



## Advances in the Theory of Nonlinear Analysis and its Applications

ISSN: 2587-2648

Peer-Reviewed Scientific Journal

# Nanofluid with nonlinear Rosseland thermal radiation and mixed convection

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### Abstract

Two dimensional flow of mixed convection nanofluid on horizontal plate with the effect of nonlinear Rosseland thermal radiation has been investigated. Mathematical model of the problem is based on partial differential equations and optimal homotopy analysis method (OHAM) is applied to sort out solutions. Moreover, comprehensive study of influence of emerging parameters is carried out via graphical interpretation and tables.

**Keywords:** Nanofluid Mixed Convection Nonlinear Rosseland Thermal Radiation

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### 1. Introduction

Nanofluids have a lot of applications in medical industry, engine cooling, detergency, pharmaceutical processes, heat exchanger and space technology. Nanofluids contain nanometer sized particles called nanoparticles. Nanofluids consist of base fluid which is usually water or oil with nanoparticles like metals, oxides, carbides and carbon. Some commonly used nanofluids are  $TiO_2$  (Titanium dioxide) in water,  $CuO$  (Copper oxide) in water,  $Al_2O_3$  (Aluminium oxide) in water,  $ZnO$  (Zinc oxide) in ethylene glycol. Choi [1] established the concept of nanofluids. Nanotechnology gain attention in the heat transfer process due to its characteristics of thermal conductivity.

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Received June 14, 2020; Accepted: May 29, 2021; Online: June 1, 2021.

Table 1: Nomenclature

Symbol	Quantity
$U_w$	Plate velocity
$T$	Fluid temperature
$T_w$	Plate temperature
$T_\infty$	Ambient temperature
$C$	Nanoparticle volume fraction
$C_w$	Nanoparticle volume fraction at plate
$C_\infty$	Nanoparticle volume fraction away from the plate
$g$	Gravitational acceleration
$\lambda$	Local buoyancy parameter
$Nt$	Thermophoresis parameter
$Nb$	Parameter of Brownian motion
$Nr$	Radiation paramter
$N_1$	Concentration buoyancy parameter
$\beta_T$	Coefficient of thermal expansion
$\beta_C$	Coefficient concentration expansion
$D_T$	Coefficient of Thermophoretic diffusion
$\lambda$	Retardation time
$Pr$	Prandtl number
$Sc$	Schmidt number
$D_B$	Coefficient of Brownian diffusion
$\theta_r$	Temperature parameter
$a_R$	Rosseland mean spectral absorption coefficient
$\sigma_{SB}$	Stefan-Boltzmann constant
$C_p$	Specific heat
$Gr_x$	Local Grashof number
$u, v$	Velocity along $x$ -axis and $y$ -axis

Mixed convection is a phenomenon occurred due to free convection and forced convection. Flow problems having mixed convection has great importance in applied perspective especially in industrial, technical processes. Pal and Mandal studied [2] three types of nanofluids along with the thermal radiation and mixed convection. Hayat et al. [3] investigated coupled stress nanofluid flow with nonlinear thermal radiation past a stretching surface. Thermal radiation effect in fluid flow problems with mixed convection and convective condition are discussed in [4]-[6].

Nonlinear thermal radiation effect has great importance in engineering, nuclear reactors, missiles, and satellites. Hayat et al. [7] studied nonlinear thermal radiation effect in viscoelastic nanofluid. Shehzad et al. [8] studied thermophoresis effect and brownian motion in Jeffrey nanofluid with thermal radiation. Pantokratoras [9] investigated natural convection on isothermal plate with the impact of linear or nonlinear Rosseland radiation convection along with radiation parameter. Work has been done in this area by researchers [10]-[14]. Farooq et al. investigated heat transfer phenomena in viscoelastic nanofluid with nonlinear radiative effects [15]. Hayat et al. [16] analyze heat transfer in nanofluid with nonlinear thermal radiation and inclined magnetic field. Many researchers pay attention towards nonlinear thermal radiation [17]-[24]. Pantokratoras and Fang [25] studies Blasius flow in the presence of nonlinear Rosseland thermal radiation. Some important phenomena regarding nonlinear thermal radiations considered by researchers [26]-[29].

In this article, nanofluid with nonlinear Rosseland thermal radiation and mixed convection has been investigated. Mathematical model involve partial differential equations. OHAM is used to investigate solutions. In addition, results are highlighted by tables and graphs.

## 2. Mathematical model

Let nanofluid flow with nonlinear Rosseland thermal radiation and mixed convection moving in the direction of a horizontal plate with components of velocity  $u$  and  $v$ . Velocity of the plate is  $U_w$  (see figure 1). Mathematical model is given below:

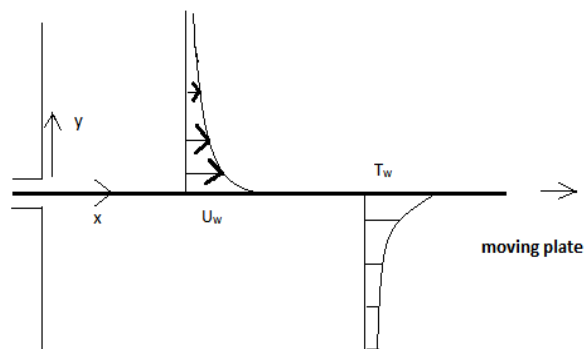


Figure 1: Model of the problem

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$uu_x + vv_y = \nu u_{yy} + g[\beta_T(T - T_\infty) + \beta_C(C - C_\infty)], \tag{2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \kappa D_B \frac{\partial T}{\partial y} \frac{\partial C}{\partial y} + \kappa \frac{D_\tau}{T_\infty} \left( \frac{\partial T}{\partial y} \right)^2 + \frac{4\sigma_{SB}}{3\rho C_p a_R} \left( \frac{\partial^2 T}{\partial y^2} \right), \tag{3}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \left( \frac{\partial^2 T}{\partial y^2} \right). \tag{4}$$

The boundary conditions are considered as:

$$u = U_w, \quad v = 0, \quad T = T_w, \quad C = C_w, \quad \text{at } y = 0, \tag{5}$$

$$u \rightarrow 0, \quad \tau \rightarrow T_\infty, \quad C \rightarrow C_\infty, \quad \text{as } y \rightarrow \infty. \tag{6}$$

Introducing similarity transformations:

$$u = U_w g'(\eta), \quad v = \frac{1}{2} \sqrt{\frac{U_w \nu}{x}} [\eta g'(\eta) - g(\eta)], \tag{7}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \sqrt{\frac{U_w}{x\nu}} y, \quad \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \tag{8}$$

By putting values of  $u$  and  $v$  in equation (1), it satisfied. Moreover substituting equations (7) and (8) into equations (2), (3) and (4), we get

$$g''' + \frac{1}{2} g g'' + \lambda(\theta + N_1 \phi) = 0, \tag{9}$$

$m$	$c_0^g$	$c_0^\theta$	$c_0^\phi$	$\varepsilon_t^m$	CPU Time
2.0	-1.27	-0.51	-0.49	$6.3 \times 10^{-3}$	19.277
4.0	-1.28	-0.57	-0.59	$3.2 \times 10^{-3}$	182.615
6.0	-1.03	-0.43	-1.05	$3.5 \times 10^{-4}$	1348.11

Table 2: Values of errors according to Optimal convergence control parameters via **BVPh2.0**

$$Nr\theta'' + NbPrNr\theta'\phi' + NtPrNr(\theta')^2 + \frac{1}{2}NrPr g\theta' + \frac{1}{3} \frac{[(\theta(\theta_r - 1) + 1)^4]''}{\theta_r - 1} = 0, \tag{10}$$

$$\phi'' + \frac{1}{2}Scg\phi' + \frac{Nt}{Nb}\theta'' = 0, \tag{11}$$

$$\begin{aligned} \phi(0) &= 1, & \phi(\infty) &= 0, \\ \theta(0) &= 1, & \theta(\infty) &= 0, \\ g(0) &= 0, & g'(0) &= 1, & g'(\infty) &= 0. \end{aligned} \tag{12}$$

Dimensionless numbers with parameters are given below:

$$\begin{aligned} Nt &= \frac{D_T(T_w - T_\infty)\kappa}{\nu T_\infty}, & Nb &= \frac{\kappa D_B(C_w - C_\infty)}{\nu}, & \alpha &= \frac{k}{\rho C_p}, \\ Sc &= \frac{\nu}{D_B}, & \lambda &= \frac{Gr_x}{Re_x^2}, & \theta_r &= \frac{T_w}{T_\infty}, & Nr &= \frac{ka_R}{4\sigma_{SB}T_\infty^3}, & Re &= \frac{U_w x}{\nu}, \\ Gr_x &= \frac{(T_f - T_\infty)x^3 g\beta_T}{\nu^2}, & N_1 &= \frac{(C_w - C_\infty)\beta_C}{(T_f - T_\infty)\beta_T}. \end{aligned} \tag{13}$$

Local Sherwood number is given below

$$Sh/Re_x^{\frac{1}{2}} = -\phi'(0), \tag{14}$$

and local Reynold is given by

$$Re_x = \frac{xU_w}{\nu}. \tag{15}$$

Assume initial approximations are

$$\begin{aligned} \phi_0(\eta) &= e^{-\eta}, \\ \theta_0(\eta) &= e^{-\eta}, \\ g_0(\eta) &= 1 - e^{-\eta}. \end{aligned} \tag{16}$$

Let auxiliary operators are

$$\begin{aligned} L_\phi &= \phi'' - \phi, \\ L_\theta &= \theta'' - \theta, \\ L_g &= g''' - g'. \end{aligned} \tag{17}$$

$m$	$\varepsilon_m^g$	$\varepsilon_m^\theta$	$\varepsilon_m^\phi$	CPU Time
6.0	$8.72 \times 10^{-6}$	$1.08 \times 10^{-4}$	$2.41 \times 10^{-4}$	12.7061
12.0	$1.68 \times 10^{-7}$	$2.82 \times 10^{-6}$	$3.79 \times 10^{-5}$	74.1165
18.0	$5.75 \times 10^{-8}$	$2.30 \times 10^{-6}$	$7.26 \times 10^{-5}$	270.534

Table 3: Error analysis from Table 2 at 6<sup>th</sup> iteration

### 3. Convergence control parameters

Convergence of solution can be control in homotopy analysis method by using different parameters denoted by  $c_0^g$ ,  $c_0^\theta$  and  $c_0^\phi$ . Values of these parameters can be obtained by minimizing error. BVPh2.0 is applied in order to get minimum error. Three arrays are selected. First array is selected at 2<sup>nd</sup> iteration, second array is selected at 4<sup>th</sup> iteration and third array is selected at 6<sup>th</sup> iteration. Table 2 shows error analysis at 6<sup>th</sup> iteration.

### 4. Results and discussion

Analysis of graphs for different parameters are examined in this section. Figure 2 depicts that there is an increase in velocity as  $\lambda$  increases. Thermal buoyancy force enhances when  $\lambda$  increases due to which velocity is enhanced. Effect of  $Nb$  on  $\theta(\eta)$  and  $\phi(\eta)$  presents in figures 3 and 4. By increasing  $Nb$ , temperature increases on the other hand concentration decreases. Figure 5 demonstrated that an increase in  $Nt$ , enhances temperature. Thermal conductivity enhances as  $Nt$  increases due to which temperature increases. Figure 6 displayed that  $\phi(\eta)$  increases as  $Nt$  increases. There is reduction in  $\theta(\eta)$  with the increasing value of  $Nr$  (see figure 7). Physically it is because of production of heat in moving fluid which is generated to increase in radiation as a result temperature raises. Figure 8 interprets the influence of  $\phi(\eta)$  for  $Sc$ . As  $Sc$  increases there is a decrease in concentration. Physically by increasing Schmidt number there is mass diffusivity become less and hence  $\phi(\eta)$  decreases. Figure 9 shows that temperature decreases as  $N_1$  increases. As  $\theta_r$  increases there is an increase in temperature (see figure 10). Further, Table 2 shows values of parameters which are responsible for convergence. Table 3 depicts error for 6th iteration. Table 4 presents values of Sherwood number corresponding to the parameters.

### 5. Conclusion

Nanofluid is considered over horizontal moving plate under the influence of nonlinear Rosseland thermal radiation with mixed convection. Fundamental observations are given below.

- $Nt$  and  $Nb$  have the same and opposite effect on  $\phi(\eta)$  and  $\theta(\eta)$  respectively.
- Increasing value of  $Nr$  accelerates the  $\theta_r$ .
- Enhancement in  $Sc$  leads to increase in  $\phi(\eta)$ .

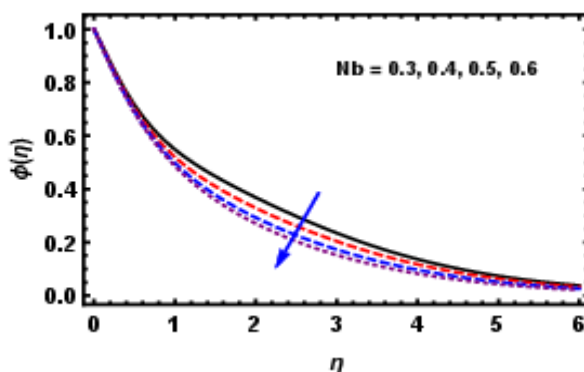


Figure 4: Influence of  $Nb$  on  $\phi(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Sc=\theta_r=1.5$  and  $\lambda=N_1=0.1$

$\lambda$	$N_1$	$Sc$	$\theta_r$	$Nt$	$Nr$	$Sh/Re_x^{\frac{1}{2}}$
0.2						0.98640
0.4						0.98833
0.6						0.99025
	0.2					0.98553
	0.4					0.98570
	0.6					0.98588
		0.1				0.85950
		0.3				0.87708
		0.5				0.89480
			0.1			0.79807
			0.3			0.78990
			0.5			0.79765
				0.0		0.62183
				0.2		0.67277
				0.4		0.73461
					0.0	0.43759
					0.2	0.51064
					0.4	0.58368

Table 4: Local Sherwood numbers for existing parameters

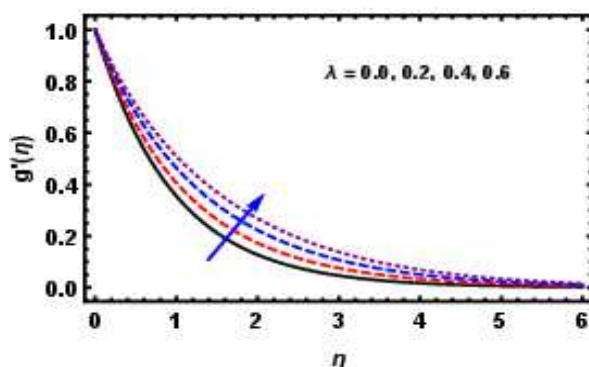


Figure 2: Influence of  $\lambda$  on  $g'(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Nb=Sc=\theta_r=1.5$  and  $N_1=0.1$

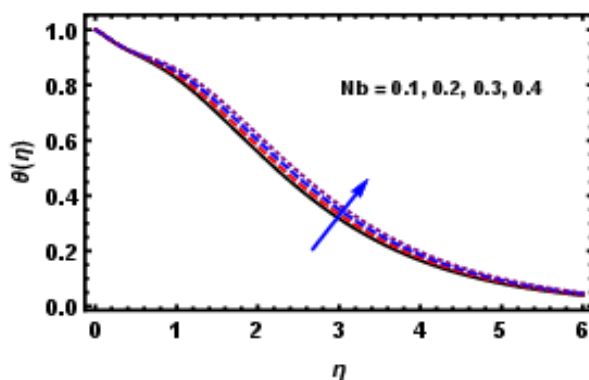


Figure 3: Influence of  $Nb$  on  $\theta(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Sc=\theta_r=1.5$  and  $\lambda=N_1=0.1$

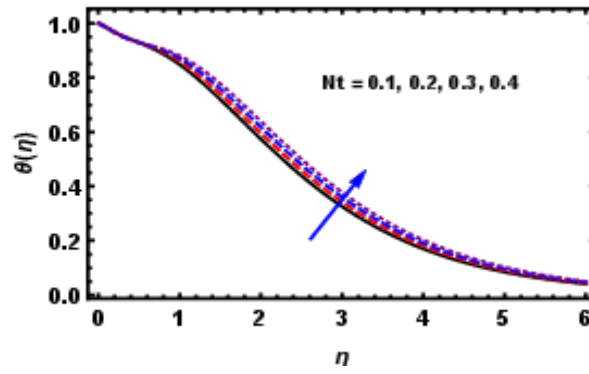


Figure 5: Influence of  $Nt$  on  $\theta(\eta)$  with  $Pr=1.0$ ,  $Nr=Nb=Sc=\theta_r=1.5$  and  $\lambda=N_1=0.1$

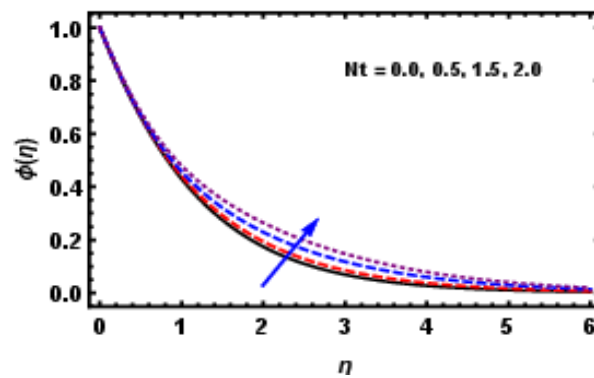


Figure 6: Influence of  $Nt$  on  $\phi(\eta)$  with  $Pr=1.0$ ,  $Nr=Nb=Sc=\theta_r=1.5$  and  $\lambda=N_1=0.1$

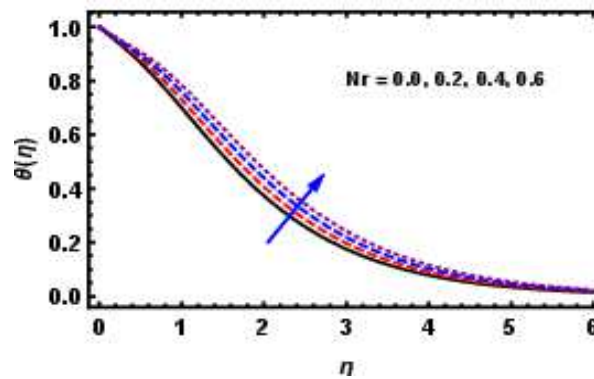


Figure 7: Influence of  $Nr$  on  $\theta(\eta)$  with  $Nt=Pr=1.0$ ,  $Nb=Sc=\theta_r=1.5$  and  $\lambda=N_1=0.1$

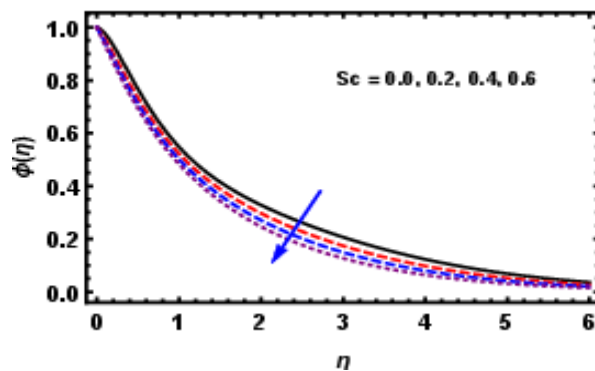


Figure 8: Influence of  $Sc$  on  $\phi(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Nb=\theta_r=1.5$  and  $\lambda=N_1=0.1$

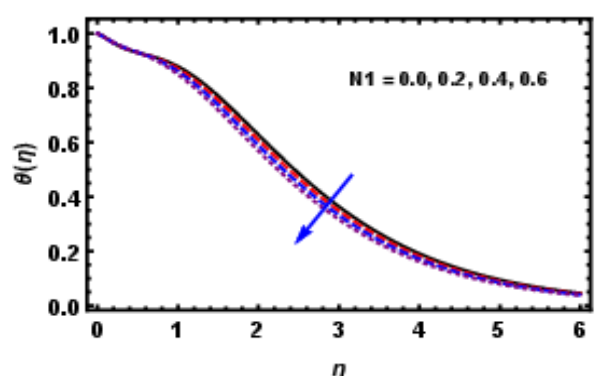


Figure 9: Influence of  $N_1$  on  $\theta(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Nb=Sc=\theta_r=1.5$  and  $\lambda=0.1$

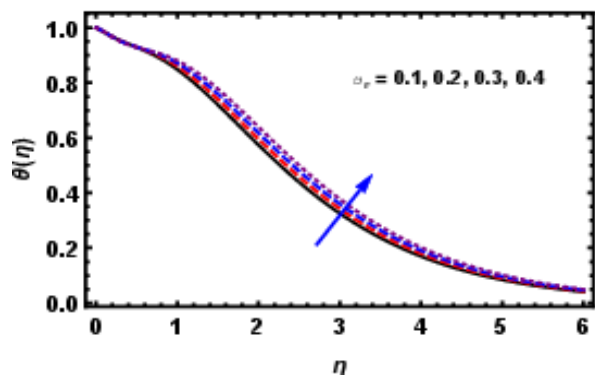


Figure 10: Influence of  $\theta_r$  on  $\theta(\eta)$  with  $Nt=Pr=1.0$ ,  $Nr=Nb=Sc=1.5$  and  $\lambda=N_1=0.1$

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