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Improving some KKM theoretic results

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Abstract

The purpose of this article is to introduce some works on the KKM theory which can be improved according to our theory on abstract convex spaces. In Section 2, we introduce some basic things on our abstract convex spaces as a preliminary. Section 3 deals with the basic results and subjects of our study on the KKM theory. In Section 4, we introduce several works that appeared since 2010. Most of these works are chosen on the basis that they can be improved by following our theory. Actually, we introduce abstracts of each work and add some comments showing how to improve them.

Keywords: Abstract convex space, KKM theorem, KKM space, mapping classes $\mathfrak{KC}$, $\mathfrak{KO}$.
2010 MSC: 46A03, 47H10, 49J53, 54C60, 54H25, 91A11, 91B02.

1. Introduction

The KKM theory, first called by the author in 1992 [17], is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences. Since the theorem and its many equivalent formulations or extensions are powerful tools in showing the existence of solutions of a lot of problems in pure and applied mathematics, many scholars have been studying its further extensions and applications.

The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan, and later to the so-called convex spaces by Lassonde, to H-spaces by Horvath and others, to G-convex spaces mainly by the present author. In 2006-09, we proposed new concepts of abstract convex spaces and partial KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Now the KKM theory becomes the study of *abstract convex spaces* due to ourselves in 2006 and we obtained a large number of new results in such frame. For the history of the KKM theory, see our recent article [28].

The purpose of this article is to introduce some works on the KKM theory which can be improved according to our accomplishment.

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In Section 2, we introduce some basic things on our abstract convex spaces as a preliminary. Section 3 deals with deals with basic results and subjects of our study on the KKM theory. In Section 4, we introduce several works on the theory appeared since 2010. Most of these works are chosen on the basis that they can be improved by following our theory. Actually, we introduce abstracts of each work, and add some comments showing how to improve them.

2. Abstract convex spaces

In order to upgrade the KKM theory, in 2006-09, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of various known types of particular spaces and adequate to establish the KKM theory.

Multimaps are also called simply *maps*. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Recall the following in [28]:

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, i.e., $\mathrm{co}_\Gamma D' \subset X$. Let $(E; \Gamma) := (E, E; \Gamma)$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E of E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, Z)$ [resp. $F \in \mathfrak{KO}(E, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property.

The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

Definition. An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

A typical example is the following type of spaces:

Definition. A ϕ_A-*space* $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\})$, any multimap $G : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset G(J) \ \text{ for each } \ A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map*.

Every ϕ_A-space is a KKM space:

Lemma. *Let $(X, D; \Gamma)$ be a ϕ_A-space and $G : D \multimap X$ a multimap with nonempty closed* [*resp. open*] *values. Suppose that G is a KKM map. Then $\{G(a)\}_{a\in D}$ has the finite intersection property.*

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{aligned} &\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde's convex space} \\ &\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longleftrightarrow \phi_A\text{-space} \\ &\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space} \\ &\Longrightarrow \text{Abstract convex space.} \end{aligned}$$

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y\in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y\in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y\in D} \overline{G(y)} = \overline{\bigcap_{y\in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have the following one of the most general KKM type theorems in [28] and others:

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, D, Z)$, and $G : D \multimap Z$ a map such that*

(1) *$\overline{G}$ is a KKM map w.r.t. F; and*

(2) *there exists a nonempty compact subset K of Z such that either*

(i) *$K = Z$;*

(ii) *$\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D\rangle$; or*

(iii) *for each $N \in \langle D\rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y\in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.*

3. The KKM theory on abstract convex spaces

In the first half of this section, we introduce some typical KKM theoretic results on abstract convex spaces appeared mainly in [21, 22, 29]. Of course we can not repeat whole works on that subject. So in the second half of this section, we state some subjects of our study on the KKM theory without mentioning the literature.

An abstract convex space $(E, D; \Gamma)$ is called a *KKM space* whenever the following holds:

(0) The KKM principle. *For any closed-valued [resp. open-valued] KKM map $G : D \multimap E$, the family $\{G(z)\}_{z\in D}$ has the finite intersection property.*

This is equivalent to each one of the following:

The Fan matching property
Another finite intersection property
The geometric property or the section property

The Fan-Browder fixed point property
Existence of maximal elements
Analytic formulation
Minimax inequality
Analytic alternative

An abstract convex space $(E, D; \Gamma)$ is called a *partial KKM space* whenever the following partial condition of (0) holds:

(0)′ The partial KKM principle. *For any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.*

This is equivalent to each partial one (that is, closed-valued case) of the above list.

Theorem. *For a compact partial KKM space $(X; \Gamma)$, the following statements hold.*

Several types of minimax inequalities
Several types of variational inequalities

Let $(X; \Gamma_1)$ and $(Y; \Gamma_2)$ be abstract convex spaces. For their product, we can define $\Gamma_{X \times Y}(A) := \Gamma_1(\pi_1(A)) \times \Gamma_2(\pi_2(A))$ for $A \in \langle X \times Y \rangle$, where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are projections.

Theorem. *For a compact product partial KKM space $(E; \Gamma) := (X \times Y; \Gamma_{X \times Y})$, the following statements hold.*

The basic minimax theorem
Generalized von Neumann-Sion minimax theorem
Collective fixed point theorem
The von Neumann-Fan intersection theorem
The Fan type analytic alternative
Generalized Nash-Fan type equilibrium theorem

Since we first named the KKM theory in 1992 [17], we have tried to improve the theory by applying such basic theory of abstract convex spaces. In fact, we have studied the KKM theory with respect to the following list of subjects:

Basic concepts of abstract convex spaces
Various types of abstract convex spaces and (partial) KKM spaces
Various properties of the KKM spaces
Improvements of topologies of the KKM spaces
Various forms of generalized KKM theorems
Some typical theorems of the KKM theory
Equivalent formulations of the KKM theorem
Results implying equivalents of the KKM theorem with some coercivity conditions
Various types of the KKM class of multimaps (subclasses of $\mathfrak{KC}$, $\mathfrak{KO}$)
The KKM theory of particular types of KKM spaces
Applications of the convex-valued KKM maps
Applications to the Hahn-Banach theory
Applications of the KKM theory to various problems (e.g., variational relations)
History of the KKM theory

For the references of each of the above subjects, the readers can consult to the publications list of our homepage: http://parksehie.com. Also see *Google Scholar* and *Research Gate.*

4. Examples of some articles to be improved

In this section we introduce some articles which can be improved by following our methods as in the previous section. In fact, we choose papers published from 2010. Most of these works are chosen on the basis that they can be improved by following our theory. Actually, we introduce abstracts of each paper, and add some comments like how to improve them.

Note that the articles denoted by "Name (year)" in the present article can be seen in the related article.

Park [20] — Commun. Korean Math. Soc. 25(2) (2010)

Abstract: Recently, some authors [Deng-Xia (2003), Deng-Yang (2006), X. P. Ding (2009), Ding-Liou-Yao (2005), J. Huang (2005)] adopted the concept of the so-called generalized R-KKM maps which are used to rewrite known results in the KKM theory. In the present paper, we show that those maps are simply KKM maps on G-convex spaces. Consequently, results on generalized R-KKM maps follow the corresponding previous ones on G-convex spaces.

Comments: Since G-convex spaces are abstract convex spaces, R-KKM maps are obsolete now.

Chebbi-Gourdel-Hammami [3] — JFPTA 9 (2011)

Abstract: We introduce a generalized coercivity type condition for set-valued maps defined on topological spaces endowed with a generalized convex structure and we extend Fan's matching theorem.

Comments: In Theorem 1, the authors assumed: "Let Z be an arbitrary set in the L-space (X, Γ), Y an arbitrary topological space, and $F : Z \to Y$ a map with quasi-compactly closed values."

From these assumptions, we judge that the authors are ignorant of recent development of the KKM theory. Note the following:

(1) The triple $(X, Z; \Gamma)$ is an abstract convex space.

(2) Let Y have the *quasi-compactly generated* topology. Then Y becomes k-space and "quasi-compactly closed" becomes simply "closed."

(3) The authors claim that L-spaces generalize many types of spaces. However, L-spaces are actually G-convex spaces as the later definition by ourselves.

(4) Their Theorems 1 and 2 seem to have no application comparing to Fan's matching theorem.

Darzi et al. [4] — Filomat 25(4) (2011)

Abstract: This paper deals with coincidence and fixed point theorems in minimal generalized convex spaces. By establishing a kind of KKM Principle in minimal generalized convex spaces, we obtain some results on coincidence point and fixed point theorems. Generalized versions of Ky Fan's lemma, Fan-Browder fixed point theorem, Nash equilibrium theorem and some Urai's type fixed point theorems in minimal generalized convex spaces are given.

Comments: Any topological space is a minimal space and not conversely. However, any minimal space can be made into a topological space.

Allmohammady et al. (2005, 2008, 2008), Darzi et al. (2008), and Delavar et al. (2011) dealt with some results in the KKM theory on generalized convex minimal spaces. By establishing a kind of the KKM principle in these spaces, they obtained some results on coincidence or fixed point theorems and others. In [25], it was shown that their results are consequences of corresponding ones for abstract convex minimal spaces in our previous work (2008) and hence, can be extended to more general setting.

Hussain-Shah [8] — CMA 62 (2011)

Abstract: In this paper we establish some topological properties of the cone b-metric spaces and then improve some recent results about KKM mappings in the setting of a cone b-metric space. We also prove some fixed point existence results for multivalued mappings defined on such spaces.

COMMENTS: Some corrections are necessary as follows: Let M be a cone b-metric space and X a nonempty subset. Then $(M, X; \mathrm{co})$ is an abstract convex space. Let Y be a topological space, $F : M \multimap Y$, $G : X \multimap Y$ two maps satisfying

$$F(\mathrm{co}\, A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle X \rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : X \multimap M$ is a KKM map with respect to the identity map 1_M of M.

A multimap $F : M \multimap Y$ is called a $\mathfrak{KC}$*-map* [resp. a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp. open-valued] KKM map $G : X \multimap Y$ with respect to F, the family $\{G(x)\}_{x \in X}$ has the finite intersection property.

This corrects the inadequate original class *KKM* given in [8].

Khamsi et al. [10] — FPTA 62 (2011)

ABSTRACT: In modular function spaces, we introduce Knaster-Kuratowski-Mazurkiewicz mappings (in short KKM-mappings) and prove an analogue to Ky Fan's fixed point theorem.

COMMENTS: In this paper, the space $(L_p, C; \mathrm{conv})$ is an abstract convex space and $G : C \multimap L_p$ is a finitely ρ-closed valued KKM map. Theorem 3.1 tells that the space is a partial KKM space. Therefore, as in Section 3, the space has several properties; e.g., Fan matching property, geometric property, the Fan-Browder fixed point property, existence of maximal elements, minimax inequality, etc.

Plubtieng-Sitthithakerngkiet [30] — FPTA 62 (2011)

ABSTRACT: This paper deals with the generalized strong vector quasiequilibrium problems without convexity in locally G-convex spaces. Using the Kakutani-Fan-Glicksberg fixed point theorem for upper semicontinuous set-valued mapping with nonempty closed acyclic values, the existence theorems for them are established. Moreover, we also discuss the closedness of strong solution set for the generalized strong vector quasiequilibrium problems.

COMMENTS: Locally G-convex spaces can be extended more general spaces. For the applications of various multimap classes in abstract convex spaces, see [23] and the more recent articles of Park.

Shabanian-Vaezpour [32] — B.Malays 39 (2011)

ABSTRACT: In this paper, a CAT(0) version of famous Fan's minimax inequality is established and as its application, we obtain some fixed point theorems and best approximation theorems in CAT(0) spaces.

COMMENTS: In 2009, Horvath [7] introduced continuous midpoint spaces as a generalization of various types of his c-spaces and gave a large number of examples. Some of them are as follows:

(1) Closed convex subsets of Banach spaces.
(2) Hyperconvex metric spaces due to Aronszajn-Panitchpackdi.
(3) Hilbert spaces.
(4) Completion of Bruhat-Tits spaces [= Hadamard spaces, that is, complete and simply connected meric spaces of nonpositive curvature (= complete CAT(0) spaces)].
(5) Complete $\mathbb{R}$-trees [= hyperconvex metric spaces with unique metric segments].
(6) Buseman midpoint spaces [includes hyperbolic metric spaces in the sense of Kirk and Reich-Shafrir].

In our recent work [29], we defined

Definition. An abstract convex space $(X, D; \Gamma)$ is called a *Horvath midpoint space* whenever X is a complete continuous midpoint metric space, D is a nonempty subset of X, and $\Gamma : \langle D \rangle \multimap X$ is a multimap such that $\Gamma(A) = \Gamma_A$ is a geodesically convex subset containing A and $\Gamma_A \subset \Gamma_B$ if $A \subset B \in \langle D \rangle$.

We showed the following in [29]:

Proposition. *Any Horvath midpoint space is a Horvath space and hence a KKM space.*

Therefore, any Horvath midpoint space satisfies all KKM theoretic results in Section 3.

Note that Shabanian-Vaezpour [32] obtained a KKM mapping principle for a complete CAT(0) space with the convex hull finite property and some of routine applications.

Simić [33] — FPTA 62 (2011)

Abstract: Recently, Ayse Sonmez [On paracompactness in cone metric spaces, Appl. Math. Lett. 23 (2010) 494–497] proved that a cone metric space is paracompact when the underlying cone is normal. Also, very recently, Kieu Phuong Chi and Tran Van An [Dugundji's theorem for cone metric spaces, Appl. Math. Lett. (2010) doi:10.1016/j.aml.2010.10.034] proved Dugundji's extension theorem for the normal cone metric space. The aim of this paper is to prove this in the frame of the tvs-cone spaces in which the cone does not need to be normal. Examples are given to illustrate the results.

Comments: The author showed that any cone normed space $(X, |||\cdot|||)$ is a partial KKM space and, hence, satisfies a large number of the KKM theoretic results in Section 3.

Fakhar et al. [6] — JGO (2012)

Abstract: In this paper, we obtain several new generalized KKM-type theorems under a new coercivity condition and the condition of intersectionally closedness which improves condition of transfer closedness. As applications, we obtain new versions of equilibrium problem, minimax inequality, coincidence theorem, fixed point theorem and an existence theorem for an 1-person game.

Comments: Let X be a convex space in the sense of Lassonde, Y a Hausdorff topological space. This paper is based on the following due to Lin (1999):

Lemma 4. *Let $T \in KKM(X,Y)$ be compact and $G : X \multimap Y$ be generalized KKM map w.r.t. T with compactly closed values. Then, $cl(T(X)) \cap (\bigcap_{x\in X} G(x)) \neq \emptyset$.*

In the recent language, $T \in \mathfrak{KC}$, G is a KKM map with respect to T, and Hausdorffness of Y is redundant. Moreover, if we assume Y has compactly generated topology or Y is a k-space, then T has simply closed values.

Therefore, this paper has to be improved.

Espinoza [5] — Bull. Assoc. Math. Venezolana 19(2) (2012)

Abstract: In this paper we obtain KKM type theorems for G-spaces, M-spaces and L-spaces which are spaces with no linear structure, these theorems are used to obtain some minimax results for these spaces. Also an intersection theorem for M-spaces is presented.

Comments: G-spaces are generalized convex spaces or G-convex spaces due to Park and Kim in 1996, and the author did not mention it. All spaces appeared in this paper are abstract convex spaces and, hence, our theory in Section 3 can be applicable.

Lu-Zhang [14] — Comp. Math. Appl. 64 (2012)

Abstract: In this paper, an improved version of section theorem is proved in FWC-spaces without any linear and convex structure under much weaker assumptions, and next as its applications, some new coincidence theorems and minimax inequalities are established in FWC-spaces. These results generalize many known theorems in the literature.

Comments: This paper introduced the concepts of FWC-spaces (short form of finite weakly convex spaces) as a unified form of many known modifications of G-convex spaces, and the better admissible class of multimaps on them. In [26], we showed that their FWC-spaces and their better admissible classes are inadequately defined and that their results can not be true.

Chaipunya-Kumam [2] — J. Ineq. Appl. (2013)

ABSTRACT: The main purpose of this paper is to study some topological nature of circular metric spaces and deduce some fixed point theorems for maps satisfying the KKM property. We also investigate the solvability of a variant of a quasi-equilibrium problem as an application.

COMMENTS: In this paper, let M be a circular metric space and X be a subadmissible subset of M. A multimap $G : X \multimap M$ is said to be a *KKM map* if for each $A \in X$ we have $ad(A) \subset G(A)$.

Note that $(M, X; ad)$ is an abstract convex space.

In this paper, as in the many metric type spaces, inadequate *KKM* class instead of $\mathfrak{KC}$ is used. Certain results can be improved by adopting current language in the KKM theory.

Mitrović [15] — MCM 57 (2013)

ABSTRACT: We establish some existence results for a generalized coupled coincidence point problem (for short, (GCCP)) in topological vector spaces. The solvability of the GCCP is presented using the Fan-KKM lemma. Also we derive the results on coupled coincidence points and coupled fixed points, which were introduced by Lakshmikantham and Ćirić and Amini-Harandi.

COMMENTS: It would be interesting to extend every thing in this article to abstract convex spaces.

Lu et al. [12] — SWJ 209 (2014)

ABSTRACT: A maximal element theorem is proved in finite weakly convex spaces (FWC-spaces, in short) which have no linear, convex, and topological structure. Using the maximal element theorem, we develop new existence theorems of solutions to variational relation problem, generalized equilibrium problem, equilibrium problem with lower and upper bounds, and minimax problem in FWC-spaces. The results represented in this paper unify and extend some known results in the literature.

COMMENTS: See the comments on Lu-Zhang [2012].

Lu et al. [13] — JIA (2014)

ABSTRACT: In this paper, noncompact CAT(0) versions of the Fan-Browder fixed point theorem are established. As applications, we obtain new minimax inequalities, a saddle point theorem, a fixed point theorem for single-valued mappings, best approximation theorems, and existence theorems of φ-equilibrium points for multiobjective noncooperative games in the setting of noncompact CAT(0) spaces. These results generalize many well-known theorems in the literature.

COMMENTS: Adopting a KKM type theorem for a complete CAT(0) space with the convex hull finite property, the authors obtain some routine results in the KKM theory on abstract convex spaces. See comments on the paper of Shabanian-Vaezpour [31] in 2011.

Shabanian-Vaezpour [32] — B.Malays 39 (2016)

ABSTRACT: In this paper, we present a modular version of KKM and generalized KKM mappings and then we establish a characterization of generalized KKM mappings in modular spaces. Also we prove an analogue to KKM principle in modular spaces. Moreover, as an application, we give some sufficient conditions which guarantee existence of solutions of minimax problems in which we get Fan's minimax inequality in modular spaces.

COMMENTS: Generalized KKM maps defined by Chang-Zhang in 1991 were extended by several authors. Finally, in [27], we showed that they are simply KKM maps in abstract convex spaces.

In 2016, Shabanian-Vaezpour [32] characterized generalized KKM map in modular spaces and obtained

Corollary 3.4. *Let ρ be a modular on Y, X be a nonempty set of Y_ρ, and $G : X \multimap Y_\rho$ be a closed-valued map. If G is KKM, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

This means that $(Y_\rho, X; \mathrm{co})$ is a partial KKM space and, hence, satisfies a large number of KKM theoretic statements in Section 3.

Jafari et al. [9] — Optimization 66(3) (2017)

Abstract: This paper deals with equilibrium problems in the setting of metric spaces with a continuous convex structure. We extend Fan's 1984 KKM theorem to convex metric spaces in order to employ some weak coercivity conditions to establish existence results for suitable local Minty equilibrium problems, where the involved bifunctions are φ-quasimonotone. By an approach which is based on the concept of the strong φ-sign property for bifunctions, we obtain existence results for equilibrium problems which generalize some results in the literature.

Comments: Fan's 1984 KKM theorem can be extended to Theorem C in Section 2 and a convex metric space is a partial KKM space. Therefore, a convex metric space satisfies many properties in Section 3.

Agarwal et al. [1] — JOTA 66(3) (2018)

Abstract: In this paper, we establish two intersection theorems which are useful in considering some optimization problems (complementarity problems, variational inequalities, minimax inequalities, saddle point problems).

Comments: It would be interesting to extend the most part of this paper to corresponding ones for abstract convex spaces as in Section 3.

Le et al. [11] — MPRA (2020)

Abstract: In characterizing the existence of general equilibrium, existing studies mainly draw on Brouwer and Kakutani fixed point theorems and, to some extent, Gale-Nikaido-Debreu lemma. In this paper, we show that Sperner lemma can play a role as an alternative powerful tool for the same purpose. Specifically, Sperner lemma can be used to prove those theorems as well as the lemma. Additionally, Kakutani theorem is shown as a corollary of Gale-Nikaido-Debreu lemma. For a demonstration of the use of Sperner lemma to prove general equilibrium existence, we consider two competitive economies marked either by production goods or financial assets. In each case, we successfully provide another proof on the existence of a general equilibrium using only Sperner lemma and without a need to call on the fixed point theorems or the lemma.

Comments: More earlier than 1999 [18], we already stated that the Sperner lemma, the Kakutani theorem, the Gale-Nikaido-Debreu lemma and many others are equivalent to the Brouwer theorem. This was shown in certain publications already.

Mitrović et al. [16] — MDPA (2020)

Abstract: In this paper we obtain a best approximations theorem for set-valued mappings in G-convex spaces. As applications, we derive results on the best approximations in hyperconvex and normed spaces. The obtain results generalize many known results in the literature.

Comments: This paper is based on a theorem of Kim and Park (2007) on G-convex spaces. Since G-convex spaces are obsolete, it can be based on Park's abstract convex spaces.

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