Erciyes Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 36, Sayı 2 , 2020

Erciyes University
Journal of Institue Of Science and Technology
Volume 36, Issue 2, 2020

Characterization of Some Special Curves in E_1^3

Beyhan Yılmaz*[1], Aykut Has[2]

*[1] Kahramanmaraş Sütçü İmam Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümü, KAHRAMANMARAŞ
[2] Kahramanmaraş Sütçü İmam Üniversitesi Fen Edebiyat Fakültesi Matematik Bölümü, KAHRAMANMARAŞ

(Alınış / Received: 12.06.2020 Kabul / Accepted: 01.07.2020, Online Yayınlanma / Published Online: 17.08.2020)

Keywords
Rectifying curve,
Normal curve,
Oskulating curve

Abstract: The planes which is spanned by $\{T(s), B(s)\}$, $\{N(s), B(s)\}$ and $\{T(s), N(s)\}$ are known as the rectifying, normal and osculating plane, respectively. The curve α is called rectifying, normal and osculating curve for which the position vector α always lie in its rectifying, normal and osculating plane, respectively. It is also known that if all rectifying planes of a non-planar curve in E^3 pass through a particular point, then the ratio of its torsion and curvature is a non-constant linear function. Rectifying, normal and osculating curves are studied many times in different spaces by many researchers. The aim of this paper is to characterize these curves from another point of view in Minkowski 3-space.

E_1^3 Uzayında Bazı Özel Eğrilerin Karakterizasyonu

Anahtar Kelimeler
Rektifiyan eğri,
Normal eğri,
Oskülatör eğri

Öz: $\{T(s), B(s)\}$, $\{N(s), B(s)\}$ ve $\{T(s), N(s)\}$ vektörleri tarafından gerilen düzlemler sırasıyla rektifiyan düzlem, normal düzlem ve oskülatör düzlem adını alırlar. Bir α eğrisi ise pozisyon vektörünün kendi rektifiyan düzleminde, normal düzleminde ya da oskülatör düzleminde yatmasıyla sırasıyla rektifiyan, normal ve oskülatör eğri olarak isimlendirilir. Ayrıca 3-boyutlu Öklid uzayında düzlemsel olmayan bütün rektifiyan eğriler için çok iyi bilinen bir karakterizasyon vardır. Bu karakterizasyon, eğrinin torsiyon ve eğriliğinin oranı sabit olmayan lineer fonksiyon olan eğriler rektifiyandır şeklindedir. Rektifiyan, normal ve oskülatör eğriler, pek çok araştırmacı tarafından farklı uzaylarda pek çok kez çalışılmıştır. Bu çalışmanın amacı 3-boyutlu Minkowski uzayında farklı bir bakış açısı ile bu eğrileri karakterize etmektir.

*İlgili Yazar, email: beyhanyilmaz@ksu.edu.tr

1. Introduction

In differential geometry, the theory of curves is one of the main study areas in Euclidean 3-space. In three-dimensional Euclidean space, to explain the geometric structure of any regular spatial curve, an orthonormal basis $T(s), N(s)$ and $B(s)$ called the Frenet frame at each point of the curve is described.

The planes which is spanned by $\{T(s), B(s)\}, \{N(s), B(s)\}$ and $\{T(s), N(s)\}$ are known as the rectifying, normal and osculating plane, respectively. The curve α is called rectifying, normal and osculating curve for which the position vector α always lie in its rectifying, normal and osculating plane, respectively [3]. These curves are studied by many authors and many characterizations are obtained [1, 4, 5, 6, 7, 9].

In this paper, we study these special curves in E_1^3 . These curves have previously been worked on in different spaces, but the importance of this study is to obtain results using a different method.

2. Material and Method

Let E_1^3 be Minkowski $3-$space with the following metric

$$\langle , \rangle_L = R^3 \times R^3 \to R$$
$$(u,v) \to \langle u,v \rangle_L = u_1v_1 + u_2v_2 - u_3v_3$$

where $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ are the usual coordinate system in E_1^3. An arbitrary vector $u \in E_1^3$ said spacelike if $\langle u,u \rangle_L > 0$ or $u = 0$, timelike if $\langle u,u \rangle_L < 0$ and null (lightlike) if $\langle u,u \rangle_L = 0$ but $u \neq 0$. This classification can be generalized for regular curve α according as the casual character of their tangent vectors. In other words, the curve α is called a spacelike (resp. timelike and lightlike) if its velocity vector $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for every $t \in I$. The norm of a vector u is given by $\|u\|_L = \sqrt{|\langle u,u \rangle_L|}$.

Assume that $\{T(s), N(s), B(s)\}$ is the moving positive directed frame along the unit speed curve α . Here $T(s) = \alpha'(s)$ is a tangent vector, $N(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ is a principal normal vector and $B(s) = T(s) \times N(s)$ is a binormal vector field along the curve α.

Frenet-Serret formulas can be given as follows, see [8]:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\varepsilon_1\varepsilon_3\kappa(s) & 0 & -\varepsilon_2\varepsilon_3\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where $\kappa(s)$ and $\tau(s)$ are curvature and torsion of α, respectively. Moreover, the Frenet-Serret vectors satisfy

$$\langle T(s), T(s) \rangle_L = \varepsilon_1, \langle N(s), N(s) \rangle_L = \varepsilon_2, \langle B(s), B(s) \rangle_L = \varepsilon_3,$$

$$\langle T(s), N(s) \rangle_L = \langle T(s), B(s) \rangle_L = \langle N(s), B(s) \rangle_L = 0,$$

and

$$T(s) \times N(s) = B(s),$$
$$B(s) \times N(s) = \varepsilon_3 T(s),$$
$$T(s) \times B(s) = \varepsilon_1 N(s).$$

Definition 2.1. A curve is congruent to a rectifying curve if and only if the ratio $\frac{\tau}{\kappa}$ is a nonconstant linear function of arclenght of parameter [2]. Also, unit speed curve with nonzero curvatures lies on a sphere if and only if $\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$ [3].

3. Results

In this section we give some new corollaries related to special curves in Minkowski 3-space. Assume that $\alpha(s)$ be a unit speed non-null curve with non-zero curvature in Minkowski 3-space. Since the rectifying plane of $\alpha(s)$ is the perpendicular plane to $N(s)$, we have

$$\langle \alpha(s) - x_0, N(s) \rangle = 0.$$

If we take the derivative of this expression, we obtain that

$$\langle T(s), N(s) \rangle + \langle \alpha(s) - x_0, N'(s) \rangle = 0.$$

By substituting from the Frenet-Serret formula, we achieve the following equation

$$-\varepsilon_1\varepsilon_3\kappa(s)\langle \alpha(s) - x_0, T(s) \rangle - \varepsilon_2\varepsilon_3\tau(s)\langle \alpha(s) - x_0, B(s) \rangle = 0. \quad (3.1)$$

So we can easily see that,

$$\langle \alpha(s) - x_0, B(s) \rangle = \frac{-\varepsilon_1 \kappa(s)}{\varepsilon_2 \tau(s)} \langle \alpha(s) - x_0, T \rangle. \tag{3.2}$$

If we take the derivative of equation (3.1), we obtain that

$$-\varepsilon_1 \varepsilon_3 \big(\langle T(s), \kappa(s) T(s) \rangle + \langle \alpha(s) - x_0, \kappa'(s) T(s) + \kappa^2(s) N(s) \rangle \big)$$
$$-\varepsilon_2 \varepsilon_3 \big(\langle T(s), \tau(s) B(s) \rangle + \langle \alpha(s) - x_0, \tau'(s) B(s) + \tau^2(s) N(s) \rangle \big) = 0.$$

If necessary arrangements are made, we can easily see that

$$-\varepsilon_1^2 \varepsilon_3 \kappa(s) - \varepsilon_1 \varepsilon_3 \kappa'(s) \langle \alpha(s) - x_0, T(s) \rangle - \varepsilon_2 \varepsilon_3 \tau'(s) \langle \alpha(s) - x_0, B(s) \rangle = 0.$$

If equation (3.2) is written in the last equation, we get

$$\varepsilon_1^2 \varepsilon_3 \kappa(s) = \varepsilon_1 \varepsilon_3 \langle \alpha(s) - x_0, T(s) \rangle \left(-\kappa'(s) + \tau'(s) \frac{\kappa(s)}{\tau(s)} \right).$$

Then using the last two equations we reach the following equations;

$$\langle \alpha(s) - x_0, T(s) \rangle = \frac{\varepsilon_1 \kappa(s)}{-\kappa'(s) + \tau'(s) \frac{\kappa(s)}{\tau(s)}} \tag{3.3}$$

and

$$\langle \alpha(s) - x_0, B(s) \rangle = \frac{-\varepsilon_1}{\varepsilon_2} \frac{\kappa(s)}{\tau(s)} \frac{\varepsilon_1 \kappa(s)}{-\kappa'(s) + \tau'(s) \frac{\kappa(s)}{\tau(s)}}. \tag{3.4}$$

Then if we work with the denominator of equation (3.3), we have

$$-\kappa'(s) + \tau'(s) \frac{\kappa(s)}{\tau(s)} = \frac{\tau'(s)\kappa(s) - \kappa'(s)\tau(s)}{\tau(s)},$$
$$= \frac{\left(\frac{\tau(s)}{\kappa(s)} \right)' \kappa^2(s)}{\tau(s)}.$$

If we using above result in equation (3.3), we can see that

$$\langle \alpha(s) - x_0, T(s) \rangle = \varepsilon_1 \frac{\frac{\tau(s)}{\kappa(s)}}{\left(\frac{\tau(s)}{\kappa(s)} \right)'}. \tag{3.5}$$

So, we obtain that

$$\langle \alpha(s) - x_0, B(s) \rangle = \frac{-1}{\varepsilon_2} \frac{1}{\left(\frac{\tau(s)}{\kappa(s)} \right)'}. \tag{3.6}$$

From the equations (3.5) and (3.6), the equation of the curve $\alpha(s)$ is obtained as follows

$$\alpha(s) - x_0 = \frac{\frac{\tau(s)}{\kappa(s)}}{\left(\frac{\tau(s)}{\kappa(s)} \right)'} T(s) - \frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{\left(\frac{\tau(s)}{\kappa(s)} \right)'} B(s). \tag{3.7}$$

Using the equality $\frac{\tau(s)}{\kappa(s)} = H(s)$ which is called the harmonic curvature function, we have

$$\alpha(s) - x_0 = \frac{H(s)}{H'(s)} T(s) - \frac{1}{\varepsilon_2 \varepsilon_3} \frac{1}{H'(s)} B(s).$$

If we take the derivative of this equation, we get

$$\left(\left(\frac{H(s)}{H'(s)}\right)' - 1\right)T(s) + \left(\frac{H(s)}{H'(s)}\kappa(s) - \frac{1}{\varepsilon_2\varepsilon_3}\frac{1}{H'(s)}\tau(s)\right)N(s) + \left(-\frac{1}{\varepsilon_2\varepsilon_3}\left(\frac{1}{H'(s)}\right)'\right)B(s) = 0.$$

Since $T(s), N(s)$ and $B(s)$ are linearly independent, we obtain that

$$\left(\frac{H(s)}{H'(s)}\right)' - 1 = 0, \tag{3.8}$$

$$\frac{H(s)}{H'(s)}\kappa(s) - \frac{1}{\varepsilon_2\varepsilon_3}\frac{1}{H'(s)}\tau(s) = 0, \tag{3.9}$$

and

$$-\frac{1}{\varepsilon_2\varepsilon_3}\left(\frac{1}{H'(s)}\right)' = 0. \tag{3.10}$$

Corollary 3.1. Let $\alpha \ : \ I \subset R \rightarrow E_1^3$ be a unit speed timelike curve with non-zero curvature in Minkowski 3-space. If every rectifying plane contains the point x_0 in R^3 , i.e, if $\alpha(s)$ is a rectifying curve, then $\frac{\tau}{\kappa}$ is a linear function.

Proof. From the equation (3.9), we obtain that $\varepsilon_2\varepsilon_3 = 1$. So we can easily say that the curve $\alpha(s)$ is a timelike in Minkowski 3-space. Easily using the equations (3.8) or (3.10), $H''(s) = 0$ and $H(s) = \frac{\tau(s)}{\kappa(s)} = cs + d$ for some constants c, d and arc length s.

With similar thought, assume that $\alpha(s)$ be a unit speed non-null curve with non-zero curvature in Minkowski 3-space. Since the normal plane of $\alpha(s)$ is orthogonal to $T(s)$, we have

$$\langle \alpha(s) - x_0, T(s) \rangle = 0.$$

If we take the derivative of this expression, we get

$$\langle T(s), T(s) \rangle + \langle \alpha(s) - x_0, T'(s) \rangle = 0.$$

Then by substituting from the Frenet-Serret formula, we have

$$\varepsilon_1 + \langle \alpha(s) - x_0, \kappa(s)N(s) \rangle = 0, \tag{3.11}$$

$$\langle \alpha(s) - x_0, N(s) \rangle = \frac{-\varepsilon_1}{\kappa(s)}. \tag{3.12}$$

If we take the derivative of equation (3.11), we can see

$$\langle \alpha(s) - x_0, \kappa'(s)N(s) \rangle - \varepsilon_1\varepsilon_3\kappa^2(s)\langle \alpha(s) - x_0, T(s) \rangle - \varepsilon_2\varepsilon_3\kappa(s)\tau(s)\langle \alpha(s) - x_0, B(s) \rangle = 0.$$

So if we write equation (3.12) in the last equation, we obtain

$$\langle \alpha(s) - x_0, B(s) \rangle = \frac{1}{\varepsilon_2\varepsilon_3}\frac{\left(\frac{\varepsilon_1}{\kappa(s)}\right)'}{\tau(s)}. \tag{3.13}$$

Using the equations (3.12) and (3.13), we can see that

$$\alpha(s) - x_0 = \frac{-\varepsilon_1}{\varepsilon_2\kappa(s)}N(s) + \frac{1}{\varepsilon_2\varepsilon_3^2}\frac{\left(\frac{\varepsilon_1}{\kappa(s)}\right)'}{\tau(s)}B(s). \tag{3.14}$$

If we say $\frac{\varepsilon_1}{\kappa(s)} = t(s)$, equation (3.14) takes the form the following equation

$$\alpha(s) - x_0 = \frac{-1}{\varepsilon_2}t(s)N(s) + \frac{1}{\varepsilon_2}\frac{t'(s)}{\tau(s)}B(s).$$

So, if we take the derivative of above equation

$$T(s)=\frac{-1}{\varepsilon_2}t^{'}(s)N(s)-\frac{1}{\varepsilon_2}t(s)(-\varepsilon_1\varepsilon_3\kappa(s)T(s)-\varepsilon_2\varepsilon_3\tau(s)B(s))$$
$$+\frac{1}{\varepsilon_2}\left(\frac{t^{'}(s)}{\tau(s)}\right)^{'}B(s)+\frac{1}{\varepsilon_2}\frac{t^{'}(s)}{\tau(s)}\tau(s)N(s)$$

and

$$(-1+\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2}\kappa(s)t(s))T(s)+\left(\frac{\varepsilon_2\varepsilon_3}{\varepsilon_2}\tau(s)t(s)+\frac{1}{\varepsilon_2}\left(\frac{t^{'}(s)}{\tau(s)}\right)^{'}\right)B(s)=0.$$

We know that $T(s)$ and $B(s)$ are linearly independent. Thus,

$$-1+\frac{\varepsilon_1\varepsilon_3}{\varepsilon_2}\kappa(s)t(s)=0, \tag{3.15}$$

$$\varepsilon_3\tau(s)t(s)+\frac{1}{\varepsilon_2}\left(\frac{t^{'}(s)}{\tau(s)}\right)^{'}=0. \tag{3.16}$$

Corollary 3.2. Let $\alpha\ :\ I\subset R\rightarrow E_1^3$ be a unit speed timelike curve with non-zero and non-constant curvature in Minkowski 3-space. If every normal plane contains the point x_0 in R^3, i.e, if $\alpha(s)$ is a normal curve, then the curve is a spherical, i.e, $\frac{\tau(s)}{\kappa(s)}=\left(\frac{\kappa^{'}(s)}{\kappa^2(s)\tau(s)}\right)^{'}$.

Proof. From the equation (3.15), we obtain $\frac{\varepsilon_3}{\varepsilon_2}=1$. So we can say that α is a timelike curve. Also using the equation (3.16), we get

$$\varepsilon_3\tau(s)t(s)+\frac{1}{\varepsilon_2}\left(\frac{t^{'}(s)}{\tau(s)}\right)^{'}=0.$$

Then using the last equation, we obtain

$$\frac{\tau(s)}{\kappa(s)}=\frac{-1}{\varepsilon_1\varepsilon_2\varepsilon_3}\left(\frac{-\varepsilon_1\kappa^{'}(s)}{\kappa^2(s)\tau(s)}\right)^{'}.$$

This completes the proof.

Corollary 3.3. Let $\alpha\ :\ I\subset R\rightarrow E_1^3$ be a unit speed non-null curve with non-zero and non-constant curvature in Minkowski 3-space. If every osculating plane contains the point x_0 in R^3, i.e, if $\alpha(s)$ is a osculating curve, then the curve is a planar curve.

Proof. Since the osculating plane of $\alpha(s)$ is the perpendicular plane to $B(s)$, we have $\langle\alpha(s)-x_0,B(s)\rangle=0$. If we take the derivative of this expression,

$$\langle T(s),B(s)\rangle+\left\langle\alpha(s)-x_0,B^{'}(s)\right\rangle=0.$$

Then by substituting from the Frenet-Serret formula we have

$$\tau(s)\langle\alpha(s)-x_0,N(s)\rangle=0,$$

$\tau=0$ is obtained from the last equation.

4. Discussion and Conclusion

Curves theory has studied in Euclidean 3 −space for a long time. Rectifying, normal and osculating curves which are special curve types have been studied many authors in different spaces. As I mentioned in the abstract part, the difference of this work is to characterize these curves from another point of view in Minkowski 3-space.

References

[1] Bozkurt, Z., Gök, I., Okuyucu, O. Z. and Ekmekci, F.N. 2013. Characterizations of Rectifying, Normal and Osculating Curves in Three Dimensional Compact Lie Groups. Life Science Journal, 10(3), 819-823.

[2] Chen, B. Y. 2003. When does the position vector of a space curve always lie in its rectifying plane?. Amer. Math. Monthly, 110, 147-152.

[3] Do Carmo, M. 1976. Differential geometry of curves and surfaces. Prentice-Hall, Upper Saddle Riv. N.J.

[4] İlarslan, K., Nesovic, E., Petrovic-Torgasev, M. 2003. Some characterizations of rectifying curves in the Minkowski 3 −space. Novi. Sad. J. Math., 33(2), 23-32.

[5] İlarslan, K. 2005. Spacelike normal curves in Minkowski space. Turk. J. Math., 29, 53-63.

[6] İlarslan, K., Nesovic´, E. 2004. Timelike and null normal curves in Minkowski space E_1^3. Indian J. Pure Appl. Math., 35, 881-888.

[7] Logan, J. and Oh, Y. M. 2017. Characterization of Rectifying and Sphere Curves in R^3. American Journal of Undergraduate Research, 14(2), 91-94.

[8] O'Neill, B. 1983. Semi-Riemannian geometry with applications to relativity. Academic Press, New York.

[9] Öztekin, H. 2016. Normal and Rectifying Curves in Galilean Space G3: Proceedings of IAM, 5(1), 98-109.