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Numerical approximation for the spread of SIQR model with Caputo fractional order derivative

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Abstract. In our paper, the spread of SIQR model with fractional order differential equation is considered. We have evaluated the system with fractional way and investigated stability of the non-virus equilibrium point and virus equilibrium points. Also, the existence of the solutions are proved. Finally, the efficient numerical method for finding solutions of system is given.

1. Introduction

Fractional calculus is a very efficient way for researchers while studying real world phenomena problems like astronomy, biology, physics also in the social sciences e.g. education, history, sociology, life sciences . In recent years, fractional order differential equations have become an important tool in mathematical modelling. The most useful way to work on modelling is considering models again with their fractional order version. The most commonly used definitions are Riemann and Caputo fractional order derivatives. The Riemann-Liouville derivative is historically the first but there are some difficulties while applying it to real life problems. In order to overcome these difficulties, the latter concept, fractional order Caputo type derivative is defined [3, 5, 6, 8, 16].

Some disease models which are an important area in mathematical modelling are discussed [1, 9, 10, 13]. In our paper, we have investigated the system of equations involving fractional derivatives. But especially we are interested in investigating the spread of fractional order SIQR model using the concept of fractional operator of Caputo differentiations. After considered SIQR model with Caputo type, disease free equilibrium and endemic equilibrium points are computed. Also we have applied the next generation matrix method to calculated the basic reproduction number R_0 [19]. The stability analysis of SIQR model and the existence and uniqueness of its solutions have been obtained. Finally a suitable iteration for the solutions of the SIQR model is obtained by Atangana-Toufik method [18].

2. Preliminaries

In this section, let us give important definitions of fractional derivatives and their useful properties [7 − 17].

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Definition 2.1. *The Gamma function* $\Gamma(x)$ *is defined by the integral as below:*

$$\Gamma(x) = e^{-t}t^{x-1}dt. \tag{1}$$

One the basic properties of the gamma function is that it satifies the following equation :

$$\Gamma(x+1) = x.\Gamma(x) = z.(z-1)! = z!. \tag{2}$$

Definition 2.2. *The Grünwald-Letnikov definition is given as*

$${}_aD_t^{\alpha} f(t) = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{(t-a)}{h}} (-1)^k \binom{\alpha}{k} f(t-kh). \tag{3}$$

Fractional derivative operator is non-local in nature and fractional equations provides an useful tool to describe phenomenas comprising memory and hereditary features. Such a phenomena can also appear in biological processes, population dinamics.

Definition 2.3. *Riemann-Liouville definition of fractional order differ-integral:*

$${}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{4}$$

where

$$n-1 < \alpha \leqslant n, n \in \mathbb{N}. \tag{5}$$

The Laplace transform of the Riemann-Liouville fractional order differ-integral is given as below:

$$L[{}_0D_t^{\alpha} f(t)] = \{ \begin{array}{lll} s^{\alpha}F(s) & for & \alpha < 0, \\ s^{\alpha}F(s) - F'(s) & for & \alpha > 0, \end{array} . \tag{6}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$.

Definition 2.4. *Caputo's definition of fractional order differ-integral:*

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{7}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ *is a fractional order of the differ-integral of the function* $f(t)$.

The Laplace transform of the Caputo fractional order differ-integral is given as follows:

$$L[{}_0^C D_t^{\alpha} f(t)] = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \tag{8}$$

where $n-1 < \alpha \leqslant n, n \in \mathbb{N}$.

Now, we give some important lemmas for Riemann–Liouville derivative and Caputo derivative as following:

Lemma 2.5. *Let us take a function* $f(x)$ *and* $m, n \geqslant 0$, *then the following equations hold.*

For $R-L$ *derivative given as:*

i. *Linearity rule:*

$$ {}_aD_t^\alpha(cf_1 + f_2) = {}_a D_t^\alpha(cf_1) + {}_a D_t^\alpha(f_2) = c\,{}_aD_t^\alpha(f_1) + {}_a D_t^\alpha(f_2). \tag{9} $$

ii. *The semi-group property does not hold. Indeed, the following equation is not always true.*

$$ D_t^\alpha D_t^\beta f = D_t^{\alpha+\beta} f. \tag{10} $$

For Caputo derivative given as:

i. *Linearity rule:*

$$ {}_a^CD_t^\alpha(cf_1 + f_2) = {}_a^C D_t^\alpha(cf_1) + {}_a^C D_t^\alpha(f_2) = c\,{}_a^CD_t^\alpha(f_1) + {}_a^C D_t^\alpha(f_2). \tag{11} $$

ii. *The semi-group property:*

$$ {}^CD_t^\alpha\,{}^C D_t^\beta f = {}^C D_t^{\alpha+\beta} f. \tag{12} $$

Fractional derivative operator is non-local in nature and fractional equations provides an useful tool to describe phenomenas comprising memory and hereditary features. Such a phenomena can also appear in biological processes, population dinamics.

Theorem 2.6. *Consider the n-dimensional system*

$$ \begin{aligned} D_a^\alpha y(t) &= f(t, y(t)),\\ y(t_0) &= y_0, \end{aligned} \tag{13} $$

where $\alpha \in (0,1)$ and D_a^α represents Caputo sense fractional derivative of order α. Let y^* be the equilibrium point of the system and $J(y^*)$ be the Jacobian matrix about the equilibrium point y^*. Then, the equilibrium point y^* is locally asymptotically stable if and only if all the eigenvalues r_i, $i = 1, 2, ..., n$ of $J(y^*)$ satify $|\arg(r_i)| > \frac{\alpha\pi}{2}$.

Theorem 2.7. *Considering the delayed fractional differential system with the Caputo fractional derivative as*

$$ \begin{aligned} D^\alpha y(t) &= My(t) + Ny(t-\tau),\\ y(t) &= \psi(t), t \in [-\tau, 0], \end{aligned} \tag{14} $$

where $\alpha \in (0,1]$, $y \in R^n$, $M, N \in R^{nxn}$, and $\psi(t) \in R_+^{nxn}$. The characteristic equation of the system (14) *is given as*

$$ \det|r^\alpha I - M - Ne^{-r\tau}| = 0. \tag{15} $$

If all the roots of (15) *have negative real parts, then the zero solution of system* (14) *is locally asymptotically stable* [12, 15]

3. Model Derivation

In this paper, we proposed a SIQR epidemic model with given first version with following form [11]:

$$ \begin{aligned} \frac{dS}{dt} &= \Lambda - \mu S - \frac{\beta SI}{N},\\ \frac{dI}{dt} &= \frac{\beta SI}{N} - (\mu + \gamma + \delta + \alpha) I,\\ \frac{dQ}{dt} &= \delta I - (\mu + \epsilon + \alpha) Q,\\ \frac{dR}{dt} &= \gamma I + \epsilon Q - \mu R, \end{aligned} \tag{16} $$

where S, I, R detone the numbers of susceptible, infective and removed, recpectively, Q detones the number of quarantined and $N = S + I + Q + R$ is the number of total population individuals. The parameter Λ is the recruitment rate of S correspoinding to births and immigration; β detones tha average number of adequate contacts; μ is the natural death rate; γ and ϵ detone the recover rates from grup I, Q to R, recpectively; δ detones the removal rate from I; α is the disease-caused death rate of I and Q. The parameters involved in the system (3) are all positive constans [11].

Fractional calculus which means fractional derivatives and fractional integrals is of increasing interest among the researchers. It is known that fractional operators describe the system behavior more accurate and efficiently than integer order derivatives. Because of great advantege of memory properties let us consider model given above, again with fractional order. Fractional order SIQR epidemic model given as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} S(t) &= \Lambda - \mu S - \frac{\beta S I}{N}, \\ {}_a^C D_t^{\alpha} I(t) &= \frac{\beta S I}{N} - (\mu + \gamma + \delta + \alpha) I, \\ {}_a^C D_t^{\alpha} Q(t) &= \delta I - (\mu + \epsilon + \alpha) Q, \\ {}_a^C D_t^{\alpha} R(t) &= \gamma I + \epsilon Q - \mu R, \end{aligned} \tag{17}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0.$$

A working on equilibrium points and their asymptotic stability:

In this part, we study stabilities of non-virus equilibrium, virus equilibrium, and basic reproduction number of our fractional model (18).

Let $\alpha \in (0, 1]$ and consider the Caputo differential equation system as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} S(t) &= F_1(t, S(t)), \\ {}_a^C D_t^{\alpha} I(t) &= F_2(t, I(t)), \\ {}_a^C D_t^{\alpha} Q(t) &= F_3(t, Q(t)), \\ {}_a^C D_t^{\alpha} R(t) &= F_4(t, R(t)). \end{aligned} \tag{18}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0. \tag{19}$$

Here,

$$\begin{aligned} F_1(t, S(t)) &= \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{N}, \\ F_2(t, I(t)) &= \frac{\beta S(t) I(t)}{N} - (\mu + \gamma + \delta + \alpha) I(t), \\ F_3(t, Q(t)) &= \delta I(t) - (\mu + \epsilon + \alpha) Q(t), \\ F_4(t, R(t)) &= \gamma I(t) + \epsilon Q(t) - \mu R(t). \end{aligned} \tag{20}$$

3.1. Analysis of the non-virus equilibrium point

A non-virus equilibrium point is the point with no virus infection. Clearly, the point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ to the non-virus equilibrium point of model (18).

Here, we examine the basic reproduction number in more detail utilizing the method given in [19]. According to the next generation matrix method, the matrices $\widetilde{F}$ and $\widetilde{W}$ are defined as:

$$\widetilde{F} = \begin{bmatrix} \frac{\beta S}{N} & 0 \\ \delta & 0 \end{bmatrix} \text{ and } \widetilde{W} = \begin{bmatrix} \mu+\gamma+\delta+\alpha & 0 \\ -\delta & \mu+\epsilon+\alpha \end{bmatrix}. \tag{21}$$

For obtaining the eigenvalues of the matrix $\widetilde{F}\,\widetilde{W}^{-1}$ at the point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$, we have to solve the following equation

$$\left|\widetilde{F}\widetilde{W}^{-1} - \lambda I\right| = 0, \tag{22}$$

where λ are the eigenvalues and I is the identity matrix. So, the reproduction number is

$$R_0 = \frac{\beta\Lambda}{N\mu(\mu+\gamma+\delta+\alpha)}. \tag{23}$$

Therefore, the disease free (non-virus) equilibrium point $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$ is locally asymptotically stable if $R_0 < 1$.

3.2. *Analysis of the virus equilibrium point*

The Jacobian matrix $J(S^*, I^*, Q^*, R^*)$ for the system given in (18) is.

$$J(S^*, I^*, Q^*, R^*) = \begin{bmatrix} -\mu - \frac{\beta I^*}{N} & \frac{\beta I^*}{N} & 0 & 0 \\ -\frac{\beta S^*}{N} & \frac{\beta S^*}{N} - (\mu+\gamma+\delta+\alpha) & \delta & \gamma \\ 0 & 0 & -(\mu+\epsilon+\alpha) & \epsilon \\ 0 & 0 & 0 & -\mu \end{bmatrix}. \tag{24}$$

We now discuss the asymptoticstability of the $E = (S^*, I^*, Q^*, R^*)$ equilibrium the system given by (18),

$$\begin{aligned} S^* &= \frac{N((\mu+\gamma+\delta+\alpha))}{\beta}, \\ I^* &= \frac{\beta - \mu N(\mu+\gamma+\delta+\alpha)}{\beta(\mu+\gamma+\delta+\alpha)}, \\ Q^* &= \frac{\delta(\beta - \mu N(\mu+\gamma+\delta+\alpha))}{\beta(\mu+\epsilon+\alpha)(\mu+\gamma+\delta+\alpha)}, \\ R^* &= \frac{(\gamma+\epsilon)(\beta - \mu N(\mu+\gamma+\delta+\alpha))}{\beta\mu(\mu+\gamma+\delta+\alpha)}. \end{aligned} \tag{25}$$

The characteristic equation of system is obtained via determination of (26)

$$K(\lambda) = \det(J - \lambda I) = 0. \tag{26}$$

The characteristic roots are obtained by solving the following equation

$$K(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0. \tag{27}$$

Here

$$\begin{aligned} a_1 &= (2\mu+\epsilon+\alpha)+\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha), \\ a_2 &= \mu(\mu+\epsilon+\alpha)+(2\mu+\epsilon+\alpha)\left[\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha)\right] \\ &\quad -\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}, \\ a_3 &= \mu(\mu+\epsilon+\alpha)\left[\mu+\frac{\beta I^*}{N}-\frac{\beta S^*}{N}+(\mu+\gamma+\delta+\alpha)\right] \\ &\quad +(2\mu+\epsilon+\alpha)\left[-\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}\right], \\ a_4 &= \mu(\mu+\epsilon+\alpha)\left[-\frac{\mu\beta I^*}{N}+\mu(\mu+\gamma+\delta+\alpha)+\frac{\beta(\mu+\gamma+\delta+\alpha)I^*}{N}\right]. \end{aligned} \tag{28}$$

For $a_1, a_2, a_3, a_4 > 0$, $a_1a_2 - a_3 > 0$ and $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$, so by Routh-Hurwitz Criterion, all characteristics roots have negative real parts. Therefore equilibrium point is asymptotic stable.

4. Working on the existence of solutions

Let $B = \Phi(q) \times \Phi(q)$ and $\Phi(q)$ be the Banach space of continuous function defined on the interval q with the norm

$$\|S, I, Q, R\| = \|S\| + \|I\| + \|Q\| + \|R\| \tag{29}$$

Here, $\|S\| = \sup\{|S(t)| : t \in q\}$, $\|I\| = \sup\{|I(t)| : t \in q\}$, $\|Q\| = \sup\{|Q(t)| : t \in q\}$ and $\|R\| = \sup\{|R(t)| : t \in q\}$.

Let us consider the classical SIQR model again by replacing the time derivative with Caputo fractional derivative:

$$\begin{aligned} {}_a^C D_t^\alpha S(t) &= F_1(t, S(t)), \\ {}_a^C D_t^\alpha I(t) &= F_2(t, I(t)), \\ {}_a^C D_t^\alpha Q(t) &= F_3(t, Q(t)), \\ {}_a^C D_t^\alpha R(t) &= F_4(t, R(t)). \end{aligned} \tag{30}$$

with initial conditions

$$S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0 \text{ and } R(t_0) = R_0. \tag{31}$$

Here,

$$\begin{aligned} F_1(t, S(t)) &= \Lambda - \mu S(t) - \frac{\beta S(t) I(t)}{N}, \\ F_2(t, I(t)) &= \frac{\beta S(t) I(t)}{N} - (\mu+\gamma+\delta+\alpha) I(t), \\ F_3(t, Q(t)) &= \delta I(t) - (\mu+\epsilon+\alpha) Q(t), \\ F_4(t, R(t)) &= \gamma I(t) + \epsilon Q(t) - \mu R(t). \end{aligned} \tag{32}$$

The above system (30)is written as below:

$$\begin{aligned} S(t)-S_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_1(\tau,S(\tau))\,d\tau, \\ I(t)-I_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_2(\tau,I(\tau))\,d\tau, \\ Q(t)-Q_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_3(\tau,Q(\tau))\,d\tau, \\ R(t)-R_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_4(\tau,R(\tau))\,d\tau. \end{aligned} \tag{33}$$

Theorem 4.1. *The kernels F_1, F_2, F_3 and F_4 satisfy the Lipschitz condition and contraction if the inequality holds as below:*

$$0 \le L_i < 1 \textit{ for } i = 1,2,3,4. \tag{34}$$

Proof. Taking S and S_1 be two functions then we have following:

$$\begin{aligned} \|F_1(t,S)-F_1(t,S_1(t))\| &= \left\|\Lambda-\mu S(t)-\frac{\beta S(t)I(t)}{N}-\Lambda+\mu S_1(t)+\frac{\beta S_1(t)I(t)}{N}\right\|, \\ &= \left\|\mu(S_1(t)-S(t))+\frac{\beta I(t)}{N}(S_1(t)-S(t))\right\|, \\ &\le \left(\mu+\frac{\beta b}{N}\right)\|S_1(t)-S(t)\|, \\ &\le L_1\|S_1(t)-S(t)\|. \end{aligned} \tag{35}$$

Taking $L_1 = \mu + \frac{\beta b}{N}$, where $a = \max_{t\in I}\|S(t)\|, b = \max_{t\in I}\|I(t)\|, c = \max_{t\in I}\|Q(t)\|, d = \max_{t\in I}\|R(t)\|$ are bounded function, then we get

$$\|F_1(t,S)-F_1(t,S_1(t))\| \le L_1\|S_1(t)-S(t)\|. \tag{36}$$

So, the Lipschitz condition and contraction are satisfied for F_1 if $0 \le L_1 < 1$ is satified. With doing same way, the other kernels also satisfy the Lipschitz condition as follows:

$$\begin{aligned} \|F_2(t,I)-F_2(t,I_1(t))\| &\le L_2\|I_1(t)-I(t)\|, \\ \|F_3(t,Q)-F_3(t,Q_1(t))\| &\le L_3\|Q_1(t)-Q(t)\|, \\ \|F_4(t,R)-F_4(t,R_1(t))\| &\le L_4\|R_1(t)-R(t)\|. \end{aligned} \tag{37}$$

Now we consider the kernels for the model, eq. (33) and ıt is rewritten as follows:

$$
\begin{aligned}
S(t) &= S_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, S(\tau))\, d\tau, \qquad (38)\\
I(t) &= I_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, I(\tau))\, d\tau,\\
Q(t) &= Q_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, Q(\tau))\, d\tau,\\
R(t) &= R_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, R(\tau))\, d\tau.
\end{aligned}
$$

Then we have the following recursive formula:

$$
\begin{aligned}
S_n(t) &= S_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_1(\tau, S_{n-1}(\tau))\, d\tau, \qquad (39)\\
I_n(t) &= I_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_2(\tau, I_{n-1}(\tau))\, d\tau,\\
Q_n(t) &= Q_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_3(\tau, Q_{n-1}(\tau))\, d\tau,\\
R_n(t) &= R_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} F_4(\tau, R_{n-1}(\tau))\, d\tau.
\end{aligned}
$$

Here initial conditions are given with $S(t_0) = S_0, I(t_0) = I_0, Q(t_0) = Q_0$ and $R(t_0) = R_0$.

The difference between the successive terms in the expression are given below:

$$
\begin{aligned}
A_n(t) &= S_n(t) - S_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S_{n-1}(\tau)) - F_1(\tau, S_{n-2}(\tau)))d\tau, \qquad (40)\\
B_n(t) &= I_n(t) - I_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_2(\tau, I_{n-1}(\tau)) - F_2(\tau, I_{n-2}(\tau)))d\tau,\\
C_n(t) &= Q_n(t) - Q_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_3(\tau, Q_{n-1}(\tau)) - F_3(\tau, Q_{n-2}(\tau)))d\tau,\\
D_n(t) &= R_n(t) - R_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_4(\tau, R_{n-1}(\tau)) - F_4(\tau, R_{n-2}(\tau)))d\tau.
\end{aligned}
$$

It is worth noticing that

$$
\begin{aligned}
S_n(t) &= \sum_{i=1}^{n} A_i(t), \\
I_n(t) &= \sum_{i=1}^{n} B_i(t), \\
Q_n(t) &= \sum_{i=1}^{n} C_i(t), \\
R_n(t) &= \sum_{i=1}^{n} D_i(t).
\end{aligned}
\tag{41}
$$

It is easy to see that the equation (40) reduces to (42),

$$
\begin{aligned}
\|A_n(t)\| &= \|S_n(t) - S_{n-1}(t)\|, \\
&\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-\tau)^{\alpha-1} \left(F_1(\tau, S_{n-1}(\tau)) - F_1(\tau, S_{n-2}(\tau))\right) d\tau \right\|.
\end{aligned}
\tag{42}
$$

So we have,

$$
\|S_n(t) - S_{n-1}(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|S_{n-1}(\tau) - S_{n-2}(\tau)\| \, d\tau, \tag{43}
$$

then we get

$$
\|A_n(t)\| \leq \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|A_{n-1}(\tau)\| \, d\tau. \tag{44}
$$

Similarly, we get the following results:

$$
\begin{aligned}
\|B_n(t)\| &\leq \frac{L_2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|B_{n-1}(\tau)\| \, d\tau, \\
\|C_n(t)\| &\leq \frac{L_3}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|C_{n-1}(\tau)\| \, d\tau, \\
\|D_n(t)\| &\leq \frac{L_4}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|D_{n-1}(\tau)\| \, d\tau.
\end{aligned}
\tag{45}
$$

After the above results, let us give a now theorem. □

Theorem 4.2. *The SIQR system* (30) *has a unique solution if we can find* $t_{\max}$ *satisfying following condition*

$$
\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_i < 1, \text{ for } i = 1, 2, 3, 4. \tag{46}
$$

Proof. *$S(t), I(t), Q(t)$ and $R(t)$ are bounded functions so from the equality* (44) *, we have the succeeding relation as follows:*

$$\begin{aligned} \|A_n(t)\| &\leq \|S_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_1\right]^n, \\ \|B_n(t)\| &\leq \|I_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_2\right]^n, \\ \|C_n(t)\| &\leq \|Q_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_3\right]^n, \\ \|D_n(t)\| &\leq \|R_0\| \left[\frac{t_{\max}^{\alpha}}{\Gamma(\alpha)} L_4\right]^n. \end{aligned} \tag{47}$$

Now let us assume that followings are satisfied

$$\begin{aligned} S(t) - S_0 &= S_n(t) - b_n(t), \\ I(t) - I_0 &= I_n(t) - c_n(t), \\ Q(t) - Q_0 &= Q_n(t) - d_n(t), \\ R(t) - R_0 &= R_n(t) - e_n(t). \end{aligned} \tag{48}$$

Now we have to show that the infinity term $\|b_\infty(t)\| \longrightarrow 0$, therefore we have

$$\begin{aligned} \|b_n(t)\| &\leq \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S) - F_1(\tau, S_{n-1})) d\tau \right\|, \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F_1(\tau, S) - F_1(\tau, S_{n-1})\| d\tau, \\ &\leq \frac{t^\alpha}{\Gamma(\alpha)} L_1 \|S - S_{n-1}\|. \end{aligned} \tag{49}$$

Repeating this process recursively, we obtain following equality

$$\|b_n(t)\| \leq \left[\frac{t^\alpha}{\Gamma(\alpha)}\right]^{n+1} L_1^n M. \tag{50}$$

Then at $t_{\max}$ we have

$$\|b_n(t)\| \leq \left[\frac{t_{\max}^\alpha}{\Gamma(\alpha)}\right]^{n+1} L_1^n M. \tag{51}$$

If we apply the limit to both sides as n tends to infinity, we have $\|b_\infty(t)\| \longrightarrow 0$. So this completes the proof. □

4.1. Uniqueness of the special solution

To prove the uniqueness of the system of solutions We assume that by contraction there exists another system of solutions of (6)*, $S_1(t), I_1(t), Q_1(t)$ and $R_1(t)$. Then we have*

$$\|S(t) - S_1(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (F_1(\tau, S) - F_1(\tau, S_1)) d\tau, \tag{52}$$

Wit applying the norm to eq. (52)*, we get*

$$\|S(t) - S_1(t)\| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F_1(\tau, S) - F_1(\tau, S_1)\| \, d\tau, \tag{53}$$

$$\|S(t) - S_1(t)\| \le \frac{1}{\Gamma(\alpha)} L_1 t^{\alpha} \|S(t) - S_1(t)\|. \tag{54}$$

Finally, this gives

$$\begin{aligned} \|S(t) - S_1(t)\| \left(1 - \frac{1}{\Gamma(\alpha)} L_1 t^{\alpha}\right) &\le 0, \\ \|S(t) - S_1(t)\| &= 0 \longrightarrow S(t) = S_1(t). \end{aligned} \tag{55}$$

It is easily showed that the equation $S(t)$ and other solutions have a unique solution.

5. Atangana-Toufik numerical scheme with Caputo derivative

First of all, it should be emphasised that the "numerical approach" is not directly equivalent to the "approach with use of computer", although we usually use numerical approach to find the solution with use of computers. Generally, analytical solutions are possible using simplifying assumptions that may not realistically reflect reality. In many applications, analytical solutions are impossible to achieve. Numerical methods makes it possible to obtain realistic solutions without the need for simplifying assumptions. There are lots of numerical methods have been used for finding the solutions of equations [2, 4, 14].

In this section, we reconsider Atangana-Toufik method for fractional differential equations with Caputo derivative as below:

$$\begin{aligned} {}_a^C D_t^{\alpha} x(t) &= f(t, x(t)), \\ x(0) &= x_0. \end{aligned} \tag{56}$$

Caputo fractional integral of this equation is given by

$$x(t) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{57}$$

If we take $t = t_{n+1}$ for $n = 0, 1, 2, ...,$ the equation (57) is rewritten as

$$x(t_{n+1}) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f(\tau, x(\tau)) d\tau. \tag{58}$$

Here, If we use the two-step Lagrange polynomial interpolation in integral then we have following

$$P_k(\tau) = f(\tau, x(\tau)) \simeq \frac{f(t_k, x_k)(\tau - t_{k-1})}{h} - \frac{f(t_{k-1}, x_{k-1})(\tau - t_k)}{h}, \tag{59}$$

where $h = t_n - t_{n-1}$. So we have

$$\begin{aligned} & x(t_{n+1}) - x(0) \\ = \; & \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \begin{pmatrix} P_k(\tau) \\ + \frac{(\tau - t_k)(\tau - t_{k-1})}{2!} \frac{\partial^2}{\partial \tau^2} \left[f(\tau, x(\tau))\right]_{\tau = \epsilon_k} \end{pmatrix} (t_{n+1} - \tau)^{\alpha-1} d\tau, \end{aligned} \tag{60}$$

or

$$x(t_{n+1}) - x(0) \tag{61}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c} \underbrace{\frac{f(t_k,x_k)}{h} \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(t_{n+1}-\tau)^{\alpha-1}\, d\tau}_{I_1} \\ -\underbrace{\frac{f(t_{k-1},x_{k-1})}{h} \int_{t_k}^{t_{k+1}} (\tau - t_k)(t_{n+1}-\tau)^{\alpha-1}\, d\tau}_{I_2} \\ + \int_{t_k}^{t_{k+1}} \frac{(\tau-t_k)(\tau-t_{k-1})}{2!} \frac{\partial^2}{\partial\tau^2} \left[f(\tau, x(\tau)) \right]_{\tau=\epsilon_k} (t_{n+1}-\tau)^{\alpha-1} d\tau \end{array} \right].$$

Finally, calculating integrals in equation above, we obtain

$$x(t_{n+1}) - x(0) \tag{62}$$

$$= \frac{f(t_k,x_k)}{h} \sum_{k=0}^{n} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\Gamma(\alpha+2)}$$

$$-\frac{f(t_{k-1},x_{k-1})}{h} \sum_{k=0}^{n} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} \\ -(n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} \\ -(n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\Gamma(\alpha+2)}$$

$$+E_n^{\alpha}$$

Above ${}_1E_n^{\alpha}$ *is error term and given by*

$$E_n^{\alpha} \tag{63}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \left(\frac{(\tau - t_k)(\tau - t_{k-1})}{2!} \frac{\partial^2}{\partial\tau^2} \left[f(\tau, x(\tau)) \right]_{\tau=\epsilon_k} \right) (t_{n+1}-\tau)^{\alpha-1} d\tau.$$

then we have

$$\left| E_n^{\alpha} \right| \tag{64}$$

$$\leq \frac{h}{2\Gamma(2+\alpha))} \max_{[0,t_{n+1}]} \left| \frac{\partial^2 f(\tau, x(\tau))}{\partial\tau^2} \right| \times$$

$$\sum_{k=0}^{n} \left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] - \\ (\alpha+1)(k-n-2) \left[\begin{array}{c} (n-k)^{\alpha} \\ -(n+1-k)^{\alpha} \end{array} \right] \end{array} \right).$$

The right-hand side converges as follows:

$$\sum_{k=0}^{n}\left(\begin{array}{c}\alpha\left[\begin{array}{c}(n-k)^{\alpha+1}\\-(n+1-k)^{\alpha+1}\end{array}\right]-\\(\alpha+1)(k-n-2)\left[\begin{array}{c}(n-k)^{\alpha}\\-(n+1-k)^{\alpha}\end{array}\right]\end{array}\right) \tag{65}$$
$$= (n^{\alpha}-(n+1)^{\alpha})\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha.$$

So we have error term as

$$\left|E_n^{\alpha}\right| \leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2 f(\tau,x(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha}) \tag{66}$$
$$\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right).$$

5.1. Application of method to system

In this part, we apply the method for fractional order Caputo system. Let us consider system with Caputo derivative.

$$\begin{aligned} {}_a^C D_t^{\alpha} S(t) &= F_1(t,S(t)), \\ {}_a^C D_t^{\alpha} I(t) &= F_2(t,I(t)), \\ {}_a^C D_t^{\alpha} Q(t) &= F_3(t,Q(t)), \\ {}_a^C D_t^{\alpha} R(t) &= F_4(t,R(t)). \end{aligned} \tag{67}$$

Then we have

$$\begin{aligned} S(t)-S_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F_1(\tau,S(\tau))\,d\tau, \\ I(t)-I_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F_2(\tau,I(\tau))\,d\tau, \\ Q(t)-Q_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F_3(\tau,Q(\tau))\,d\tau, \\ R(t)-R_0 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}F_4(\tau,R(\tau))\,d\tau. \end{aligned} \tag{68}$$

At a given point $t = t_{n+1}$, following formula is written

$$S_{n+1} - S_0 \tag{69}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c} \frac{F_1(t_k,S_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\ -\frac{F_1(t_{k-1},S_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} \\ -(n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \end{array} \right] +_1 R_n^{\alpha},$$

$$I_{n+1} - I_0$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c} \frac{F_2(t_k,I_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\ -\frac{F_2(t_{k-1},I_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} - \\ (n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \end{array} \right] +_2 R_n^{\alpha},$$

$$Q_{n+1} - Q_0$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} \left[\begin{array}{c} \frac{F_3(t_k,Q_k)}{h} \frac{\left(\begin{array}{c} \alpha \left[\begin{array}{c} (n-k)^{\alpha+1} \\ -(n+1-k)^{\alpha+1} \end{array} \right] \\ -(\alpha+1)(n-k+2) \left[\begin{array}{c} (n-k)^{\alpha+1} - \\ (n+1-k)^{\alpha+1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \\ -\frac{F_3(t_{k-1},Q_{k-1})}{h} \frac{\left(\begin{array}{c} (\alpha+1)(k-n-1) \left[\begin{array}{c} (n+2-k)^{\alpha} - \\ (n-k+1)^{\alpha+1} \end{array} \right] \\ -\alpha \left[\begin{array}{c} (n-k+2)^{\alpha-1} - \\ (n+1-k)^{\alpha-1} \end{array} \right] \end{array} \right)}{\alpha(\alpha+1)} \end{array} \right] +_3 R_n^{\alpha},$$

$$R_{n+1}-R_0$$
$$= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{n}\left[\begin{array}{c}\frac{F_4(t_k,R_k)}{h}\frac{\left(\begin{array}{c}\alpha\left[\begin{array}{c}(n-k)^{\alpha+1}\\-(n+1-k)^{\alpha+1}\end{array}\right]\\-(\alpha+1)(n-k+2)\left[\begin{array}{c}(n-k)^{\alpha+1}-\\(n+1-k)^{\alpha+1}\end{array}\right]\end{array}\right)}{\alpha(\alpha+1)}\\-\frac{F_4(t_{k-1},R_{k-1})}{h}\frac{\left(\begin{array}{c}(\alpha+1)(k-n-1)\left[\begin{array}{c}(n+2-k)^{\alpha}-\\(n-k+1)^{\alpha+1}\end{array}\right]\\-\alpha\left[\begin{array}{c}(n-k+2)^{\alpha-1}-\\(n+1-k)^{\alpha-1}\end{array}\right]\end{array}\right)}{\alpha(\alpha+1)}\end{array}\right]+_4R_n^{\alpha}.$$

Where

$$\begin{aligned}\left|{}_1R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_1(\tau,S(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\&\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),\end{aligned} \tag{70}$$

$$\begin{aligned}\left|{}_2R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_2(\tau,I(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\&\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),\end{aligned}$$

$$\begin{aligned}\left|{}_3R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_3(\tau,Q(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\&\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right),\end{aligned}$$

$$\begin{aligned}\left|{}_4R_n^{\alpha}\right| &\leq \frac{h}{2\Gamma(2+\alpha))}\max_{[0,t_{n+1}]}\left|\frac{\partial^2F_4(\tau,R(\tau))}{\partial\tau^2}\right|(n^{\alpha}-(n+1)^{\alpha})\\&\times\left(\left(\frac{(n+1)(\alpha n-n-4(\alpha+1))}{2}\right)-(n+1)^{\alpha+1}\alpha\right).\end{aligned}$$

6. Conclusion

In this paper fractional order SIQR model is considered. Here, we generalize the previous model by considering the order as fractional order. As we saw that, the fractional order model is much more efficient in modeling than its integer order version. We have applied the next generation matrix method to calculated the basic reproduction number R_0. Also, the detailed analysis such as existence ande uniqueness results of the solution and efficient numerical scheme for model are presented.

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