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Central Automorphisms of Semidirect Product of p-Groups

Özge ÖZTEKİN

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Özet

Bu çalışmada, p asal sayı olmak üzere $\mathbb{Z}_{p^2}$ ile $\mathbb{Z}_p$,nin yarı-direkt çarpımının merkezi otomorfizmlerinin formu belirlenmiştir..

Anahtar Kelimeler: Merkezi otomorfizm, yarı- direkt çarpım, p-grup.

Abstract

In this paper, we determine the form of central automorphisms of semi-direct product of $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p$, where p is odd prime number.

Keywords: Central automorphism, semi-direct product, p-group.

Introduction

A finite cyclic group of order n will be denoted Z_n. If the elements of group are integers we denote Z_n by $\mathbb{Z}_n$ and use additive notation. We know that any group of order p, where p is a prime is isomorphic to the cyclic group $\mathbb{Z}_p$. Generally, the term p-group is used for finite p-groups.

Let $\mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p$ be the semi-direct product of $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p$ with respect to φ, where φ is homomorphism from $\mathbb{Z}_p$ to automorphism group of $\mathbb{Z}_{p^2}$ and p is odd prime number. The center of

Gaziantep University, Faculty of Science, Department of Mathematics. Gaziantep, Türkiye, ozgeoztekin@gantep.edu.tr

a group $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$, denoted by $C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$ is the subgroup of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ of largest order that commutes with every element in $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$. By $Aut(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$ we denote the group of all automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$. An automorphism θ of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ is called central if $g^{-1}\theta(g) \in C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$, for all $g \in \mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$. The set of all central automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$, denoted by $Aut_C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$ and it is normal subgroup of $Aut(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$.

In literature, there are some well-known results about central automorphisms of finite groups [1,2,3]. First, Adney and Yen [1] studied the central automorphisms of p-groups and they proved that if G is a purely non-abelian finite group, then there exists a bijection between $Aut_C(G)$ and $Hom(G/G', Z(G))$.

The form of automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ was given by Stahl in [4]. But the form of central automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ hasn't given yet. In this paper we give the form of such automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$.

1. Preliminaries

Definition 1.1. Let H and K be non-trivial finite groups and a group homomorphism $\theta:K\rightarrow Aut(H)$, we can construct a new group $H \rtimes_\theta K$, called the semidirect product of H and K with respect to θ, defined as follows:

i) As a set, $H \rtimes_\theta K$ is the cartesian product H and K.

ii) Multiplication of elements $H \rtimes_\theta K$ is determined by the homomorphism θ. The operation is

$$*: (H \times K) \times (H \times K) \rightarrow H \rtimes_\theta K$$

defined by

$$(h_1, k_1) * (h_2, k_2) = (h_1\, \theta_{k1}(h_2), k_1 k_2)$$

for $h_1,h_2 \in H$ and $k_1,k_2 \in K$.

Theorem 1.2. If H, K and θ are as in the above definition then $H \rtimes_\theta K$ is a group of order $|G|=|H||K|$.

Theorem 1.3. [4] Let $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$, where φ is the unique homomorphism from $\mathbb{Z}_p$ to $Aut(\mathbb{Z}_{p^2})$ and it is determined by $\varphi:\mathbb{Z}_p\rightarrow Aut(\mathbb{Z}_{p^2})$ such that $\varphi(a)=1+pa$.Therefore, the operation $*$ of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ is determined as:

For all $(a,b),(c,d) \in \mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$

$$\begin{aligned}(a,b)*(c,d) &= (a+\varphi(b)c,b+d)\\ &= (a+(1+pb)c,b+d)\end{aligned}$$

Theorem 1.4. [4] Let $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$, where p is prime number. Any automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ has the form

$$\{a \to a^i b^j \ \ i \in \mathbb{Z}_{p^2} \ \ i \neq 0(\mathrm{mod} p),\ b \to a^{pm} b^l \ \ m,j,l \in \mathbb{Z}_p \ \ l \neq 0(\mathrm{mod} p)\}$$

Theorem 1.5. $|Aut(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)| = p^3(p-1)$.

2. Main Results

Let $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$, where p is odd prime number. Two generators of the group $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ are $a=(1,0)$ and $b=(0,1)$. First we find the center of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ to determine the central automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ by using this generators.

Theorem 2.1. The center $C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$ of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ is $C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)=\{(1\cdot p,0),(2\cdot p,0),...,(p\cdot p,0)\}$.

Proof. If $(a,b)\in C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$ then for every $(c,d)\in(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)$,

$$(a,b)*(c,d)=(c,d)*(a,b).$$

by using operation of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ we get

$$(a+\varphi(b)c,b+d) = (c+\varphi(d)a,b+d)$$

$$(a+(1+pb)c,b+d) = (c+(1+pd)a,b+d).$$

If we apply this equation on generators, we get;

If $(c,d)=(1,0)$ then,

$$(a+(1+pb)1,b+0) = (1+(1+p.0)a,b+0)$$

We get $b\equiv 0(\mathrm{mod} p)$ and we know that $b\in\mathbb{Z}_p$ therefore $b=0$.

If $(c,d)=(0,1)$ then,

$$(a+(1+pb)0,b+1) = (0+(1+p1)a,b+1)$$

We get $pa\equiv 0(\mathrm{mod} p)$ and we know that $a\in \mathbb{Z}_{p^2}$ therefore $a \in \{p,2p....,pp\}$.

So we have

$C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)=\{(a,0)|a\in\mathbb{Z}_{p^2} \text{ and } p|a\}$

Corollary. $|C(\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p)|=p$.

Now we can give the main theorem of this paper.

Theorem 2.2. Any central automorphisms of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$ has the form

$$\{a \to a^{kp+1}\ ,\ b \to b\ \}$$

where $k\in\mathbb{N}$ and p is odd prime number.

Proof. Let θ be an automorphism of $\mathbb{Z}_{p^2} \rtimes_\varphi \mathbb{Z}_p$. So from Theorem 1.4;

$$\theta(a,b)=(a^i b^j, a^{pm} b^l) \qquad (1)$$

for all $(a,b) \in \mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p$, where $i \in \mathbb{Z}_{p^2}$, $j,m,l \in \mathbb{Z}_p$ and $i,l \not\equiv 0 (mod p)$. If $\theta \in Aut_C(\mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p)$ then for all $g=(a,b) \in \mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p$, θ satisfy $g^{-1} * \theta(g) \in C(\mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p)$, where g^{-1} is the inverse of g. If $g=(a,b) \in \mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p$ then $g^{-1}=(bpa,p-b)$. By using the operation $*$ rule we get

$$g^{-1} * \theta(g) = (bpa-a,p-b) * (ai+bj,bl)$$
$$=(bpa-a+ \varphi(p-b)(ai+bj),(p-b)+bl)$$
$$=(bpa-a+ (1+p(p-b))(ai+bj),(p-b)+bl)$$

For $(bpa-a+ (1+p(p-b))(ai+bj),(p-b)+bl) \in C(\mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p)$

$$(bpa-a+ (1+p(p-b))(ai+bj),p-b+bl) =(k \cdot p,0)$$

where $k \in \{1,2,...,p\}$. If we apply this equation on generators, we get;

If (a,b)=(1,0) then,

$$(-1+ (1+p(p))(i),p) =(k \cdot p,0)$$
$$(-1+(i),p) =(k \cdot p,0)$$

Therefore $i=kp+1$.

If (a,b)=(0,1) then,

$$((1+p(p-1))(j),p-1+l) =(k \cdot p,0).$$

$j=kp$ and we know that $j \in \mathbb{Z}_p$ therefore $j=0$. So the conditions $i=1$ and $j=0$ satisfy this equation for all g.

For the other values of l, i and j the θ automorphism is not central. (In [5], the other conditions are investigated for case p=3). We put this conditions at (1) we get the general form of central automorphisms as:

$$\theta(a,b)=(a^{kp+1},b).$$

Corollary. $|Aut_C(\mathbb{Z}_{p^2} \rtimes_{\varphi} \mathbb{Z}_p)|=p$.

REFERENCES

A. R. Jamali and H. Mousavi, On the central automorphism group of finite p-groups, Algebra Coolog. 9(1) (2002), 7-14.

G. Saeden Stahl, J. Laine, G. Behm, On p-groups of low power order, Master Thesis, KTH, (2010).

J. E. Adney, T.Yen, Automorphisms of p-groups, Illinois J. Math. 9(1965), 137-143.

M. J. Curran and D. J . McCaughan, Central automorphisms of finite groups, Bull. Austral. Math. Soc. 34(2), (1986), 191-198.

Ö. Öztekin and Z. Gürbüz, Central Automorphism Groups For Semidirect Product of p-Groups The Eurasia Proceedings of Science, Technology, Engineering & Mathematics (EPSTEM), 2, (2018), 288-289.