Research Article GU J Sci 33(3): 737-749 (2020) DOI: 10.35378/gujs.590435

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Common Solution for Nonlinear Operators in Banach Spaces

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Highlights

- Construction of a hybrid inertial algorithm.
- Theoretical proof of the constructed algorithm.
- Numerical illustration of the applicability of the algorithm.

Article Info

Received: 10/07/2019
Accepted: 03/04/2020

Keywords

Bregman nonexpansive-type operator
Common element
Continuous monotone operator
Inertial component

Abstract

This paper formulates a hybrid approximation process involving inertial component and demonstrates a convergence results for it. The formulated scheme converges faster and finds a common solution for some nonlinear operators in Banach spaces. The method of our proof and results obtained is well involved and significant.

1. INTRODUCTION

Throughout, the Banach space X is taken to be real and reflexive with X^* as the dual. We let $\|.\|: X \to R$ represent the norm function. Let $d_h : dom h \times \text{int}(dom h) \to R^+$ represent a bifunctions induced by a convex function. Let $dom h = \{u \in X : h(u) < +\infty\}$ and $\text{int}(dom h)$ represent the domain and interior domain of a convex function, $h : X \to (-\infty, +\infty]$ respectively. The convex function h is Gâteaux differentiable at u if $\lim_{s \to 0^+} \dfrac{(h(u+sz) - h(u))}{s} = h^\circ(u,z)$ exists for any z in X. By this, $h^\circ(u,z) = \nabla h(u)$, as the gradient of h.

Let the convex function h be Gâteaux differentiable at u, then $d_h : dom h \times \text{int}(dom h) \to R^+$ defined by

$$d_h(z,u) = h(z) - h(u) - \langle \nabla h(u), z \rangle + \langle \nabla h(u), u \rangle, \tag{1}$$

is the Bregman function induce by h.

This function $d_h : dom h \times \text{int}(dom h) \to R^+$ defined by (1) has some nice properties like:

P1. The function $d_h(.,u)$ is convex.

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P2. $d_h(u,u)=0.$

P3. $d_h(z,u)>0.$

P4. $d_h(z,u)=d_h(z,v)+d_h(v,u)+\langle z-v,\nabla h(v)\rangle-\langle z-v,\nabla h(u)\rangle.$

P5. $d_h(u,v)+d_h(v,u)=\langle u-v,\nabla h(u)\rangle-\langle u-v,\nabla h(v)\rangle.$

P6. $d_h(u,v) \le \|u\|\|\nabla h(u)-\nabla h(v)\| + \|v\|\|\nabla h(u)-\nabla h(v)\|.$

Observe that P4 implies P5 and P6 if $u=z.$ For proof of (P1 – P3), see [1,2].

Let K represent a non-void, closed, convex subset of $\operatorname{int}(domh).$ Let $G:K\to K$ represent a map. $G:K\to K$ is nonexpansive if $\|Gu-Gz\|\le\|u-z\|,\ \forall u,z\in K;$ $G:K\to K$ is (quasi)-nonexpansive if $\|Gu-z^0\|\le\|u-z^0\|,$ and $Fix(G)=\{z^0\in K:Gu=u\}$ is the collection of fixed point of $G:K\to K.$ An element $u^*\in K$ is asymptotic fixed point of $G:K\to K$ when $\{u_n\}$ is contained in K and converging weakly to u so that $\|u_n-Gu_n\|=0.$ We denote the set by $Fi\hat{x}(G).$

A map $G:K\to \operatorname{int} domh,$ is Bregman relatively nonexpansive (BRNE) [3] if

$$d_h(z^0,Gu)\le d_h(z^0,u),\ \forall u\in K,\forall z^0\in Fix(G)$$
$$Fi\hat{x}(G)=Fix(G).$$

For a differentiable function $h:X\to R^+$ and for all $u\in X,$ [4,5] gives

$$d_h(u_0,u)=\min\{d_h(z,u):u\in X\}\Leftrightarrow\langle\nabla h(u),z-u_0\rangle-\langle\nabla h(u_0),z-u_0\rangle\le 0,\ \ \forall z\in K. \tag{2}$$

In addition, if $K\subset X,$ then for $u\in\operatorname{int} domh,$ we have a unique $u_0\in K$ such that the mapping $P_K^h:\operatorname{int} domh\to K$ which satisfy

$$d_h(u_0,u)=\min\{d_h(z,u):z\in K\} \tag{3}$$

is the Projection of $u\in\operatorname{int} domh$ onto the set $K\subset domh,$ where $P_K^h(u)=u_0.$ The Bregman Projection mapping in view of [4,5] satisfy:

$$d_h(z,P_K^h(u))+d_h(P_K^h(u),u)\le d_h(z,u),\ \ z\in K. \tag{4}$$

Given h a norm square with $u\in X,$ then we see that $\nabla h(u)=2Ju,$ where $J:X\to X^*$ is defined and (1) reduce to $\phi(z,u)=\|z\|^2-2\langle Ju,z\rangle+\|u\|^2$ known as the Lyapunov functional [6].

The function $h:X\to(-\infty,+\infty]$ is Legendre [7], if the following hold

(i) $\operatorname{int} domh$ is non-void, h is differentiable on $\operatorname{int} domh$ with $domh=\operatorname{int} domh,$

(ii) $\operatorname{int} domh^*$ is non-void, h^* is differentiable on $\operatorname{int} domh^*$ with $domh^*=\operatorname{int} domh^*.$

With $h:X\to(-\infty,+\infty]$ a Legendre function, and X reflexive, then ∇h is a bijection which satisfies $\nabla h=(\nabla h^*)^{-1},\ range\nabla h=domain\nabla h^*=\operatorname{int} domainh^*.$ If $h:X\to(-\infty,+\infty]$ is single-valued and X

is smooth and strictly convex, then $J = \nabla h.$ Given $h(u) = t^{-1} \| u \|^2,\ t \in (1, \infty)$, then we have a Legendre function [8-12].

The modulus of total convexity of h at $u \in \operatorname{int} dom h$ $W_h(u,.) : \operatorname{int} dom h \times R^+ \to R^+$ is defined by

$$W_h(u,s) = \inf\{d_h(z,u) : z \in dom h, \| z - u \| = s\}. \tag{5}$$

If $W_h(x,s)$ is positive, then $h : X \to (-\infty, +\infty]$ becomes totally convex at u for positive values of $s.$ For more information, see [7].

A mapping $a : K \to X^*$ is monotone if $\forall u, z \in K,$ we have

$$\langle au, u - z \rangle - \langle az, u - z \rangle \geq 0. \tag{6}$$

Note that a point $z^0 \in K$ such that

$$\langle az^0, z \rangle - \langle az^0, z^0 \rangle \geq 0 \ \ \forall z \in K, \tag{7}$$

solves a variational inequality problem. The collection of solution of (7) is represented by

$$VI(K,a) = \{z^0 \in K : \langle az^0, z \rangle - \langle az^0, z^0 \rangle \geq 0 \ \ \forall z \in K\}.$$

Suppose in addition $a : K \to X^*$ is continuous, we have $VI(K,a) = \{z^0 \in K : \langle az^0, z \rangle - \langle az^0, z^0 \rangle \geq 0 \ \ \forall z \in K\}$ is closed and convex.

Over the years, smooth convex minimization problem involving generalized nonexpansive and Bregman nonexpansive operators has attracted the interest of many researchers and authors seeking for existence of solutions. It is a fact that most published works on these operators has been the iterative approximation of (common) solution of operators. Furthermore, most of these results only focused on the strong convergence of the formulated schemes to the fixed point sets (see [13-15]). However, very few authors have paid attention to the speed or the rate of convergence of sequence of iterates of Bregman nonexpansive-type operators to their (common) fixed point sets when they exists. Thus, a two-step iterative method to increase the rate of convergence was used in the works of [16-18] and which is defined as

$$u_{n+1} = u_n + \beta_n (u_n - u_{n-1}) \tag{8}$$

for all non-negative integers $n,$ where $\beta_n \in (0,1).$ We note here that the inertial is represented by the component,
$\beta_n (u_n - u_{n-1})$.

Using Lyapunov functional, [14] formulated the hybrid method as given below:

$$\begin{cases} u_0 \in K_1 = K \\ \hat{y}^n = J^*\big(b_n J(u_n) + (1-b_n) J\, G J_{r_n} u_n\big), \\ C_n = \{u^* \in K : \phi(u^*, \hat{y}^n) \le \phi(u^*, u_n)\} \\ Q_n = \{u^* \in K : \langle u_n - u^*, Ju_0\rangle - \langle u_n - u^*, Ju_n\rangle \ge 0\}, \\ u_{n+1} = \Pi_{C_n \cap Q_n}(u_0), \end{cases} \tag{9}$$

where $G: K \to K$ is relatively nonexpansive map. $J_r = (J + rM)^{-1} J$ is a resolvent for maximal map $M: K \to X^*$ with r positive. They showed that their method converge strongly to mutual element of $Fix(G) \bigcap M^{-1}(0)$ nearest u_0.

In 2018, [16], formulated and studied the following methods generated by $\{u_n\}$ as follows: $u_0, u_1 \in X$ and

$$\begin{cases} K_0 = X \\ z_n = u_n - \alpha_n (u_{n-1} - u_n), \\ \hat{y}^n = J^*\big((1-\beta) J(z_n) + \beta\, J\, G z_n\big), \\ C_{n+1} = \{u^* \in K : \phi(u^*, \hat{y}^n) \le \phi(u^*, z_n)\}, \\ u_{n+1} = \Pi_{C_{n+1}}(u_0), \end{cases} \tag{10}$$

where $G: X \to X$ is relatively nonexpansive map expressed as $Gu = J^{-1}(\sum_{i=1}^{\infty} \eta_i (\beta_i JGx + (1-\beta_i) JG_i u))$. They showed that their method converge strongly to a mutual element of $Fix(G) = \bigcap_{i=1}^{\infty} Fix(G_i)$.

Both algorithms (9) and (10) were formulated in uniformly convex and smooth Banach spaces. We observe also that algorithm (9) combined the intersection of two half sets and at each iteration, it is used and taken as the next Projection which is not easily done in application. Secondly, it has no inertial component that could speed up the convergence of their algorithm. On the other hand, algorithm (10) has the inertial component.

Question. Can we formulate iterative scheme following hybrid method with inertial component without the intersection of two half sets? Can our algorithm converge to common element of our non-void set faster in reflexive and real Banach space?

Our motivation for study is the results of [14] and [16]. We aim to study an iterative scheme with inertial component for our operator. The formulated scheme converges faster and finds a common solution for some nonlinear operators in Banach spaces. The method of our proof and results obtained is well involved and significant.

2. MAIN RESULTS

Let K be a non-void, closed and convex subset of reflexive Banach space X. Let the function $h: X \to (-\infty, +\infty)$ represent bounded Legendre, uniform Fréchet differentiable, totally convex. Let the map $G: K \to K$ represent a Bregman relatively nonexpansive, $a: K \to X^*$ represent a continuous monotone map. We assume $F = Fix(G) \bigcap VI(K, a)$ to be non-void. For u element of X, define the mapping $T_r^a : X \to K$ as follows:

$$T_r^a x = \{z^0 \in K : \langle az^0, z\rangle r_n - \langle az^0, z^0\rangle r_n + \left\langle \nabla h(z^0) - \nabla h(u), z - z^0 \right\rangle \geq 0,\ \forall z \in K.$$

Set $x_0, x_1 \in K.$ Then define $\{x_n\}$ by the manner below:

$$\begin{cases} x_0 \in K_0 = K \\ z_n = \nabla h^*\left(\nabla h(x_n) + \alpha_n\left(\nabla h(x_n) - \nabla h(x_{n-1})\right)\right), \\ y_n = \nabla h^*\left((1-\eta)\nabla h(z_n) + \eta\, \nabla h(Gz_n)\right), \\ w_n = T_{r_n}^a y_n, \\ K_{n+1} = \left\{u \in K_n : d_h\left(u, w_n\right) \leq d_h\left(u, z_n\right)\right\}, \\ x_{n+1} = P_{K_{n+1}}^h\left(x_0\right), \end{cases} \tag{11}$$

where $\{r_n\} \subset (0,\infty),\ n \in N,\ \alpha_n \in (0,1),\ \eta \in (0,1).$

Lemma 2.1. The scheme (11) is well defined.

Proof.

First, we demonstrate that $F = Fix(G) \bigcap VI(K,a)$ is closed, convex. Note in [2], $Fix(G)$ is closed, convex. Also note in [13], $VI(a,\ K)$ is closed, convex. So $F = Fix(T) \bigcap VI(K,a)$ is closed, convex. Secondly, we demonstrate that K_n is closed, convex for each non-negative integer.

To realize this, from our setting in (11), K_n is closed. Moreover, since $d_h\left(u, w_n\right) \leq d_h\left(u, z_n\right)$ is equivalent of $\left\langle \nabla h(z_n) - \nabla h(w_n), u\right\rangle + \left\langle \nabla h(z_n) - \nabla h(w_n), w_n - z_n\right\rangle \leq h(w_n) - h(z_n),$

it follows that K_n is a half space and hence convex for each nonnegative integer.

In addition, we demonstrate that $F \subset K_n$ for each nonnegative integer. Clearly, from our setting, $F \subset K_0 = K.$ If $F \subset K_t$ for some $t > 0,$ then with $q \in F,$ and using P1 together with [13], we obtain

$$\begin{aligned} d_h(q, w_t) &= d_h(q, T_{r_n}^a y_t) \\ &\leq d_h(q, y_t). \end{aligned} \tag{12}$$

Furthermore,

$$\begin{aligned} d_h(q, y_t) &= d_h\left(q, \nabla h^*\left((1-\eta)\nabla h(z_t) + \eta\, \nabla h(Gz_t)\right)\right) \\ &\leq (1-\eta) d_h\left(q, z_t\right) + \eta\, d_h\left(q, Gz_t\right) \\ &\leq (1-\eta) d_h\left(q, z_t\right) + \eta\, d_h\left(q, z_t\right) \\ &= d_h\left(q, z_t\right). \end{aligned}$$

Thus,

$$d_h(q, y_t) \leq d_h\left(q, z_t\right). \tag{13}$$

Using (13) in (12) gives

$$d_h(p, w_h) \le d_h(p, z_h).$$

So $q \in K_{t+1}$ and $K_{t+1} \subset K_t$. This implies $F \subset K_n$. Thus, (11) become well defined

Lemma 2.2. Let K be a non-void, closed and convex subset of reflexive Banach space X. Let the function $h: X \to (-\infty, +\infty)$ represent bounded Legendre, uniform Fr$\acute{e}$chet differentiable, totally convex. Let the map $G: K \to K$ represent a Bregman relatively nonexpansive, $a: K \to X^*$ represent a continuous monotone map. We assume $F = Fix(G) \bigcap VI(K, a)$ to be non-void. Let $\{x_n\}$ be produced by (11). Then the following holds

(i) $\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0,$

(ii) $\lim_{n\to\infty} \| x_n - z_n \| = 0,$

(iii) $\lim_{n\to\infty} \| x_{n+1} - w_n \| = 0,$

(iv) $\lim_{n\to\infty} \| z_n - w_n \| = 0,$

(v) $\lim_{n\to\infty} \| w_n - y_n \| = 0,$

(vi) $\lim_{n\to\infty} \| x_n - y_n \| = 0,$

(vii) $\lim_{n\to\infty} \| z_n - Tz_n \| = 0.$

Proof.

Seeing that $x_n = P^h_{K_n}(x_0)$ with $x_{n+1} = P^h_{K_{n+1}}(x_0) \in K_{n+1} \subset K_n$, gives

$$\begin{aligned} &d_h(x_n, x_0) \le d_h(x_{n+1}, x_0) - d_h(x_{n+1}, x_n) \\ &d_h(x_{n+1}, x_0) \ge d_h(x_n, x_0). \end{aligned} \tag{14}$$

This shows that $\{d_h(x_n, x_0)\}$ is monotone nondecreasing sequence. Besides, from (4),

$$d_h(x_n, x_0) = d_h(P^h_{K_n}(x_0), x_0) \le d_h(q, x_0) - d_h(q, P^h_{K_n}(x_0)) \le d_h(q, x_0) \quad \forall n \in N \cup \{0\},\ q \in F.$$

$$\Rightarrow \quad d_h(x_n, x_0) \le d_h(q, x_0). \tag{15}$$

This demonstrates boundedness of $\{d_h(x_n, x_0)\}$. From [4], boundedness of $\{x_n\}$ hold. But (14) combined with (15), shows that $\lim_{n\to\infty} d_h(x_n, x_0)$ exist. Now wlog, let

$$\lim_{n\to\infty} d_h(x_n, x_0) = l. \tag{16}$$

In addition to (16) and (4), we get

$$\begin{aligned} d_h(x_{n+\mu}, x_n) &= d_h(x_{n+\mu}, P^h_{K_n}(x_0)) \\ &\le d_h(x_{n+\mu}, x_0) - d_h(x_n, x_0) \to 0 \text{ as } n \to \infty. \end{aligned}$$

So that

$$\lim_{n\to\infty} d_h(x_{n+\mu}, x_n) = 0,\ \mu > 0.$$

In particular,

$$\lim_{n\to\infty} d_h(x_{n+1}, x_n) = 0. \tag{17}$$

Therefore, we get from [7]

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{18}$$

This establishes (i).

Now, since ∇h is uniformly continuous, we get

$$\lim_{n\to\infty} \| \nabla h(x_{n+1}) - \nabla h(x_n) \| = 0. \tag{19}$$

Furthermore, from the definition of z_n, and together with (19), we obtain

$$\begin{aligned} \| \nabla h(x_n) - \nabla h(z_n) \| &= \| \nabla h(x_n) - \nabla h(x_n) - \alpha_n \nabla h(x_n - x_{n-1}) \| \\ &= \| \alpha_n \nabla h(x_{n-1} - x_n) \| \\ &\le \| \nabla h(x_{n-1} - x_n) \| \to 0 \qquad \text{when n} \to \infty. \\ &\Rightarrow \lim_{n\to\infty} \| \nabla h(x_n) - \nabla h(z_n) \| = 0. \end{aligned} \tag{20}$$

Invoking [4], we obtain

$$\lim_{n\to\infty} \| x_n - z_n \| = 0. \tag{21}$$

This establishes (ii).

Moreso, since $\{ z_n \}$ is bounded and using P6, (21) we obtain

$$\lim_{n\to\infty} d_h(x_n, z_n) = 0. \tag{22}$$

In addition, since $x_{n+1} \in K_{n+1} \subset K_n$, from the definition of the half space, we obtain

$$d_h(x_{n+1}, w_n) \le d_h(x_{n+1}, z_n). \tag{23}$$

Moreover, using P4, (17), (20), and (2), we get

$$\lim_{n\to\infty} d_h(x_{n+1}, z_n) = 0. \tag{24}$$

This implies that

$$\lim_{n\to\infty} d_h(x_{n+1}, w_n) = 0. \tag{25}$$

Thus, [7] suggest that

$$\lim_{n\to\infty}\| x_{n+1} - z_n \| = 0$$

and

$$\lim_{n\to\infty}\| x_{n+1} - w_n \| = 0. \tag{26}$$

This establishes (iii).

Combining (18) and (26), we get

$$\| x_n - w_n \| \to 0 \text{ as n} \to \infty.$$

Also, from (26) we get

$$\lim_{n\to\infty}\| x_n - w_n \| = 0. \tag{27}$$

and

$$\lim_{n\to\infty}\| z_n - w_n \| = 0. \tag{28}$$

This establishes (iv).

Next, using [13], this implies

$$d_h(y_n, w_n) = d_h(y_n, T^a_{r_n} y_n) \le d_h(u, w_n) - d_h(u, y_n)$$
$$\le d_h(u, y_n) - d_h(u, y_n) \to 0 \text{ as n} \to \infty.$$

$$\Rightarrow \lim_{n\to\infty} d_h(y_n, w_n) = 0.$$

Using [7] gives

$$\lim_{n\to\infty}\| w_n - y_n \| = 0. \tag{29}$$

This establishes (v).

Now, from the uniform continuity of $\nabla h,$ (29) becomes

$$\| \nabla h(w_n) - \nabla h(y_n) \| \to 0 \ as \ n \to \infty. \tag{30}$$

Thus using (29) with (27) gives

$$\lim_{n\to\infty}\| x_n - y_n \| = 0. \tag{31}$$

This establishes (vi).

Using (26) with (29) gives

$$\lim_{n\to\infty}\| x_{n+1} - y_n \| = 0.$$

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With this and P6 we get

$$d_h(x_{n+1}, y_n) \to 0 \ as \ n \to \infty.$$

Thus combining (29) and (28) we get

$$\lim_{n\to\infty}\| z_n - y_n \| = 0. \tag{32}$$

Since ∇h is uniformly continuous, we obtain

$$\|\nabla h(z_n)-\nabla h(y_n)\|\to 0 \ as \ n\to\infty. \tag{33}$$

Since, $y_n=\nabla h^*\left((1-\eta)\nabla h(z_n)+\eta\nabla h(Gz_n)\right)$, we obtain

$$\|\nabla h(z_n)-\nabla h(y_n)\|=\|\nabla h(z_n)-\nabla h(z_n)+\eta(\nabla h(Gz_n)-\nabla h(z_n))\|=\eta\|\nabla h(Gz_n)-\nabla h(z_n)\|.$$

Using (33) gives

$$\|\nabla h(Gz_n)-\nabla h(z_n)\|\to 0 \ as \ n\to\infty.$$

So that

$$\|z_n-Gz_n\|\to 0 \ as \ n\to\infty. \tag{34}$$

This establishes (vii).

Theorem 2.3. Let K be a non-void, closed and convex subset of reflexive Banach space X. Let the function $h:X\to(-\infty,+\infty)$ represent bounded Legendre, uniform Fr$\acute{e}$chet differentiable, totally convex. Let the map $G:K\to K$ represent a Bregman relatively nonexpansive, $a:K\to X^*$ represent a continuous monotone map. We assume $F=Fix(G)\bigcap VI(K,a)$ to be non-void. Let $\{x_n\}$ be produced by (11). Then $\{x_n\}$ converge strong to a point of $F=Fix(G)\bigcap VI(K,a)$.

Proof.

From Lemma 2.2, $\{x_n\}$ is bounded. Since X is reflexive, there exist $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges weakly to $\hat{u}$ as $i\to\infty.$ We now show that $\hat{u}=P_F^h(x_0).$ Setting $\hat{q}=P_F^h(x_0),$ then from (4), we get

$$d_h(\hat{u},x_0)\le d_h(u,x_0)\ \ \forall u\in F \text{ and } d_h(\hat{q},x_0)\le d_h(u,x_0)\ \ \forall u\in F, \tag{35}$$

in addition to (35), we get

$$\begin{aligned} d_h(\hat{u},\hat{q}) &\le d_h(\hat{u},x_0)-d_h(\hat{q},x_0)\\ &\le d_h(u,x_0)-d_h(u,x_0)\\ &=0. \end{aligned} \tag{36}$$

This implies

$$d_h(\hat{u},\hat{p})\le 0.$$

By the uniqueness of $P_F^h(x_0)$, $\hat{u}=\hat{p}.$ So, we have shown that $\hat{u}=P_F^h(x_0).$ Next, we show that $x_n\to\hat{u}=P_F^h(x_0)$ as $n\to\infty.$ To do this, recall from Lemma 2.1, that the half space K_n is closed and convex and $F\subset K_n$, implying that K_n is weakly closed and $\hat{u}\in K_n,\ \ \forall n\ge 0.$ Recall from definition that $x_{n_i}=P^f_{K_{n_i}}(x_0),$ so that $d_h(x_{n_i},x_0)\le d_h(\hat{u},x_0)$. Using the weakly lower semi-continuity of h on the convex set, we get

$$
\begin{aligned}
d_h(\hat{u}, x_0) &= h(\hat{u}) - h(x_0) - \left\langle \nabla h(x_0), \hat{u} - x_0 \right\rangle \\
&\leq \liminf_{i\to\infty} \left\{ h(x_{n_i}) - h(x_0) - \left\langle \nabla h(x_0), x_n - x_0 \right\rangle \right\} \\
&= \liminf_{i\to\infty} d_h(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} d_h(x_{n_i}, x_0) \\
&\leq d_h(\hat{u}, x_0).
\end{aligned}
$$

This implies

$$\lim_{i\to\infty} d_h(x_{n_i}, x_0) = d_h(\hat{u}, x_0). \tag{37}$$

This implies that $\lim_{i\to\infty} h(x_{n_i}) = h(\hat{u}).$ Since h is uniformly continuous, we get

$$\lim_{i\to\infty} x_{n_i} = \hat{u}.$$

Since $\{x_n\}$ is convergent, invoking Lemma 2.2 gives

$$x_n \to \hat{u} \ as \ n \to \infty. \tag{38}$$

Now from Lemma 2.2, $\{z_n\}$ is bounded implying there exists a subsequence $\{z_{n_i}\}$ such that z_{n_i} converges weakly to $\hat{u}$ as $i \to \infty.$ Applying condition (ix) in Lemma 2.2, we obtain $\lim_{i\to\infty} \| z_{n_i} - Gz_{n_i} \| = 0.$ Since our map is Bregman relatively nonexpansive, we have $\hat{u} \in F(G).$ Next, we show that $\hat{u} \in VI(K, a).$ From the definition of w_n, we get

$$\langle aw_n, z \rangle r_n - \langle aw_n, w_n \rangle r_n + \left\langle \nabla h(w_n) - \nabla h(z_n), z - w_n \right\rangle \geq 0, \ \forall z \in K,$$

using (39), the fact that w_n converges to $\hat{u}$ as $n \to \infty$, and the continuity of a, we have

$$\left\langle a\hat{u}, z \right\rangle - \langle a\hat{u}, \hat{u} \rangle \geq 0 \quad \forall z \in K.$$

Thus $\hat{u} \in VI(K, a).$ Therefore, $\hat{u} \in F = Fix(G) \bigcap VI(K, a).$ ■

Remark 2.4. Our result in particular, extends the mappings and results of [14] to a more general mapping corresponding to Bregman distance function in reflexive Banach space. Our scheme has the inertial term known to speed up convergence of sequences. Our scheme is applicable in Hilbert spaces when we consider $\nabla h = I$, the identity mapping and $X = H.$

3. Numerical Example

A direct application of Theorem 2.3 is in this section given to demonstrate convergence of sequences generated by it.

Let $X = R,$ $K = [0, 4],$ $h(x) = x^2,$ $\nabla h(x) = 2x.,$ $h^*(u^*) = \sup\{\langle u^*, x \rangle - h(x)\} = \frac{1}{4}(u^*)^2,$

$\nabla h^*(u^*) = \frac{1}{2} u^*.$ $\langle az, y - z \rangle = zy - z^2.$

Let $G : [0, 4] \to [0, \infty)$ be defined by

$$G(x) = \begin{cases} 0 \ if \ x \neq 4 \\ 2 \ if \ x = 4. \end{cases}$$

It is clear that $Fix(G) = \{0\}$ since for $x \neq 4$, $G(x) = x \Rightarrow 0 = x.$ Thus, for any $x \neq 4, x = 0.$ Again, for $x = 4$, $G(x) = x \Rightarrow 2 = x.$ Thus for $x = 4$, $2 = 4$, which is not possible. So $Fix(G) = \{0\}$. Next, we observe that x_n converges weakly to 0, hence $x_n - Gx_n \to 0$ and $Fi\hat{x}(G) = Fix(G) = \{0\}$ [15]. From the definition of Bregman relatively nonexpansive mapping, one can easily demonstrate that

$$d_h(0, Gx) \leq d_h(0, x) \tag{39}$$

Infact

$$\begin{aligned} d_h(0, Gx) &= h(0) - h(Gx) - \langle \nabla h(Gx), 0 - Gx \rangle \\ &= 0 - 4 - \langle 0, 0 - 2 \rangle \\ &= -4 \end{aligned} \tag{40}$$

$$\begin{aligned} d_h(0, x) &= h(0) - h(x) - \langle \nabla h(x), 0 - x \rangle \\ &= 0 - x^2 - \langle 2x, 0 - x \rangle \\ &= -x^2 + 2x^2 \\ &= x^2. \end{aligned} \tag{41}$$

Thus, using (40) and (41) we have

$$d_h(0, Gx) \leq d_h(0, x), \quad for\,all\, x \in [0, 4].$$

Furthermore, setting $x_0 = \frac{1}{2}$, $x_1 = 1$, $\alpha_n = \frac{n+1}{4n}$, $\eta = \frac{1}{2}$, $r = 1$, we get

$$z_n = \nabla h^* \left(\nabla h(x_n) + \frac{n+1}{4n} \left(\nabla h(x_n) - \nabla h(x_{n-1}) \right) \right)$$

$$y_n = \nabla h^* \left((1-\eta) \nabla h(z_n) + \eta \nabla h(Gz_n) \right) = \frac{1}{2} z_n,$$

$$w_n = T_{r_n}^a y_n = \frac{1}{3} z_n.$$

$$K_{n+1} = \{ u \in K_n : d_h(u, w_n) \leq d_h(u, z_n) \} = \left\{ u \in K_n : u \leq -\frac{13}{24} z_n \right\}.$$

Therefore, we have our scheme (11) now simplified thus:

$$\begin{cases} x_0 \in [0,4], \ \ Chosen \ \ arbitrarily, \\ z_n = \nabla h^*\left(\nabla h(x_n) + \dfrac{n+1}{4n}\left(\nabla h(x_n) - \nabla h(x_{n-1})\right)\right), \\ y_n = \dfrac{1}{2} z_n, \\ w_n = \dfrac{1}{3} z_n, \\ K_{n+1} = \left\{ u \in K_n : u \le -\dfrac{13}{24} z_n \right\}, \\ x_{n+1} = P^h_{K_{n+1}}\left(x_0\right) = u, \quad \forall n \ge 1. \end{cases}$$

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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