON EQUATIONS

Now, we will present Hamilton equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on flat manifold. Let (T^*M,W^*) be on flat manifold. Suppose that the complex structures, a Liouville form and a 1-form on flat manifold are shown by W^*, Ω and ω, respectively. Consider a 1-form ω such that

$$\omega=(1/2)[x_1dx_1+x_2dx_2+x_3dx_3-y_1dy_1-y_2dy_2-y_3dy_3]. \tag{12}$$

Then, we obtain the Liouville form as follows:

$$\begin{aligned}
\Omega=-W^*(\omega)=&-(1/2)[x_1W^*(dx_1)+x_2W^*(dx_2)+x_3W^*(dx_3)-y_1W^*(dy_1)-y_2W^*(dy_2)\\
&-y_3W^*(dy_3)]\\
=&-(1/2)[x_1(e^{2f}\cos(x_3)dy_1+e^{2f}\sin(x_3)dy_2)+x_2(-e^{2f}\sin(x_3)dy_1+e^{2f}\cos(x_3)dy_2)+x_3e^{2f}dy_3\\
&+y_1(e^{-2f}\cos(x_3)dx_1-e^{-2f}\sin(x_3)dx_2)+y_2(e^{-2f}\sin(x_3)dx_1+e^{-2f}\cos(x_3)dx_2)+y_3e^{-2f}dx_3].
\end{aligned} \tag{13}$$

It is well known that if Φ is a closed on flat manifold, then Φ is also a symplectic structure on (T^*M,W^*). Therefore the 2-form $\Phi=-d\Omega$ indicates the canonical symplectic form and derived from the 1-form Ω to find to mechanical equations. Then the 2-form Φ is calculated as below:

$\Phi=d(\Omega)$

$=(1/2)[e^{2f}\cos(x_3)dy_1\wedge dx_1+e^{2f}\sin(x_3)dy_2\wedge dx_1+x_1 2((\partial f)/(\partial x_1))e^{2f}\cos(x_3)dy_1\wedge dx_1$

$+x_1 2((\partial f)/(\partial x_1))e^{2f}\sin(x_3)dy_2\wedge dx_1-e^{2f}\sin(x_3)dy_1\wedge dx_2+e^{2f}\cos(x_3)dy_2\wedge dx_2$

$-x_2 2((\partial f)/(\partial x_2))e^{2f}\sin(x_3)dy_1\wedge dx_2+x_2 2((\partial f)/(\partial x_2))e^{2f}\cos(x_3)dy_2\wedge dx_2$

$+x_1 2((\partial f)/(\partial x_3))e^{2f}\cos(x_3)dy_1\wedge dx_3+x_1 2((\partial f)/(\partial x_3))e^{2f}\sin(x_3)dy_2\wedge dx_3$

$-x_1 e^{2f}\sin(x_3)dy_1\wedge dx_3+x_1 e^{2f}\cos(x_3)dy_2\wedge dx_3-x_2 2((\partial f)/(\partial x_3))e^{2f}\sin(x_3)dx_3\wedge dy_1$

$+x_2 2((\partial f)/(\partial x_3))e^{2f}\cos(x_3)dy_2\wedge dx_3-x_2 e^{2f}\cos(x_3)dy_1\wedge dx_3-x_2 e^{2f}\sin(x_3)dy_2\wedge dx_3$

$+e^{2f}dy_3\wedge dx_3+x_3 2((\partial f)/(\partial x_3))e^{2f}dy_3\wedge dx_3-2y_1((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_1\wedge dx_3$

$+2y_1((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_2\wedge dx_3-y_1 e^{-2f}\sin(x_3)dx_1\wedge dx_3-y_1 e^{-2f}\cos(x_3)dx_2\wedge dx_3$

$-y_2 2((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_1\wedge dx_3-y_2 2((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_2\wedge dx_3$

$+y_2 e^{-2f}\cos(x_3)dx_1\wedge dx_3-y_2 e^{-2f}\sin(x_3)dx_2\wedge dx_3+e^{-2f}\cos(x_3)dx_1\wedge dy_1-e^{-2f}\sin(x_3)dx_2\wedge dy_1$

$-y_1 2((\partial f)/(\partial y_1))e^{-2f}\cos(x_3)dx_1\wedge dy_1+y_1 2((\partial f)/(\partial y_1))e^{-2f}\sin(x_3)dx_2\wedge dy_1$

$+e^{-2f}\sin(x_3)dx_1\wedge dy_2+e^{-2f}\cos(x_3)dx_2\wedge dy_2-y_2 2((\partial f)/(\partial y_1))e^{-2f}\sin(x_3)dx_1\wedge dy_2$

$-y_2 2((\partial f)/(\partial y_2))e^{-2f}\cos(x_3)dx_2\wedge dy_2+e^{-2f}dx_3\wedge dy_3-y_3 2((\partial f)/(\partial y_3))e^{-2f}dx_3\wedge dy_3].$

(14)

Take a vector field X_H so that called to be Hamiltonian vector field associated with Hamiltonian energy H and determined by

$$X_H=\sum_{i=1}^{3}\left(X^i\frac{\partial}{\partial x_i}+Y^i\frac{\partial}{\partial y_i}\right). \tag{15}$$

$\Phi(X_H)$ will be calculated using Φ and X_H. Calculations use external product feature. These properties are

$$f\wedge g=-g\wedge f$$

$$f\wedge g(v)=f(v)g-g(v)f.$$

(16)

We have

$$i_{X_H}\Phi = \Phi\left(X_H\right)$$

$$=(1/2)X^1(-e^{2f}\cos(x_3))dy_1-e^{2f}\sin(x_3)dy_2-x_1 2((\partial f)/(\partial x_1))e^{2f}\cos(x_3)dy_1$$

$$-x_1 2((\partial f)/(\partial x_1))e^{2f}\sin(x_3)dy_2)+(1/2)X^1(-2y_1((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_3$$

$$-y_1e^{-2f}\sin(x_3)dx_3-y_2 2((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_3+y_2e^{-2f}\cos(x_3)dx_3)$$

$$+(1/2)X^1(-e^{-2f}\cos(x_3)dy_1-y_1 2((\partial f)/(\partial y_1))e^{-2f}\cos(x_3)dy_1+e^{-2f}\sin(x_3)dy_2$$

$$-y_2 2((\partial f)/(\partial y_2))e^{-2f}\sin(x_3)dy_2)+(1/2)X^2(e^{2f}\sin(x_3)dy_1-e^{2f}\cos(x_3)dy_2$$

$$+x_2 2((\partial f)/(\partial x_2))e^{2f}\sin(x_3)dy_1-x_2 2((\partial f)/(\partial x_2))e^{2f}\cos(x_3)dy_2)$$

$$+(1/2)X^2(2y_1((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_3-y_1e^{-2f}\cos(x_3)dx_3$$

$$-y_2 2((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_3-y_2e^{-2f}\sin(x_3)dx_3)+(1/2)X^2(-e^{-2f}\sin(x_3)dy_1$$

$$+y_1 2((\partial f)/(\partial y_1))e^{-2f}\sin(x_3)dy_1+e^{-2f}\cos(x_3)dy_2-y_2 2((\partial f)/(\partial y_2))e^{-2f}\cos(x_3)dy_2)$$

$$+(1/2)X^3(-e^{2f}dy_3-x_3 2((\partial f)/(\partial x_3))e^{2f}dy_3+2y_1((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_1$$

$$-2y_1((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_2+y_1e^{-2f}\sin(x_3)dx_1)+(1/2)X^3(y_1e^{-2f}\cos(x_3)dx_2$$

$$+y_2 2((\partial f)/(\partial x_3))e^{-2f}\sin(x_3)dx_1+y_2 2((\partial f)/(\partial x_3))e^{-2f}\cos(x_3)dx_2-y_2e^{-2f}\cos(x_3)dx_1)$$

$$+(1/2)X^3(y_2e^{-2f}\sin(x_3)dx_2-e^{-2f}dy_3+y_3 2((\partial f)/(\partial y_3))e^{-2f}dy_3)+(1/2)Y^1(e^{2f}\cos(x_3))dx_1$$

$$+x_1 2((\partial f)/(\partial x_1))e^{2f}\cos(x_3)dx_1-e^{2f}\sin(x_3)dx_2-x_2 2((\partial f)/(\partial x_2))e^{2f}\sin(x_3)dx_2)$$

$$+(1/2)Y^1(e^{-2f}\cos(x_3)dx_1+e^{-2f}\sin(x_3)dx_2+y_1 2((\partial f)/(\partial y_1))e^{-2f}\cos(x_3)dx_1$$

$$-y_1 2((\partial f)/(\partial y_1))e^{-2f}\sin(x_3)dx_2)+(1/2)Y^2(e^{2f}\sin(x_3)dx_1+x_1 2((\partial f)/(\partial x_1))e^{2f}\sin(x_3)dx_1$$

$$+e^{2f}\cos(x_3)dx_2+x_2 2((\partial f)/(\partial x_2))e^{2f}\cos(x_3)dx_2)+(1/2)Y^2(-e^{-2f}\sin(x_3)dx_1$$

$$-e^{-2f}\sin\cos(x_3)dx_2+y_2 2((\partial f)/(\partial y_2))e^{-2f}\sin(x_3)dx_1+y_2 2((\partial f)/(\partial y_2))e^{-2f}\cos(x_3)dx_2)$$

$$+(1/2)Y^3(e^{2f}dx_3+x_3 2((\partial f)/(\partial x_3))e^{2f}dx_3+e^{-2f}dx_3-y_3 2((\partial f)/(\partial y_3))e^{-2f}dx_3).$$

(17)

Furthermore, the differential of Hamiltonian energy H is obtained by

$$dH=\sum_{i=1}^{3}\left(\frac{\partial H}{\partial x_i}dx_i+\frac{\partial H}{\partial y_i}dy_i\right) \tag{18}$$

X^1, X^2, X^3, Y^1, Y^2 and Y^3 are obtained using the $i_{X_H}\Phi = dH$. Consider the curve and its velocity vector fields;

$$\begin{aligned} \sim : I \subset R \to M,\ \ \sim(t) &= \sum_{i=1}^{3}\left(x_i(\mathrm{t}), \mathrm{y}_i(t)\right), \\ &= \sum_{i=1}^{3}\left(\frac{dx_i}{dt}\frac{\partial}{\partial x_i}+\frac{dy_i}{dt}\frac{\partial}{\partial y_i}\right), \end{aligned} \tag{19}$$

such that an integral curve of the Hamiltonian vector field X_H, i.e.,

$$X_H(\sim(t))=\frac{\partial \sim}{\partial t}=\sum_{i=1}^{3}\left(\frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_i}+\frac{\partial H}{\partial y_i}\frac{\partial}{\partial y_i}\right),\ \ t\in I. \tag{20}$$

Then, we find the following equations;

$$\begin{aligned}
&(1)\ Y^1\cos(x_3)(e^{2f}+x_1 2((\partial f)/(\partial x_1))e^{2f}-e^{-2f}-y_1 2((\partial f)/(\partial y_1))e^{-2f}) \\
&\quad +Y^2\sin(x_3)(e^{2f}+x_1 2((\partial f)/(\partial x_1))e^{2f}+e^{-2f}-y_2 2((\partial f)/(\partial y_1))e^{-2f})=2((\partial H)/(\partial x_1)), \\
&(2)\ Y^1\sin(x_3)(-e^{2f}-x_2 2((\partial f)/(\partial x_2))e^{2f}-e^{-2f}+y_1 2((\partial f)/(\partial y_1))e^{-2f}) \\
&\quad +Y^2\cos(x_3)(e^{2f}+x_2 2((\partial f)/(\partial x_2))e^{2f}+e^{-2f}-y_2 2((\partial f)/(\partial y_2))e^{-2f})=2((\partial H)/(\partial x_2)), \\
&(3)\ Y^3(e^{2f}+x_3 2((\partial f)/(\partial x_3))e^{2f}+e^{-2f}-y_3 2((\partial f)/(\partial y_3))e^{-2f})=2((\partial H)/(\partial x_3)), \\
&(4)\ X^1\cos(x_3)(-e^{2f}-x_1 2((\partial f)/(\partial x_1))e^{2f}+e^{-2f}+y_1 2((\partial f)/(\partial y_1))e^{-2f}) \\
&\quad +X^2\sin(x_3)(e^{2f}+x_2 2((\partial f)/(\partial x_2))e^{2f}+e^{-2f}-y_1 2((\partial f)/(\partial y_1))e^{-2f})=2((\partial H)/(\partial y_1)), \\
&(5)\ X^1\sin(x_3)(-e^{2f}-x_1 2((\partial f)/(\partial x_1))e^{2f}-e^{-2f}+y_2 2((\partial f)/(\partial y_1))e^{-2f}) \\
&\quad +X^2\cos(x_3)(-e^{2f}-x_2 2((\partial f)/(\partial x_2))e^{2f}-e^{-2f}+y_2 2((\partial f)/(\partial y_2))e^{-2f})=2((\partial H)/(\partial y_2)), \\
&(6)\ X^3(-e^{2f}-x_3 2((\partial f)/(\partial x_3))e^{2f}-e^{-2f}+y_3 2((\partial f)/(\partial y_3))e^{-2f})=2((\partial H)/(\partial y_3))dy_3,
\end{aligned} \tag{21}$$

or

$$A=\cos(x_3)(e^{2f}+x_1 2((\partial f)/(\partial x_1))e^{2f}-e^{-2f}-y_1 2((\partial f)/(\partial y_1))e^{-2f})$$

$$B=\sin(x_3)(e^{2f}+x_1 2((\partial f)/(\partial x_1))e^{2f}+e^{-2f}-y_2 2((\partial f)/(\partial y_1))e^{-2f})$$

$$C=\sin(x_3)(-e^{2f}-x_2 2((\partial f)/(\partial x_2))e^{2f}-e^{-2f}+y_1 2((\partial f)/(\partial y_1))e^{-2f})$$

$$D=\cos(x_3)(e^{2f}+x_2 2((\partial f)/(\partial x_2))e^{2f}+e^{-2f}-y_2 2((\partial f)/(\partial y_2))e^{-2f})$$

$$E=\cos(x_3)(-e^{2f}-x_1 2((\partial f)/(\partial x_1))e^{2f}+e^{-2f}+y_1 2((\partial f)/(\partial y_1))e^{-2f})$$

$$F=\sin(x_3)(e^{2f}+x_2 2((\partial f)/(\partial x_2))e^{2f}+e^{-2f}-y_1 2((\partial f)/(\partial y_1))e^{-2f})$$

$$G=\sin(x_3)(-e^{2f}-x_1 2((\partial f)/(\partial x_1))e^{2f}-e^{-2f}+y_2 2((\partial f)/(\partial y_1))e^{-2f})$$

$$H=\cos(x_3)(-e^{2f}-x_2 2((\partial f)/(\partial x_2))e^{2f}-e^{-2f}+y_2 2((\partial f)/(\partial y_2))e^{-2f}),$$

(22)

so,

(PDE1) $((dx_1)/(dt))=(2/((EH-FG)))[H((\partial H)/(\partial y_1))-F((\partial H)/(\partial y_2))]$

(PDE2) $((dx_2)/(dt))=(2/((GF-EH)))[G((\partial H)/(\partial y_1))-E((\partial H)/(\partial y_2))]$

(PDE3) $((dx_3)/(dt))=(2/((-e^{2f}-x_3 2((\partial f)/(\partial x_3))e^{2f}-e^{-2f}$

$+y_3 2((\partial f)/(\partial y_3))e^{-2f})))((\partial H)/(\partial y_3))$

(PDE4) $((dy_1)/(dt))=(2/((DA-BC)))[D((\partial H)/(\partial x_1))-B((\partial H)/(\partial x_2))]$

(PDE5) $((dy_2)/(dt))=(2/((BC-AD)))[C((\partial H)/(\partial x_1))-A((\partial H)/(\partial x_2))]$

(PDE6) $((dy_3)/(dt))=(2/((e^{2f}+x_3 2((\partial f)/(\partial x_3))e^{2f}+e^{-2f}$

$-y_3 2((\partial f)/(\partial y_3))e^{-2f})))((\partial H)/(\partial x_3)).$

(23)

Hence, the equations introduced in (23) are named Weyl-Hamilton equations on flat manifold (T^*M,W^*) and then the triple (T^*M,Φ,X_H) is said to be a Weyl-Hamiltonian mechanical system on flat manifold.

13. EQUATİONS SOLVING WITH COMPUTER

These found (23) are partial differential equation and there are seven independent variables. Obtained from equation (PDE3) as an example will be solved with symbolic computational program software. The software codes of these equation,

Codes of Equations (PDE3)

PDE3:=diff(x_1(t),t)=2∗diff(H(x_1,x_2,x_3,y_1,y_2,y_3,t),y_3)/(-exp(2∗f)

-x_1(t)∗2∗diff(f,c)∗exp(2∗f)-exp(-2∗f)+y_3(t)∗2∗diff(f,c)∗exp(-2∗f));

for f=sin(t), y_3(t)=cost, x_1(t)=t.

(24)

Closed Solution (PDE3)

H(x_1,x_2,x_3,y_1,y_2,y_3,t)=(-1/2∗exp(2∗sin(t))-1/2∗exp(2∗sin(t)))∗cost

+1/2∗F_1(x_1,x_2,x_3,y_1,y_2,t)∗exp(2∗sin(t))+1/2∗F_1(x_1,x_2,x_3,y_1,y_2,t)∗exp(-2∗sin(t)).

(25)

14. DISCUSSION

In this study, Hamilton Equations raised in (23) on flat manifold for mechanical systems. Today, it is well-known Hamiltonian models have emerged as a very important tool for mechanical systems. classical field theory utilizes traditionally the language of Hamiltonian dynamics. Also, this theory has extended to time-dependent classical mechanics. A Hamiltonian space has been certified as an excellent model for some important problems in relativity, gauge theory and electromagnetism. Hamiltonian gives a model for both the gravitational and electromagnetic field in a very natural blending of the geometrical structures of the space with the characteristic properties of these physical fields. Hamiltonian dynamics is used as a model for field theory, quantum physics, optimal control, biology and fluid dynamics. Most important advantage of flat manifold is to allow the calculation of linear distance. Since Weyl's unified theory, the metrics have been thought as the gravitational potential, as in general relativity, and the corresponding forms are thought as the electromagnetic potentials. Hence, the differential equations found are considered Weyl-Hamilton equations on conformally flat manifold such that they could be used in modelling the problems in various physical, relativistic and mechanical areas for geodesics [28,31,32]. In addition, we, using symbolic computational program that these equations closed solutions (25), were found.

REFERENCES

[1] Olszak K., Olszak Z., **(2011),** On 4-Dimensional, Conformally Flat, Almost Kählerian Manifolds, arXiv:1110.3775v1, 1-7.
[2] Schwartz F., **(2011),** Volumetric Penrose Inequality For Conformally Flat Manifolds, Annales Henri Poincaré, Vol.12, Issue 1, 67-76.
[3] Ge Y., Wang G., Wu J., **(2012),** The Mass of an Asymptotically Flat Manifold, arXiv:1211.3645v2, 1-32.
[4] Gonzalez M.D.M., **(2005),** Singular Sets of A Class of Locally Conformally

Flat Manifolds, Duke Mathematical Journal, Vol. 129, No.3, 551-572.
[5] Upadhyay S., **(2000),** A Bounding Question For Almost Flat Manifolds, Transactions of The American Mathematical Society, Vol. 353, No: 3, 963-972.
[6] Kapovich M., **(1993),** Flat Conformal Structures On 3-Manifolds, I: Uniformization of Closed Seifert Manifolds, JDG, 38, 191-215.
[7] Zhu S., **(1994),** The Classi.cation of Complete Locally Conformally Flat Manifolds of Nonnegative Ricci Curvature, Pacific Journal of Mathematics, Vol. 163, No. 1, 189-199.
[8] Kulkarni R.S., **(1972),** Conformally Flat Manifolds, Proc. Nat. Acad. Sci. USA, Vol. 69, No. 9, 2675-2676.
[9] Dotti I.G., Miatello R.J., **(2005),** On The Cohomology Ring of Flat Manifolds with a Special Structure, Revista De La Uni On Matematica Argentina, Vol. 46, Numero 2, 133-147.
[10] Szczepanski A., **(2005),** Kähler at Manifolds of Low Dimensions, Institut des Hautes Etudes Scientifiques, 1-8.
[11] Akbulut S., Kalafat M., **(2008),** A Class of Locally Conformally .at 4-Manifolds, arXiv:0807.0837v5, 1-29.
[12] Abood H.M., **(2010),** Almost Hermitian Manifold with Flat Bochner Tensor, European Journal of Pure and Applied Mathematics, Vol. 3, No. 4, 730-736.
[13] Lutowski R., **(2009),** On Symmetry of Flat Manifolds, Institute of Mathematics, University of Gdansk, Conference on Algebraic Topology CAT'09, 1-29.
[14] Kasap Z., Tekkoyun M., **(2013),** Mechanical Systems on Almost Para/Pseudo-Kähler Weyl Manifolds, IJGMMP, Vol. 10, No.5, 1-8.
[15] Kasap Z., **(2013),** Weyl-Mechanical Systems on Tangent Manifolds of Constant W-sectional Curvature, IJGMMP, Vol. 10, No.10, 1-13.
[16] Wikipedia1, http://en.wikipedia.org/wiki/Conformal_class.
[17] Wikipedia2, http://mathworld.wolfram.com/FlatManifold.html.
[18] Wikipedia3, http://en.wikipedia.org/wiki/Conformally_.at_manifold.
[19] McDu D., Salamon D., **(1995),** J-holomorphic Curves and Quantum Cohomology, http://www.math.sunysb.edu/~dusa/jholsm.pdf, 1-217.
[20] Kadosh L., **(1996),** Topics in Weyl Geometry. Dissertationial, University of California.
[21] Folland G.B., **(1997),** Weyl manifolds, J. Differential Geometry, 4, 145-153.
[22] Pedersen H., Poon Y.S., Swann A., **(1993),** The Einstein-Weyl Equations in Complex and Quaternionic Geometry, Diferential Geom. Appl. 3, no.4, 309-321.
[23] Gilkey P., Nikcevic S., **(2003),** Kähler and para-Kähler Curvature Weyl manifolds, http://arxiv.org/pdf/1011.4844.pdf, 1-11.
[24] Gilkey P., Nikcevic S., **(2011),** Kähler.Weyl manifolds dimensional 4, http://arxiv.org/pdf/1109.4532.pdf, 1-10.
[25] Wikipedia4, http://en.wikipedia.org/wiki/Weyl_curvature_tensor.
[26] Newlander A., Nirenberg L., **(1957),** Complex Aanalytic Coordinates in Almost Complex Manifolds. Ann. of Math., 65, 391-404.
[27] Brozos-Vazquez M., Gilkey P., Merino E., **(2010),** Geometric Realizations of Kaehler and of Para-Kaehler Curvature Models, IJGMMP, Vol.7, No.3, 505-515.
[28] Miron R., Hrimiuc D., Shimada H., Sabau S.V., **(2002),** The Geometry of

Hamilton and Lagrange Spaces, Kluwer Academic Publishers.
[29] Klein J., **(1962),** Escapes variationnals et Mécanique, Ann. Inst. Fourier, Grenoble, 12, 1-124.
[30] Abraham R., Marsden J.E., Ratiu T., **(2001),** Manifolds, Tensor Analysis and Applications, Springer, 483-542.
[31] Weyl H., **(1922),** Space-Time-Matter, Dover Publ. Translated From the 4th German Edition by H. Brose. London: Methuen. Reprint New York: Dover (1952).
[32] De Leon M., Rodrigues P.R., **(1989),** Methods of Differential Geometry in Analytical Mechanics, North-Hol. Math.St.,152, Elsevier Sc. Pub. Com., Inc., Amsterdam.