Second-order half-linear delay differential equations: Oscillation tests

Omar Bazighifan[a], Shyam Sundar Santra[b]

[a]*Department of Mathematics, Faculty of Science, Hadhramout University, 50512 Hadhramout, Yemen.*
[b]*Department of Mathematics, JIS College of Engineering, Kalyani, 741235, India.*

Abstract

Differential equations of second-order appear in physical applications such as fluid dynamics, electromagnetism, acoustic vibrations and quantum mechanics. In this paper, necessary and sufficient conditions are established of the solutions to second-order helf-linear delay differential equations of the form

$$\left(r(w')^{\mu_1}\right)'(\iota) + q(\iota)w^{\mu}(\kappa(\iota)) = 0 \,,$$

under the assumption $\int_{\iota}^{\infty} \left(r(s)\right)^{-1/\mu_1} ds = \infty$. We consider two cases when $\mu_1 > \mu$ and $\mu_1 < \mu$. Some examples are given to show effectiveness and applicability of the main result, and state an open problem.

Keywords: Oscillation; non-oscillation; delay; half-linear; Lebesgue's dominated convergence theorem.
2010 MSC: 34C10; 34C15; 34K11.

1. Introduction

The main feature of this article is to establish necessary and sufficient condition for the oscillation of solution of second-order half-linear delay differential equation

$$\left(r(w')^{\mu_1}\right)'(\iota) + q(\iota)w^{\mu}(\kappa(\iota)) = 0, \tag{1}$$

by considering two cases when $\mu_1 > \mu$ and $\mu_1 < \mu$. We suppose that the following assumptions hold:

Email addresses: o.bazighifan@gmail.com (Omar Bazighifan), shyam01.math@gmail.com (Shyam Sundar Santra)

(A1) μ_1 and μ are the quotient of two odd positive integers, $r, q \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $r(\iota) > 0$ and q is not identically zero eventually, $\kappa \in C([\iota_0, \infty), R_+)$ such that $\kappa(\iota) \leq \iota$ for $\iota \geq \iota_0$, $\kappa(\iota) \to \infty$ as $\iota \to \infty$.

(A2) $r(\iota) > 0$ and $\int_0^\infty (r(s))^{-1/\mu_1} \, ds = \infty$. Letting $\nabla(\iota) = \int_0^\iota (r(s))^{-1/\mu_1} \, ds$, we have $\lim_{\iota \to \infty} \nabla(\iota) = \infty$.

Initially, we consider a single delay, but in a later section, we can see the effect of having several delays. As an example of a function satisfying (A2), we have $r(\iota) = e^{-\iota}$ or $r(\iota) = \iota^{\mu_1}$.

The interest in the study of functional differential equations comes from their applications to engineering and natural sciences. Equations involving arguments that are delayed, advanced or a combination of both arise in models such as the lossless transmission lines in high speed computers that interconnect switching circuits. Moreover, delay differential equations play an important role in modeling virtually every physical, technical, and biological process, from celestial motion, to bridge design, to interactions between neurons.

Below, we review some key results in the oscillation of second-order differential equations which motivated our study.

In very recent paper [7], Bazighifan et al. have studied a second-order differential equations with several delays and several sub-linear neutral coefficients and established several sufficient conditions for oscillation of solution of the considered equation. Brands [11] showed that for bounded delays, the solutions to

$$w''(\iota) + q(\iota)w(\iota - \kappa(\iota)) = 0$$

are oscillatory if and only if the solutions to $u''(\iota) + q(\iota)u(\iota) = 0$ are oscillatory.

Recently, Chatzarakis et al. [12] have established sufficient conditions for the oscillation and asymptotic behavior of all solutions of the second-order half-linear differential equations of the form,

$$\left(r(w')^\mu \right)'(\iota) + q(\iota)w^\mu(\kappa(\iota)) = 0. \tag{2}$$

In an another paper, Chatzarakis et al. [13] have considered (2) and established improved oscillation criteria for (2).

Fisnarova and Marik [16] considered the half-linear differential equation

$$\left(r(\iota)\Phi(z'(\iota)) \right)' + c(\iota)\Phi(w(\kappa(\iota))) = 0, \quad z(\iota) = w(\iota) + b(\iota)w(\kappa(\iota)),$$

where $\Phi(\iota) = |\iota|^{p-2}\iota$, $p \geq 2$.

Karpuz and Santra [19] have established several sufficient conditions for the oscillation and asymptotic behavior of the solutions to the equation

$$\left[r(\iota)(w(\iota) + b(\iota)w(\sigma(\iota)))' \right]' + \sum_{i=1}^m c_i(\iota)g_i(w(\kappa_i(\iota))) = 0,$$

for different ranges of the neutral coefficient b. In [24, 25], Pinelas et al. have considered the first order non-linear neutral delay differential equations and obtained necessary and sufficient conditions for the oscillation of solution of the considered equation for the various ranges of the neutral coefficient.

Oscillation criteria for second-order delay differential equations have been reported in [1, 2, 3, 4, 14, 15, 18, 22, 23, 26, 27, 29, 30, 31, 32, 33]. Note that most publications consider only sufficient conditions, and just a few of them consider necessary and sufficient conditions. Thus, interested to find the necessary and sufficient conditions for the oscillation of the solution of (1).

By a solution to equation (1), we mean a function $w \in C([T_w, \infty), \mathbb{R})$, where $T_w \geq \iota_0$, such that $rw' \in C^1([T_w, \infty), \mathbb{R})$, and w satisfies (1) on the interval $[T_w, \infty)$. A solution w of (1) is said to be proper if w is not identically zero eventually, i.e., $\sup\{|w(\iota)| : \iota \geq T\} > 0$ for all $\iota \geq T_w$. We assume that (1) possesses such solutions. A solution of (1) is called *oscillatory* if it has arbitrarily large zeros on $[T_w, \infty)$; otherwise, it is said to be *non-oscillatory*. (1) itself is said to be oscillatory if all of its solutions are oscillatory.

2. Main Results

Lemma 2.1. *Assume that (A1) and (A2) hold. If w is an eventually positive solution of* (1), *then*

$$w'(\iota) > 0 \quad and \quad \big(r(w')^{\mu_1}\big)'(\iota) < 0 \quad for\ all\ large\ t. \tag{3}$$

Proof. Since $w(\iota)$ is an eventually positive solution of (1), there exists $\iota_0 \geq 0$ such that $w(\iota) > 0$ and $w(\kappa(\iota)) > 0$ for $\iota \geq \iota_0$. By (1), we have

$$\big(r(w')^{\mu_1}\big)'(\iota) = -q(\iota)w^{\mu}(\kappa(\iota)) \leq 0 \quad for\ \iota \geq \iota_0$$

which means that $\big(r(w')^{\mu_1}\big)(\iota)$ is non-increasing. We claim that $\big(r(w')^{\mu_1}\big)(\iota) > 0$ for $\iota \geq \iota_0$. On the contrary, assume that $\big(r(w')^{\mu_1}\big)(\iota) \leq 0$ for some $\iota \geq \iota_0$. Then, we can find $\iota^* \geq \iota_0$ and $\epsilon_1 > 0$ such that $\big(r(w')^{\mu_1}\big)(\iota) \leq -\epsilon_1$ for all $\iota \geq \iota^*$. Integrating the inequality $w'(\iota) \leq (-\epsilon_1/r(\iota))^{1/\mu_1}$ from ι^* to ι $(\iota > \iota^*)$, by (A2) we obtain

$$w(\iota) \leq w(\iota^*) - \epsilon_1^{1/\mu_1} \int_{\iota^*}^{\iota} (r(s))^{-1/\mu_1}\, ds \to -\infty, \quad as\ \iota \to \infty.$$

This contradicts $w(\iota)$ being a positive solution. So, $\big(r(w')^{\mu_1}\big)(\iota) > 0$ for $\iota \geq \iota_0$. Consequently, since $r(\iota) \geq 0$, then $w'(\iota) \geq 0$ for $\iota \geq \iota_0$. □

2.1. The Case $\mu_1 > \mu$.

In what follows, we assume that there exists a constant ν such that $0 < \mu < \nu < \mu_1$ and

$$u^{\mu-\nu} \quad is\ non\text{-}increasing\ for\ 0 < u. \tag{4}$$

Lemma 2.2. *Assume that all conditions of Lemma 2.1 hold. Then there exists $\iota_1 \geq \iota_0$ and $\kappa > 0$ such that for $\iota \geq \iota_1$, the following holds:*

$$w(\iota) \leq \epsilon^{1/\mu_1} \nabla(\iota) \tag{5}$$

$$\big(\nabla(\iota) - \nabla(\iota_1)\big)\Big[\int_t^{\infty} q(\zeta)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\zeta))\big)^{\mu-\nu} w^{\nu}(\kappa(\zeta))\, d\zeta \Big]^{1/\mu_1} \leq w(\iota). \tag{6}$$

Proof. By Lemma 2.1, $\big(r(w')^{\mu_1}\big)(\iota)$ is positive and non-increasing. Then there exists $\kappa > 0$ and $\iota_1 \geq \iota_0$ such that $\big(r(w')^{\mu_1}\big)(\iota) \leq \epsilon$. Integrating the inequality $w'(\iota) \leq (\epsilon/r(\iota))^{1/\mu_1}$, we have

$$w(\iota) \leq w(\iota_1) + \epsilon^{1/\mu_1}\big(\nabla(\iota) - \nabla(\iota_1)\big).$$

Since $\lim_{\iota\to\infty} \nabla(\iota) = \infty$, the last inequality becomes

$$w(\iota) \leq \epsilon^{1/\mu_1}\nabla(\iota) \quad for\ \iota \geq \iota_1,$$

which is (5).

By (5) and assumption (4), we have

$$w^{\mu}(\kappa(\iota)) = w^{\mu-\nu}(\kappa(\iota))w^{\nu}(\kappa(\iota)) \geq \big(\epsilon^{1/\mu_1}\nabla(\kappa(\iota))\big)^{\mu-\nu} w^{\nu}(\kappa(\iota)).$$

Integrating (1) from ι to ∞, we have

$$\lim_{A\to\infty}\big[(r(w')^{\mu_1})(s)\big]_{\iota}^{A} + \int_{\iota}^{\infty} q(s)\big(\epsilon^{1/\mu_1}\nabla(\kappa(s))\big)^{\mu-\nu} w^{\nu}(\kappa(s))\, ds \leq 0.$$

Using that $\big(r(w')^{\mu_1}\big)(\iota)$ is positive and non-increasing, we have

$$\int_\iota^\infty q(s)\big(\epsilon^{1/\mu_1}\nabla(\kappa(s))\big)^{\mu-\nu}w^\nu(\kappa(s))\,ds \le \big(r(w')^{\mu_1}\big)(\iota) \quad \text{for } \iota \ge \iota_1\,.$$

Therefore,

$$w'(\iota) \ge \Big[\frac{1}{r(\iota)}\int_\iota^\infty q(s)\big(\epsilon^{1/\mu_1}\nabla(\kappa(s))\big)^{\mu-\nu}w^\nu(\kappa(s))\,ds\Big]^{1/\mu_1}\,. \tag{7}$$

Since $w(\iota) \ge 0$, integrating (7) from ι_1 to ι, we obtain

$$w(\iota) \ge \int_{\iota_1}^\iota \Big[\frac{1}{r(s)}\int_s^\infty q(\zeta)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\zeta))\big)^{\mu-\nu}w^\nu(\kappa(\zeta))\,d\zeta\Big]^{1/\mu_1}\,ds$$
$$\ge \big(\nabla(\iota)-\nabla(\iota_1)\big)\Big[\int_t^\infty q(\zeta)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\zeta))\big)^{\mu-\nu}w^\nu(\kappa(\zeta))\,d\zeta\Big]^{1/\mu_1}\,,$$

which is (6).   □

Theorem 2.1. *Assume that* (A1) *and* (A2) *hold. Then every solution of* (1) *is oscillatory if and only if*

$$\int_{\iota_0}^\infty q(s)\nabla^\mu(\kappa(s))ds = +\infty\,. \tag{8}$$

Proof. To prove sufficiency by contradiction, assume that w is a non-oscillatory solution of (1). Without loss of generality we may assume that $w(\iota)$ is eventually positive. Then Lemmas 2.1 and 2.2 hold for $\iota \ge \iota_1$. So,

$$w(\iota) > \big(\nabla(\iota)-\nabla(\iota_1)\big)w^{1/\mu_1}(\iota) \quad \text{for all} \quad \iota \ge \iota_1,$$

where

$$w(\iota) = \int_t^\infty q(\zeta)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\zeta))\big)^{\mu-\nu}w^\nu(\kappa(\zeta))\,d\zeta \ge 0\,.$$

Since $\lim_{\iota\to\infty}\nabla(\iota) = \infty$, there exists $\iota_2 \ge \iota_1$, such that $\nabla(\iota)-\nabla(\iota_1) \ge \frac{1}{2}\nabla(\iota)$ for $\iota \ge \iota_2$. Then

$$w(\iota) > \frac{1}{2}\nabla(\iota)w^{1/\mu_1}(\iota) \quad \text{for } \iota \ge \iota_2$$

and $w^\nu/(\epsilon^{1/\mu_1}\nabla)^\nu \ge w^{\nu/\mu_1}/(2\epsilon^{1/\mu_1})^\nu$. Taking the derivative of w we have

$$w'(\iota) = -q(\iota)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\iota))\big)^{\mu-\nu}w^\nu(\kappa(\iota))$$
$$\le -q(\iota)\big(\epsilon^{1/\mu_1}\nabla(\kappa(\iota))\big)^\mu w^{\nu/\mu_1}(\kappa(\iota))(2\epsilon^{1/\mu_1})^{-\nu} \le 0\,.$$

Therefore, $w(\iota)$ is non-increasing so $w^{\nu/\mu_1}(\kappa(\iota))/w^{\nu/\mu_1}(\iota) \ge 1$, and

$$\big(w^{1-\nu/\mu_1}(\iota)\big)' = (1-\nu/\mu_1)w^{-\nu/\mu_1}(\iota)w'(\iota) \le -(1-\nu/\mu_1)2^{-\nu}\epsilon^{(\mu-\nu)/\mu_1}q(\iota)\nabla^\mu(\kappa(\iota))\,.$$

Integrating this inequality form ι_2 to ι, we have

$$\big[w^{1-\nu/\mu_1}(s)\big]_{\iota_2}^\iota \le -(1-\nu/\mu_1)2^{-\nu}\epsilon^{(\mu-\nu)/\mu_1}\int_{\iota_2}^\iota q(s)\nabla^\mu(\kappa(s))\,ds\,.$$

Since $\nu/\mu_1 < 1$ and $w(\iota)$ is positive and non-increasing, we have

$$\int_{\iota_2}^\iota q(s)\nabla^\mu(\kappa(s))\,ds \le \frac{2^\nu\epsilon^{(\nu-\mu)/\mu_1}}{(1-\nu/\mu_1)}w^{1-\nu/\mu_1}(\iota_2)\,.$$

This contradicts (8) and proves the oscillation of all solutions.

Next, we show that (8) is necessary. Suppose that (8) does not hold; so for each $\lambda > 0$, there exists $\iota \geq \iota_0$ such that

$$\int_\iota^\infty q(s)\nabla^\mu(\kappa(s))ds \leq \frac{\lambda^{1-\mu/\mu_1}}{2}. \tag{9}$$

Let us consider the closed subset of continuous functions

$$M = \Big\{ w \in C([\iota_0, +\infty), \mathbb{R}) : w(\iota) = 0 \text{ for } \iota_0 \leq \iota < T \text{ and}$$

$$\Big(\frac{\lambda}{2}\Big)^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)] \leq w(\iota) \leq \lambda^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)] \text{ for } T \leq \iota \Big\}.$$

We define the operator $\Phi : M \to C([\iota_0, +\infty), \mathbb{R})$ by

$$(\Phi w)(\iota) = \begin{cases} 0, & \iota_0 \leq \iota < T \\ \int_\iota^\iota \big[\frac{1}{r(s)}\big[\frac{\lambda}{2} + \int_s^\infty q(\zeta)w^\mu(\kappa(\zeta))d\zeta\big]\big]^{1/\mu_1}ds, & T \leq \iota. \end{cases}$$

For $w \in M$ and $\iota \geq T$, we have

$$(\Phi w)(\iota) \geq \int_\iota^\iota \Big[\frac{1}{r(s)}\frac{\lambda}{2}\Big]^{1/\mu_1} ds = \Big(\frac{\lambda}{2}\Big)^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)].$$

For $w \in M$ and $\iota \geq T$, we have $w(\iota) \leq \lambda^{1/\mu_1}\nabla(\iota)$ and $w^\mu(\kappa(\iota)) \leq \big(\lambda^{1/\mu_1}\nabla(\kappa(\iota))\big)^\mu$. Then using (9) we have

$$(\Phi w)(\iota) \leq \int_\iota^\iota \Big[\frac{1}{r(s)}\Big(\frac{\lambda}{2} + \frac{\lambda}{2}\Big)\Big]^{1/\mu_1} ds = \lambda^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)].$$

Thus, $\Phi w \in M$. Let us define now a sequence of continuous function $v_n : [\iota_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$v_1(\iota) = \begin{cases} 0, & t \in [\iota_0, T) \\ \big(\frac{\lambda}{2}\big)^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)], & \iota \geq T. \end{cases}$$

$$v_n(\iota) = (\Phi v_{n-1})(\iota) \quad \text{for } n > 1.$$

By induction, it is easy to verify that for $n > 1$,

$$\Big(\frac{\lambda}{2}\Big)^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)] \leq v_{n-1}(\iota) \leq v_n(\iota) \leq \lambda^{1/\mu_1}[\nabla(\iota) - \nabla(\iota)].$$

Therefore, the point-wise limit of the sequence exists. Let $\lim_{n\to\infty} v_n(\iota) = v(\iota)$ for $\iota \geq \iota_0$. By Lebesgue's dominated convergence theorem $v \in M$ and $(\Phi v)(\iota) = v(\iota)$, where $v(\iota)$ is a solution of equation (1) on $[T, \infty)$. Hence, (8) is a necessary condition. This completes the proof. □

Example 2.1. Consider the delay differential equation

$$(e^{-\iota}(w'(\iota))^{3/5})' + (\iota + 1)(w(\iota - 2))^{1/3} = 0, \quad \iota \geq 0. \tag{10}$$

Here $\mu_1 = 3/5$, $\mu = 1/3$, $r(\iota) = e^{-\iota}$, $\kappa(\iota) = \iota - 2$, $\nabla(\iota) = \int_{\iota_0}^\iota e^{5s/3} ds = \frac{3}{5}\big(e^{5\iota/3} - 1\big)$. For $\nu = 1/2$, we have $0 < \mu < \nu < \mu_1$ and $u^{\mu-\nu} = u^{-1/6}$ which is a decreasing function. To check (8) we have

$$\int_{\iota_0}^\infty q(s)\nabla^\mu(\kappa(s))ds = \int_{\iota_0}^\infty (s+1)\Big(\frac{3}{5}\big(e^{5(s-2)/3} - 1\big)\Big)^{1/3} ds = \infty,$$

because the integral approaches $+\infty$ as $s \to +\infty$. So that all the assumptions in Theorem 2.1 hold. Thus, every solution of (10) oscillates.

2.2. The Case $\mu_1 < \mu$.

In what follows, we assume that there exists $\mu > \nu > \mu_1 > 0$ such that

$$u^{\mu-\nu} \quad \text{is non-decreasing for } 0 < u. \tag{11}$$

Lemma 2.3. *Assume that all conditions of Lemma 2.1 hold. Then there exists $\iota_1 \geq \iota_0$ and $\kappa > 0$ such that for $\iota \geq \iota_1$, the following holds:*

$$x^\mu(\kappa(\iota))) \geq \epsilon^{\mu-\nu} w^\nu(\kappa(\iota)). \tag{12}$$

Proof. By Lemma 2.1, it follows that $w'(\iota) > 0$, so w is increasing and $w(\iota) \geq w(\iota_0)$ for $\iota \geq \iota_0$. Thus

$$w(\kappa(\iota)) \geq w(\kappa(\iota_0)) := \epsilon > 0 \quad \text{for } \iota \geq \iota_1 := \iota_0.$$

From (11), we have

$$w^\mu(\kappa(\iota))) = w^{\mu-\nu}(\kappa(\iota))w^\nu(\kappa(\iota)) \geq \epsilon^{\mu-\nu} w^\nu(\kappa(\iota)) \text{ for } \iota \geq \iota_1,$$

which is (12). $\qquad\square$

Theorem 2.2. *Assuming* (A1), (A2) *and $r(\iota)$ is non-decreasing, every solution of* (1) *is oscillatory if and only if*

$$\int_T^\infty \left[\frac{1}{r(s)} \int_s^\infty q(\zeta)\,d\zeta \right]^{1/\mu_1} ds = +\infty \quad \text{for all} \quad T > 0. \tag{13}$$

Proof. To prove sufficiency by contradiction, assume that w is a non-oscillatory solution of (1). Without loss of generality we may assume that $w(\iota)$ is eventually positive. Then Lemmas 2.1 and 2.3 hold for $\iota \geq \iota_1$. Using (12) in (1) and integrating the final inequality from ι to ∞, we have

$$\lim_{A\to\infty} \left[(r(w')^{\mu_1})'(s) \right]_\iota^A + \epsilon^{\mu-\nu} \int_\iota^\infty q(s)w^\nu(\kappa(s))\,ds \leq 0.$$

Using that $(r(w')^{\mu_1})(\iota)$ is positive and nonincreasing, and $r'(\iota) \geq 0$, we have

$$\epsilon^{\mu-\nu} \int_\iota^\infty q(s)w^\nu(\kappa(s))\,ds \leq (r(w')^{\mu_1})(\iota) \leq (r(w')^{\mu_1})(\kappa(\iota)) \leq r(\iota)(w'(\kappa(\iota)))^{\mu_1}$$

for all $\iota \geq \iota_1$. Therefore,

$$\epsilon^{(\mu-\nu)/\mu_1} \left[\frac{1}{r(\iota)} \int_\iota^\infty q(s)w^\nu(\kappa(s))\,ds \right]^{1/\mu_1} \leq w'(\kappa(\iota))$$

which implies that

$$\epsilon^{(\mu-\nu)/\mu_1} \left[\frac{1}{r(\iota)} \int_\iota^\infty q(s)\,ds \right]^{1/\mu_1} \leq \frac{w'(\kappa(\iota))}{w^{\nu/\mu_1}(\kappa(\iota))}. \tag{14}$$

Integrating (14) from ι_1 to ∞, we have

$$\epsilon^{(\mu-\nu)/\mu_1} \int_{\iota_1}^\infty \left[\frac{1}{r(s)} \int_s^\infty q(\zeta)\,d\zeta \right]^{1/\mu_1} ds \leq \frac{w^{1-\nu/\mu_1}(\kappa(\iota_1))}{\nu/\mu_1 - 1} < \infty.$$

This contradicts (13) and proves the oscillation of all solutions.

Next, we show that (13) is necessary. Suppose that (13) does not hold; so for each $\lambda > 0$, there exists $\iota \geq \iota_0$ such that

$$\int_\iota^\infty \left[\frac{1}{r(s)} \int_s^\infty q(\zeta)d\zeta \right]^{1/\mu_1} ds \leq \frac{\lambda^{1-\mu/\mu_1}}{2}. \tag{15}$$

Let us consider the closed subset of continuous functions

$$M = \left\{ w \in C([\iota_0, +\infty), \mathbb{R}) : w(\iota) = \frac{\lambda}{2} \text{ for } \iota \in [\iota_0, T) \text{ and } \frac{\lambda}{2} \le w(\iota) \le \lambda \text{ for } \iota \ge T \right\}.$$

Then we define the operator $\Phi : M \to C([\iota_0, +\infty), \mathbb{R})$ by

$$(\Phi w)(\iota) = \begin{cases} \lambda/2, & \iota_0 \le \iota < T \\ \lambda/2 + \int_\iota^\iota \left[\frac{1}{r(s)} \int_s^\infty q(\zeta) w^\mu(\kappa(\zeta)) d\zeta \right]^{1/\mu_1} ds & T \le \iota. \end{cases}$$

Note that for $w \in M$, we have $(\Phi w)(\iota) \ge \lambda/2$. Also for $w \in M$ and $\iota \ge T$, we have $w(\iota) \le \lambda$ and by (15), $(\Phi w)(\iota) \le \lambda$. Therefore, $\Phi w \in M$. Similarly to the proof of Theorem 2.1, the mapping Φ has a fixed point $v \in M$; that is, $(\Phi v)(\iota) = v(\iota)$ for $\iota \ge \iota_0$. Easily, we can verify that $u(\iota)$ is a solution of (1), such that $\lambda/2 \le v(\iota) \le \lambda$ for $\iota \ge T$. Thus we have a non-oscillatory solution to (1). This completes the proof. $\qquad \square$

Example 2.2. Consider the delay differential equation

$$(\iota^{\mu_1}(w'(\iota))^{1/5})' + (\iota + 1)(w(\iota - 2))^{5/3} = 0, \quad \iota \ge 0. \tag{16}$$

Here $\mu_1 = 1/5$, $\mu = 5/3$, $r(\iota) = \iota^{\mu_1}$, $\kappa(\iota) = \iota - 2$, $\nabla(\iota) = \log(\iota) - 1$. For $\nu = 4/3$, we have $\mu > \nu > \mu_1 > 0$ and $u^{\mu - \nu} = u^{1/3}$ which is an increasing function. The integral in (13) is equal to

$$\int_{\iota_2}^\infty \left[\frac{1}{\iota^{1/5}} \int_s^\infty (\zeta + 1) d\zeta \right]^5 ds = \infty.$$

So, all the assumptions in Theorem 2.2 hold. Thus, every solution of (16) oscillates.

3. Conclusion

The aim of this work is to establish necessary and sufficient conditions for the oscillation of solution to second-order half-linear differential equation. The obtained oscillation theorems complement the well-known oscillation results present in the literature.

In this section, we state one remark and illustrate it by two examples.

Remark 3.1. *The results of this paper also hold for equations of the form*

$$\left(r(w')^{\mu_1} \right)'(\iota) + \sum_{i=1}^m q_i(\iota) w^{\mu_i}(\kappa_i(\iota)) = 0,$$

where r, q_i, μ_i, κ_i $(i = 1, 2, \ldots, m)$ satisfy the assumptions (A1)–(A2), (4) and (11). In order to extend Theorem 2.1 and Theorem 2.2, there exists an index i such that q_i, μ_i, κ_i fulfills (8) and (13).

We conclude the paper by two examples that show how Remark 3.1 can be applied.

Example 3.1. Consider the delay differential equation

$$(e^{-\iota}(w'(\iota))^{3/5})' + \frac{1}{\iota + 1}(w(\iota - 2))^{1/3} + \frac{1}{\iota + 2}(w(\iota - 1))^{1/5} = 0, \quad \iota \ge 0. \tag{17}$$

Here $\mu_1 = 3/5$, $r(\iota) = e^{-\iota}$, $\kappa_1(\iota) = \iota - 2$, $\kappa_2(\iota) = \iota - 1$, $\nabla(\iota) = \int_{\iota_0}^\iota e^{5s/3} ds = \frac{3}{5} \left(e^{5\iota/3} - 1 \right)$ and $i = 1, 2$. $\mu_1 = 1/3$ and $\mu_2 = 1/5$. For $\nu = 1/2$, we have $0 < \max\{\mu_1, \mu_2\} < \nu < \mu_1$ and $u^{\mu_1 - \nu} = u^{-1/6}$ and $u^{\mu_2 - \nu} = u^{-3/10}$ which both are decreasing functions. To check (8) we have

$$\int_{\iota_0}^\infty \sum_{i=1}^m q_i(s) \nabla^{\mu_i}(\kappa_i(s)) ds \ge \int_{\iota_0}^\infty q_1(s) \nabla^{\mu_1}(\kappa_1(s))) ds$$

$$= \int_{\iota_0}^\infty \frac{1}{s+1} \left(\frac{3}{5} \left(e^{5(s-2)/3} - 1 \right) \right)^{1/3} ds = \infty,$$

because the integral approaches $+\infty$ as $s \to +\infty$. So that all the assumptions of Theorem 2.1 hold. Thus, every solution of (17) oscillates.

Example 3.2. Consider the delay differential equation

$$(\iota^{\mu_1}(w'(\iota))^{3/5})' + \iota(w(\iota - 2))^{5/3} + (\iota + 1)(w(\iota - 1))^{7/3} = 0, \quad \iota \geq 0. \tag{18}$$

Here $\mu_1 = 3/5$, $r(\iota) = \iota^{\mu_1}$, $\kappa_1(\iota) = \iota - 2$, $\kappa_2(\iota) = \iota - 1$, $\nabla(\iota) = \log(\iota) - 1$ and $i = 1, 2$. $\mu_1 = 5/3$ and $\mu_2 = 7/3$. For $\nu = 4/3$, we have $\min\{\mu_1, \mu_2\} > \nu > \mu_1 > 0$ and $u^{\mu_1 - \nu} = u^{1/3}$ and $u^{\mu_2 - \nu} = u$ which both are increasing functions. Clearly, all the assumptions of Theorem 2.2 hold. Thus, every solution of (18) oscillates.

Open problem

This work, as well as [3, 4, 7, 13, 14, 15, 19, 20, 26, 27, 29], lead us to pose an open problem: Can we find necessary and sufficient conditions for the oscillation of the solutions to second-order neutral differential equation with several delays and several half-linear neutral coefficients for different ranges of the neutral coefficients.

Acknowledgments

References

[1] R.P. Agarwal, L. Berzansky, E. Braverman and A. Domoshnitsky, Nonoscillation theory of functional differential equations with applications, *Springer*, 2012.

[2] B. Baculikova, T. Li and J. Dzurina, Oscillation theorems for second order neutral differential equations, *Electron. J. Quali. Theo. diff. equa.*, (74): (2011), 1–13.

[3] B. Baculikova and J. Dzurina, Oscillation theorems for second order neutral differential equations, *Comput. Math. Appl.*, **61** (2011), 94-99.

[4] B. Baculikova and J. Dzurina, Oscillation theorems for second-order nonlinear neutral differential equations, *Comput. Math. Appl.*, **62** (2011), 4472–4478.

[5] C. Cesarano, O. Bazighifan, Qualitative behavior of solutions of second order differential equations, *Symmetry*, **11** (2019), 1–8.

[6] O. Bazighifan, C. Cesarano, Some New Oscillation Criteria for Second-Order Neutral Differential Equations with Delayed Arguments, *Mathematics*, **7** (2019), 1–8.

[7] O. Bazighifan, M. Ruggieri, S.S. Santra, A. Scapellato, Qualitative Properties of Solutions of Second-Order Neutral Differential Equations, *Symmetry*, 12(9), (2020), 1520; DOI:10.3390/sym12091520.

[8] O. Bazighifan, Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations, *Adv. Difference Equ.*, **201** (2020), 1–12.

[9] O. Moaaz, E.M. Elabbasy, B. Qaraad, An improved approach for studying oscillation of generalized Emden Fowler neutral differential equation, *J. Inequal. Appl.* 2020, 2020, 69.

[10] O. Moaaz, New criteria for oscillation of nonlinear neutral differential equations, *Adv. Differ. Equ.* 2019, 2019, 484.

[11] J.J.M.S. Brands, Oscillation theorems for second-order functional-differential equations, *J. Math. Anal. Appl.* **63**(1): (1978), 54–64.

[12] G.E. Chatzarakis, J. Dzurina and I. Jadlovska, New oscillation criteria for second-order half-linear advanced differential equations, *Appl. Math. Comput.*, **347** (2019), 404–416.

[13] G.E. Chatzarakis and I. Jadlovska, Improved oscillation results for second-order half-linear delay differential equations, *Hacet. J. Math. Stat.*, **48**(1): (2019), 170–179.

[14] G.E. Chatzarakis, J. Dzurina and I. Jadlovska, A remark on oscillatory results for neutral differential equations, *Appl. Math. Lett.*, **90** (2019), 124–130.

[15] J. Dzurina, Oscillation theorems for second order advanced neutral differential equations, *Tatra Mt. Math. Publ.*, DOI: 10.2478/v10127-011-0006-4, **48** (2011), 61–71.

[16] S. Fisnarova and R. Marik, Oscillation of neutral second order half-linear differential equations without commutativity in delays, *Math. Slovaca*, **67** No. 3 (2017), 701–718.

[17] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, *Clarendon*, Oxford, 1991.

[18] M. Hasanbulli and Y.V. Rogovchenko, Oscillation criteria for second order nonlinear neutral differential equations, *Appl. Math. Comput.*, **215** (2010), 4392–4399.

[19] B. Karpuz and S.S. Santra, Oscillation theorems for second-order nonlinear delay differential equations of neutral type, *Hacet J. Math. Stat.*, Doi: 10.15672/HJMS.2017.542 (in press)

[20] B. Karpuz and S. .S Santra; *New criteria for the oscillation and asymptotic behavior of second-order neutral differential equations with several delays*, Turk J Math., (2020) 44: 1990 −2003. doi:10.3906/mat-2006-103

[21] G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, *Marcel Dekker*, New York and Basel, 1987.

[22] Q. Li, R. Wang, F. Chen and T. Li, Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients, *Adv. Diff. Equ.* (2015) 2015:35. DOI 10.1186/s13662-015-0377-y.

[23] Y. Liu, J. Zhanga and J. Yan, Existence of oscillatory solutions of second order delay differential equations, *J. Comput. Appl. Math.*, **277** (2015), 17–22.

[24] S. Pinelas, S.S. Santra, Necessary and sufficient condition for oscillation of nonlinear neutral first-order differential equations with several delays, *J. Fixed Point Theory Appl.*, **20**(27): (2018), 1–13.

[25] S. Pinelas and S.S. Santra, Necessary and sufficient conditions for oscillation of nonlinear first order forced differential equations with several delays of neutral type, *Analysis*, **39**(3): (2019), 97–105.

[26] S.S. Santra, Oscillation analysis for nonlinear neutral differential equations of second order with several delays, *Mathematica*, **59(82)**(1-2): (2017), 111–123.

[27] S.S. Santra, Oscillation analysis for nonlinear neutral differential equations of second order with several delays and forcing term, *Mathematica*, **61**(84)(1): (2019), 63–78.

[28] S.S. Santra, Necessary and sufficient condition for the solutions of first-order neutral differential equations to be oscillatory or tend to zero, *KYUNGPOOK Math. J.*, **59** (2019), 73-82.

[29] S.S. Santra, Necessary and sufficient condition for oscillatory and asymptotic behaviour of second-order functional differential equations, *Krag. J. Math.*, **44**(3): (2020), 459–473.

[30] S.S. Santra, O. Bazighifan, H. Ahmad and Yu-Ming Chu, Second-order differential equation: oscillation theorems and applications, *Mathematical Problems in Engineering*, Volume 2020, Article ID 8820066, 6 pages. https://doi.org/10.1155/2020/8820066.

[31] S.S. Santra, I. Dassios, and T. Ghosh, On the asymptotic behavior of a class of second-order non-linear neutral differential equations with multiple delays, *Axioms* 2020, 9, 134; doi:10.3390/axioms9040134.

[32] S.S. Santra, O. Bazighifan, H. Ahmad and Shao-Wen Yao, Second-order differential equation with multiple delays: oscillation theorems and applications, *Complexity*, Volume 2020, Article ID 8853745, 6 pages. https://doi.org/10.1155/2020/8853745

[33] S.S. Santra, T. Ghosh and O. Baghifan, Explicit criteria for the oscillation of second-order differential equations with several sub-linear neutral coefficients, *Advances in Difference Equations* (2020) 2020:643 https://doi.org/10.1186/s13662-020-03101-1

[34] J.S.W. Wong, Necessary and suffcient conditions for oscillation of second order neutral differential equations, *J. Math. Anal. Appl.*, **252**(1): (2000), 342–352.