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# Some Structures on Pythagorean Fuzzy Topological Spaces

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**Abstract** — In this paper, we introduced some operations such as Pythagorean fuzzy interior, Pythagorean fuzzy closure, Pythagorean fuzzy boundary, Pythagorean fuzzy basic on Pythagorean fuzzy topological spaces. Also, the notions of Pythagorean fuzzy open (closed) functions and Pythagorean fuzzy homeomorphism are introduced and their basic properties are investigated.

***Keywords*** — *Pythagorean fuzzy topological spaces, Pythagorean fuzzy interior, Pythagorean fuzzy closure, Pythagorean fuzzy boundary, Pythagorean fuzzy open (closed) functions, Pythagorean fuzzy homeomorphism.*

## 1. Introduction

In 1965, to dispose uncertain or vaguee information in decision making, fuzzy set theory was first introduced by Zadeh [1]. Fuzzy set theory was characterized by a membership function which assigns to each target a membership value ranging between 0 and 1. Further, Chang [2] introduced the fuzzy topological spaces and studied some basic notions of topology such as open set, closed set and continuity. Later, Lowen [3, 4] also made different studies on fuzzy topological spaces. Intuitionistic fuzzy set (IFS), initially proposed by Atanassov [5], is incorporated the degree of non-membership $\gamma$ into the analysis along with the membership degree $\mu$ in such a way that $\mu+\gamma \leq 1$. Coker [6] introduced the concept of intuitionistic fuzzy topological spaces and studied some notions such as continuity and compactness. Then, different studies were carried out on intuitionistic fuzzy topological spaces [7–9]. Although intuitionistic fuzzy set theory is popular, in some pactical decision-making processes the sum of degree of membership and the degree of non-membership, in which an alternative that meets the criteria of expertise is given, can be larger than 1; but their square sum is 1 or less.
Yager developed pythagorean fuzzy set (PFS) [10] characterized by a membership degree and non-membership degree which satisfies the condition that the square sum of its membership and non-membership degree is less than or equal to 1. Obviously, PFS is more effective than IFS. Yager [11] showed this situation with an example. An expert gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his support against membership is $\frac{1}{2}$. Since the sum of the two values is bigger than 1. They are not avaliable for IFS. But they are avaliable for PFS since $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \leq 1$. The PFS also has been studied from different perspectives such as in decision-making technologies [11–14], aggregation operator [15–18], information measure [19, 20], the extensions of PFS [21–24] and basic properties [10, 25, 26]. Besides, In 2019, Olgun et al. [27] introduce pythagorean fuzzy topological spaces and studied some properties. After that, In 2020, Naeem et. al. studied Pythagorean m-polar Fuzzy Topology with TOPSIS Approach in Exploring Most Effectual Method for Curing from COVID-19 [28].

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In this study, we investigated some basic notions of pythagorean fuzzy topological spaces such as pythagorean fuzzy interior, pythagorean fuzzy closure, pythagorean fuzzy boundary and pythagorean fuzzy basic. Finally, we also defined pythagorean fuzzy open (closed) function and pythagorean fuzzy homeomorphism.

## 2. Preliminaries

In this section, we will give some preliminary information for the present study.

**Definition 2.1.** [1] Let $X$ be an universe. A fuzzy set ($FS$ for short) $A$ in $X$, $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A : X \to [0,1]$ is the membership function of the fuzzy set $A$; $\mu_A(x) \in [0,1]$ is the membership of $x \in X$ in $A$.

**Definition 2.2.** [5] Let $X$ be a non-empty fixed set. An intuitionistic fuzzy set ($IFS$ for short) $A$ in $X$ is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

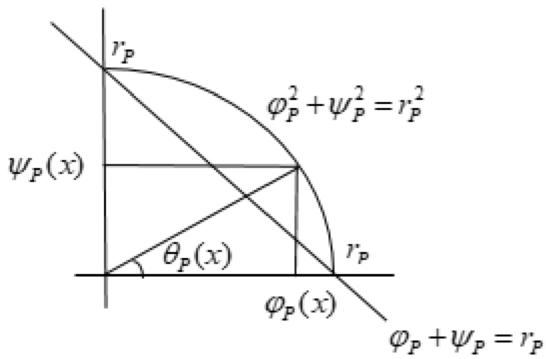
The degree of indeterminacy $I_A = 1 - \mu_A(x) - \gamma_A(x)$.

**Definition 2.3.** [10] Let $X$ be a universe of discourse. A pythagorean fuzzy set $P$ in $X$ is given by $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ where the functions $\varphi_P(x) : X \to [0,1]$ denotes the degree of membership and $\psi_P(x) : X \to [0,1]$ denotes the degree of non-membership of the element $x \in X$ to the set $P$, respectively, with the condition that $0 \leq (\varphi_P(x))^2 + (\psi_P(x))^2 \leq 1$.

The degree of indeterminacy $I_P = \sqrt{1 - (\varphi_P(x))^2 - (\psi_P(x))^2}$.

**Remark 2.4.** It is easy to check that $PFS$s generalize $IFS$s. That is, all intuitionistic fuzzy degrees are part of the Pythagorean fuzzy degrees. In actual decision-making problems, the $PFS$ characterizes a larger membership space than the $IFS$. Namely, the $PFS$ a higher capability than the $IFS$ to model vagueness in real decision-making problems. Yager [10] proposed a novel concept of $PFS$ to model the condition that the sum of the degree to which an alternative $x_i$ satisfies and dissatisfies with respect to the attribute $C_j$ is bigger than 1, while the $IFS$ cannot deal with it.

**Fig. 1.** Comparison of intuitionistic fuzzy subsets and Pythagorean fuzzy subsets




**Definition 2.5.** [10] Let $P_1 = \{\langle x, \varphi_{P_1}(x), \psi_{P_1}(x)\rangle : x \in X\}$ and $P_2 = \{\langle x, \varphi_{P_2}(x), \psi_{P_2}(x)\rangle : x \in X\}$ be two pythagorean fuzzy sets over $X$. Then,

**a** the pythagorean fuzzy complement of $P_1$ is defined by

$$P_1^c = \{\langle x, \psi_{P_1}(x), \varphi_{P_1}(x)\rangle : x \in X\},$$

**b** the pythagorean fuzzy intersection of $P_1$ and $P_2$ is defined by

$$P_1 \cap P_2 = \{\langle x, \min\{\varphi_{P_1}(x), \varphi_{P_2}(x)\}, \max\{\psi_{P_1}(x), \psi_{P_2}(x)\}\rangle : x \in X\},$$

**c** the pythagorean fuzzy union of $P_1$ and $P_2$ is defined by

$$P_1 \cup P_2 = \{\langle x, \max\{\varphi_{P_1}(x), \varphi_{P_2}(x)\}, \min\{\psi_{P_1}(x), \psi_{P_2}(x)\}\rangle : x \in X\},$$

**d** we say $P_1$ is a pythagorean fuzzy subset of $P_2$ and we write $P_1 \subseteq P_2$ if $\varphi_{P_1}(x) \leq \varphi_{P_2}(x)$ and $\psi_{P_1}(x) \geq \psi_{P_2}(x)$ for each $x \in X$,

**e** $0_X = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_X = \{\langle x, 1, 0\rangle : x \in X\}$.

**Definition 2.6.** [27] Let $X \neq \emptyset$ be a set and $\tau$ be a family of pythagorean fuzzy subsets of $X$. If

**T1** $0_X$, $1_X \in \tau$,

**T2** for any $P_1$, $P_2 \in \tau$, we have $P_1 \cap P_2 \in \tau$,

**T3** for any $\{P_i\}_{i \in I} \subseteq \tau$, we have $\underset{i \in I}{\cup} P_i \in \tau$

then $\tau$ is called a pythagorean fuzzy topology on $X$ and the pair $(X, \tau)_p$ is said to be a pythagorean fuzzy topological space ($PFTS$ for short).Each member of $\tau$ is called a pythagorean fuzzy open set ($PFOS$ for short). The complement of a pythagorean fuzzy open set is called a pythagorean fuzzy closed set ($PFCS$ for short).

**Remark 2.7.** As any fuzzy set or intuitionistic fuzzy set can be considered as a pythagorean fuzzy set, we observe that any fuzzy topological space or intuitionistic fuzzy topological space is a pythagorean fuzzy topological space as well. Conversely, it is obvious that pythagorean fuzzy topological space needs not to be a fuzzy topological space or intuitionistic fuzzy topological space. Even a pythagorean fuzzy open set may be neither a fuzzy set nor an intuitionistic fuzzy set (see following example).

**Example 2.8.** [27] Let $X = \{x_1, x_2\}$. Consider the following family of pythagorean fuzzy subsets $\tau = \{0_X, 1_X, P_1, ..., P_5\}$ where

$$\begin{aligned} P_1 &= \{\langle x_1, 0.5, 0.7\rangle, \langle x_2, 0.2, 0.4\rangle\}, \\ P_2 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.3, 0.9\rangle\}, \\ P_3 &= \{\langle x_1, 0.4, 0.8\rangle, \langle x_2, 0.1, 0.95\rangle\}, \\ P_4 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.3, 0.4\rangle\}, \\ P_5 &= \{\langle x_1, 0.5, 0.7\rangle, \langle x_2, 0.2, 0.9\rangle\}. \end{aligned}$$

Observe that $(X, \tau)_p$ is a pythagorean fuzzy topological space.

**Definition 2.9.** [27] Let $X$ and $Y$ be two non-empty sets, let $f : X \to Y$ be a function and let $A$ and $B$ be Pythagorean fuzzy subsets of $X$ and $Y$, respectively. Then, the membership and non-membership functions of image of $A$ with respect to $f$ that is denoted by $f[A]$ are defined by

$$\mu_{f[A]}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_{f[A]}(y) = \begin{cases} \inf\limits_{z \in f^{-1}(y)} v_A(z), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

respectively. The membership and non-membership functions of pre-image of $B$ with respect to $f$ that is denoted by $f^{-1}[B]$ are defined by

$\mu_{f^{-1}[B]}(x) = \mu_B(f(x))$ and $v_{f^{-1}[B]}(x) = v_B(f(x))$ respectively.

In the study [27], they showed that $\mu^2_{f[A]} + v^2_{f[A]} \leq 1$ pythagorean fuzzy membership condition is provide for pythagorean fuzzy image and pre-image.

**Proposition 2.10.** [27] Let $X$ and $Y$ be two non-empty sets and $f : X \to Y$ be a pythagorean fuzzy function. Then, we have

1. $f^{-1}[B^c] = \left(f^{-1}[B]\right)^c$ for any pythagorean fuzzy subset $B$ of $Y$.
2. $(f[A])^c \subseteq f[A^c]$ for any pythagorean fuzzy subset $A$ of $X$.
3. If $B_1 \subseteq B_2$ then $f^{-1}[B_1] \subseteq f^{-1}[B_2]$ where $B_1$ and $B_2$ are pythagorean fuzzy subset of $Y$.
4. If $A_1 \subseteq A_2$ then $f[A_1] \subseteq f[A_2]$ where $A_1$ and $A_2$ are pythagorean fuzzy subset of $X$.
5. $f\left[f^{-1}[B]\right] \subseteq B$ for any pythagorean fuzzy subset $B$ of $Y$.
6. $A \subseteq f^{-1}[f[A]]$ for any pythagorean fuzzy subset $A$ of $X$.

**Definition 2.11.** [27] Let $(X,\tau_1)_p$ and $(Y,\tau_2)_p$ be two pythagorean fuzzy topological spaces and $f : X \to Y$ be a pythagorean fuzzy function. Then, $f$ is said to be pythagorean fuzzy continuous if for any pythagorean fuzzy subset $A$ of $X$ and for any neighbourhood $V$ of $f[A]$ there exists a neighbourhood $U$ of $A$ such that $f[U] \subseteq V$.

**Theorem 2.12.** [27] Let $(X,\tau_1)_p$ and $(Y,\tau_2)_p$ be two pythagorean fuzzy topological spaces. A function $f : X \to Y$ is pythagorean fuzzy continuous iff for each open (closed) pythagorean fuzzy subset $B$ of $Y$ we have $f^{-1}[B]$ is an open (closed) pythagorean fuzzy subset of $X$.

## 3. Basic Results

**Definition 3.1.** Let $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$ be a family of pythagorean fuzzy sets over $X$. Then,

**a** $\underset{i\in I}{\cap} P_i = \{\langle x, \inf\{\varphi_{P_i}(x)\}, \sup\{\psi_{P_i}(x)\}\rangle : x \in X\}$,

**b** $\underset{i\in I}{\cup} P_i = \{\langle x, \sup\{\varphi_{P_i}(x)\}, \inf\{\psi_{P_i}(x)\}\rangle : x \in X\}$.

Note that $\underset{i\in I}{\cap} P_i$ and $\underset{i\in I}{\cup} P_i$ are pythagorean fuzzy sets over $X$. We shall $\underset{i\in I}{\cap} P_i$ define $\underset{i\in I}{\cap} P_i = \left\{\left\langle x, \alpha_{\underset{i\in I}{\cap} P_i}, \beta_{\underset{i\in I}{\cap} P_i}\right\rangle : x \in X\right\}$ such that $\alpha_{\underset{i\in I}{\cap} P_i} = \inf\{\varphi_{P_i}(x)\}$ and $\beta_{\underset{i\in I}{\cap} P_i} = \sup\{\psi_{P_i}(x)\}$. In order to for $\underset{i\in I}{\cap} P_i$ to be pythagorean fuzzy set we must have that $\alpha^2_{\underset{i\in I}{\cap} P_i}(x) + \beta^2_{\underset{i\in I}{\cap} P_i}(x) \leq 1$. We see since $\beta^2_{\underset{i\in I}{\cap} P_i}(x) = \sup\{\psi^2_{P_i}(x)\}$, then

$$\begin{aligned}\beta^2_{\underset{i\in I}{\cap} P_i}(x) &= \sup\{\psi^2_{P_i}(x)\} = \sup\{r_i^2 - \varphi^2_{P_i}, r_i^2 - \psi^2_{P_i}\} \\ &\leq \sup\{r_i^2 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}, r_i^2 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\}\end{aligned}$$

$$\begin{aligned}\beta^2_{\underset{i\in I}{\cap} P_i}(x) &\leq \sup\{1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\}, 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\} \\ &\leq 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\}\end{aligned}$$

where $\varphi^2_{P_i} + \psi^2_{P_i} = r_i^2$ for every $i \in I$. From this we see that $\alpha^2_{\underset{i\in I}{\cap} P_i}(x) + \beta^2_{\underset{i\in I}{\cap} P_i}(x) \leq \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\} + 1 - \inf\{\varphi^2_{P_i}, \psi^2_{P_i}\} \leq 1$. Thus, $\underset{i\in I}{\cap} P_i$ is a pythagorean fuzzy set.

The prof is trivial for $\underset{i\in I}{\cup} P_i$.

**Theorem 3.2.** Let $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$ be a family of pythagorean fuzzy sets over $X$. Then,

**i** $\overline{\underset{i\in I}{\cap} P_i} = \underset{i\in I}{\cup} \overline{P_i}$,

**ii** $\overline{\underset{i\in I}{\cup} P_i} = \underset{i\in I}{\cap} \overline{P_i}$.

PROOF. i) We have $\underset{i\in I}{\cap} P_i = \{\langle x, \inf\{\varphi_{P_i}(x)\}, \sup\{\psi_{P_i}(x)\}\rangle : x \in X\}$. Then

$$\overline{\underset{i\in I}{\cap} P_i} = \{\langle x, \sup\{\psi_{P_i}(x), \inf\{\varphi_{P_i}(x)\}\}\rangle : x \in X\}$$

and $\overline{P_i} = \{\langle x, \psi_{P_i}(x), \varphi_{P_i}(x)\rangle : x \in X\}$ and so $\underset{i\in I}{\cup} \overline{P_i} = \{\langle x, \sup\{\psi_{P_i}(x), \inf\{\varphi_{P_i}(x)\}\}\rangle : x \in X\}$. That is, $\overline{\underset{i\in I}{\cap} P_i} = \underset{i\in I}{\cup} \overline{P_i}$.

ii) It is proved similar to (i) □

**Definition 3.3.** Let $(X,\tau)_p$ be a $PFTS$ and $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ be a $PFS$ over $X$. Then the pythagorean fuzzy interior, pythagorean fuzzy closure and pythagorean fuzzy boundary of $P$ are defined by;

**a** $int(P) = \cup\{G : G \text{ is a } PFOS \text{ in } X \text{ and } G \subseteq P\}$,

**b** $cl(P) = \cap\{K : K \text{ is a } PFCS \text{ in } X \text{ and } P \subseteq K\}$,

**c** $Fr(P) = cl(P) \cap cl(P^c)$.

It is clear that,

**a** $int(P)$ is the biggest pythagorean fuzzy open set contained $P$,

**b** $cl(P)$ is the smallest pythagorean fuzzy closed set containing $P$.

**Remark 3.4.** From the definition pythagorean fuzzy union and intersection, it is obvious that pythagorean fuzzy interior, closure and boundary is a pythagorean fuzzy set.

**Example 3.5.** Let $X = \{x_1, x_2, x_3\}$. Consider the following family of pythagorean fuzzy sets $\tau = \{1_X, 0_X, P_1, P_2, P_3, P_4,\}$ where

$$\begin{aligned} P_1 &= \{\langle x_1, 0.6, 0.8\rangle, \langle x_2, 0.7, 0.6\rangle, \langle x_3, 0.3, 0.2\rangle\}, \\ P_2 &= \{\langle x_1, 0.7, 0.9\rangle, \langle x_2, 0.2, 0.5\rangle, \langle x_3, 0.1, 0.9\rangle\}, \\ P_3 &= \{\langle x_1, 0.7, 0.8\rangle, \langle x_2, 0.7, 0.5\rangle, \langle x_3, 0.3, 0.2\rangle\}, \\ P_4 &= \{\langle x_1, 0.6, 0.9\rangle, \langle x_2, 0.2, 0.6\rangle, \langle x_3, 0.1, 0.9\rangle\}. \end{aligned}$$

It is clear that $(X,\tau)_p$ is a pythagorean fuzzy topological space. Now, assume that,

$$P = \{\langle x_1, 0.8, 0.5\rangle, \langle x_2, 0.9, 0.3\rangle, \langle x_3, 0.4, 0.1\rangle\}$$

is a pythagorean fuzzy subset over $X$. Then

$$\begin{aligned} int(P) &= 0_X \cup P_1 \cup P_2 \cup P_3 \cup P_4 \\ &= P_3 = \{\langle x_1, 0.7, 0.8\rangle, \langle x_2, 0.7, 0.5\rangle, \langle x_3, 0.3, 0.2\rangle\}. \end{aligned}$$

On the other hand, in order to find the pythagorean fuzzy closure of $P$, it necessary to determine the pythagorean fuzzy closed sets over $X$. Then

$$\begin{aligned} P_1^c &= \{\langle x_1, 0.8, 0.6\rangle, \langle x_2, 0.6, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\}, \\ P_2^c &= \{\langle x_1, 0.9, 0.7\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.9, 0.1\rangle\}, \\ P_3^c &= \{\langle x_1, 0.8, 0.7\rangle, \langle x_2, 0.5, 0.7\rangle, \langle x_3, 0.2, 0.3\rangle\}, \\ P_4^c &= \{\langle x_1, 0.9, 0.6\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.9, 0.1\rangle\}. \end{aligned}$$

Hence,

$$cl(P) = 1_X$$

Similarly to find the pythagorean fuzzy boundary of $P$,

$$P^c = \{\langle x_1, 0.5, 0.8\rangle, \langle x_2, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.4\rangle\}$$

$$\begin{aligned} cl\left(P^c\right) &= 1_X \cap P_1^c \cap P_2^c \cap P_3^c \cap P_4^c \\ &= P_3^c = \left\{\left\langle x_1, 0.8, 0.7\right\rangle, \left\langle x_2, 0.5, 0.7\right\rangle, \left\langle x_3, 0.2, 0.3\right\rangle\right\} \end{aligned}$$

$$\begin{aligned} Fr(P) &= cl(P) \cap cl\left(P^c\right) \\ &= 1_X \cap P_3^c \\ &= \left\{\left\langle x_1, 0.8, 0.7\right\rangle, \left\langle x_2, 0.5, 0.7\right\rangle, \left\langle x_3, 0.2, 0.3\right\rangle\right\}. \end{aligned}$$

**Proposition 3.6.** Let $(X,\tau)_p$ be a $PFTS$ and $P$, $P_1$, $P_2$ be $PFS$s over $X$. Then the following properties hold;

**i** $int\left(P\right) \subseteq P$,

**ii** $int\left(int\left(P\right)\right) = int\left(P\right)$,

**iii** $P_1 \subseteq P_2 \Rightarrow int\left(P_1\right) \subseteq int\left(P_2\right)$,

**iv** $int\left(P_1 \cap P_2\right) = int\left(P_1\right) \cap int\left(P_2\right)$,

**v** $int\left(1_X\right) = 1_X$, $int\left(0_X\right) = 0_X$.

Proof. (i), (ii), (iii) and (v) can be easily obtained from the definition of the pythagorean fuzzy interior.

(iv) From $int\left(P_1 \cap P_2\right) \subseteq int\left(P_1\right)$ and $int\left(P_1 \cap P_2\right) \subseteq int\left(P_2\right)$ we obtain $int\left(P_1 \cap P_2\right) \subseteq int\left(P_1\right) \cap int\left(P_2\right)$. On the other hand, from the facts $int\left(P_1\right) \subseteq P_1$ and $int\left(P_2\right) \subseteq P_2 \Rightarrow int\left(P_1\right) \cap int\left(P_2\right) \subseteq P_1 \cap P_2$ and $int\left(P_1\right) \cap int\left(P_2\right) \in \tau$ we have $int\left(P_1\right) \cap int\left(P_2\right) \subseteq int\left(P_1 \cap P_2\right)$. So, proof of the axioms (iv) is obtained from these two inequalities. □

**Theorem 3.7.** Let $J : PFS(X) \to PFS(X)$ be a mapping. The family $\tau = \{P \in PFS(X) : J(P) = P\}$ is a pythagorean fuzzy topology over $X$, if the mapping $J$ satisfies the following conditions:

**i** $J(P) \subseteq P$,

**ii** $J(1_X) = 1_X$,

**iii** $J\left(J(P)\right) = J(P)$,

**iv** $J\left(P_1 \cap P_2\right) = J\left(P_1\right) \cap J\left(P_2\right)$.

Also, $J(P) = int\left(P\right)$ for each pythagorean fuzzy set $P$ in this pythagorean fuzzy topological space.

Proof. Straightforward. □

**Proposition 3.8.** Let $(X,\tau)_p$ be a $PFTS$ and $P$, $P_1$, $P_2$ be $PFS$s over $X$. Then the following properties hold;

**i** $P \subseteq cl\left(P\right)$,

**ii** $cl\left(cl\left(P\right)\right) = cl\left(P\right)$,

**iii** $P_1 \subseteq P_2 \Rightarrow cl\left(P_1\right) \subseteq cl\left(P_2\right)$,

**iv** $cl\left(P_1 \cup P_2\right) = cl\left(P_1\right) \cup cl\left(P_2\right)$,

**v** $cl\left(1_X\right) = 1_X$, $cl\left(0_X\right) = 0_X$.

Proof. (i), (ii), (iii) and (v) can be easily obtained from the definition of the pythagorean fuzzy closure.

(iv) From $cl\left(P_1\right) \subseteq cl\left(P_1 \cup P_2\right)$ and $cl\left(P_2\right) \subseteq cl\left(P_1 \cup P_2\right)$ we obtain $cl\left(P_1\right) \cup cl\left(P_2\right) \subseteq cl\left(P_1 \cup P_2\right)$. On the other hand, from the facts $P_1 \subseteq cl\left(P_1\right)$ and $P_2 \subseteq cl\left(P_2\right) \Rightarrow P_1 \cup P_2 \subseteq cl\left(P_1\right) \cup cl\left(P_2\right)$ and $cl\left(P_1\right) \cup cl\left(P_2\right) \in PFCS$ we have $cl\left(P_1 \cup P_2\right) \subseteq cl\left(P_1\right) \cup cl\left(P_2\right)$. Thus, proof of the axioms (iv) is obtained from these two inequalities. □

**Theorem 3.9.** Let $C : PFS(X) \to PFS(X)$ be a mapping. The family $\tau = \{P \in PFS(X) : C(P^c) = P^c\}$ is a pythagorean fuzzy topology over $X$, if the mapping $C$ satisfies the following conditions:

**i** $P \subseteq C(P)$,

**ii** $C(0_X) = 0_X$,

**iii** $C(C(P)) = C(P)$,

**iv** $C(P_1 \cup P_2) = C(P_1) \cup C(P_2)$.

Also, $C(P) = cl(P)$ for each pythagorean fuzzy set $P$ in this pythagorean fuzzy topological space.

Proof. Straightforward. □

**Theorem 3.10.** Let $(X, \tau)_p$ be a $PFTS$ and $P$ be a $PFS$ over $X$. Then,

**a** $cl(P^c) = (int(P))^c$,

**b** $int(P^c) = (cl(P))^c$.

Proof. (a) Let $P = \{\langle x, \varphi_P(x), \psi_P(x)\rangle : x \in X\}$ and assume that the family of $PFS$s contained in $P$ are indexed by the family $\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$. Then we see that $int(P) = \{\langle x, \sup\{\varphi_{P_i}(x)\}, \inf\{\psi_{P_i}(x)\}\rangle : x \in X\}$ and hence $(int(P))^c = \{\langle x, \inf\{\psi_{P_i}(x)\}, \sup\{\varphi_{P_i}(x)\}\rangle : x \in X\}$. Since $P^c = \{\langle x, \psi_P(x), \varphi_P(x)\rangle : x \in X\}$ and $\varphi_{P_i}(x) \leq \varphi_P(x)$, $\psi_{P_i}(x) \geq \psi_P(x)$ for each $i \in I$, we obtain that

$\{P_i = \{\langle x, \varphi_{P_i}(x), \psi_{P_i}(x)\rangle : x \in X\}\}_{i\in I}$ is the family of $PFS$s containing $P^c$, i.e. $cl(P^c) = \{\langle x, \inf\{\psi_{P_i}(x)\}, \sup\{\varphi_{P_i}(x)\}\rangle : x \in X\}$. Therefore, $cl(P^c) = (int(P))^c$ immediately.

(b) This analagous to (a). □

**Proposition 3.11.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then, the following are equivalent to each other;

**a** $f$ is a pythagorean fuzzy continuous function,

**b** $f[cl(P)] \subseteq cl(f[P])$ for each $PFS$ $P$ in $X$,

**c** $cl\left(f^{-1}[K]\right) \subseteq f^{-1}[cl(K)]$ for each $PFS$ $K$ in $Y$,

**d** $f^{-1}[int(K)] \subseteq int\left(f^{-1}[K]\right)$ for each $PFS$ $K$ in $Y$.

Proof. a)⇒b) Let $f : X \to Y$ be a pythagorean fuzzy continuous function and $P$ be a $PFS$ over $X$. Then, $f[P] \subseteq cl(f[P])$ and $P \subseteq f^{-1}[cl(f[P])]$. Since $cl(f[P])$ is a pythagorean fuzzy closed set in $Y$ and $f$ is a pythagorean fuzzy continuous function, $f^{-1}[cl(f[P])]$ is a pythagorean fuzzy closed set in $X$. On the other hand, if $cl(P)$ is the smallest pythagorean fuzzy closed set containing $P$, then $cl(P) \subseteq f^{-1}[cl(f[P])]$ and so, $f[cl(P)] \subseteq cl(f[P])$.

b)⇒c) Suppose that $P = f^{-1}[K]$. From (b), $f[cl(P)] = f\left[cl\left(f^{-1}[K]\right)\right] \subseteq cl(f[P]) = cl\left(f\left[f^{-1}[K]\right]\right) \subseteq cl(K)$. Then, $cl\left(f^{-1}[K]\right) = cl(P) \subseteq f^{-1}[f[cl(P)]] \subseteq f^{-1}[cl(K)]$.

c)⇒d) Since $int(K) = (cl(K^c))^c$, then $cl\left(f^{-1}[K]\right) = cl(P) \subseteq f^{-1}[f[cl(P)]] \subseteq f^{-1}[cl(K)]$.

Assume that, $G$ is a pythagorean fuzzy open set in $Y$. Then, $int(G) = G$. From (d), $f^{-1}[G] = f^{-1}[int(G)] \subseteq int\left(f^{-1}[G]\right) \subseteq f^{-1}[G]$. Therefore, $f$ is a pythagorean fuzzy continuous function. □

**Definition 3.12.** Let $(X, \tau)_p$ be a $PFTS$.

**a** A subfamily $\Gamma$ of $\tau$ is called a pythagorean fuzzy basic ($PFB$ for short) for $\tau$, if for each $P \in \tau$, $P = 0_X$ or there exists $\Gamma' \subseteq \Gamma$ such that $P = \cup\Gamma'$.

**b** A subfamily $\Phi$ of $\tau$ is called a pythagorean fuzzy subbase ($PFSB$ for short) for $\tau$, if the family $\Gamma = \left\{\cap\Phi' : \Phi' \text{ is a finite subset of } \Phi\right\}$ is a pythagorean fuzzy basic for $\tau$.

**Example 3.13.** Considering the pythagorean fuzzy topology in Example 1, the family

$$\Phi = \{P_1, P_2\}$$

is a pythagorean fuzzy subbase for $\tau$ and

$$\Gamma = \{P_1, P_2, P_4\}$$

is a pythagorean fuzzy basic for $\tau$.

**Theorem 3.14.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

**i** $f$ is a pythagorean fuzzy continuous function iff for each $B \in \Gamma$ we have $f^{-1}[B]$ is a pythagorean fuzzy open subset of $X$ such that $\Gamma$ is a pythagorean fuzzy basic for $\tau_2$.

**ii** $f$ is a pythagorean fuzzy continuous function iff for each $K \in \Phi$ we have $f^{-1}[K]$ is a pythagorean fuzzy open subset of $X$ such that $\Phi$ is a pythagorean fuzzy subbase for $\tau_2$.

Proof. i) Let $f$ be a pythagorean fuzzy continuous function. Since each $B \in \Gamma \subseteq \tau_2$ and $f$ is a pythagorean fuzzy continuous function, then $f^{-1}[B] \in \tau_1$.

Concersely, suppouse that $\Gamma$ is a pythagorean fuzzy basic for $\tau_2$ and $f^{-1}[B] \in \tau_1$ for each $B \in \Gamma$. Then for arbitrary a pythagorean fuzzy open set $P \in \tau_2$,

$$f^{-1}[P] = f^{-1}\left[\underset{B\in\Gamma}{\cup} B\right] = \underset{B\in\Gamma}{\cup} f^{-1}[B] \in \tau_1.$$

That is, $f$ is a pythagorean fuzzy continuous function.

ii) Let $f$ be a pythagorean fuzzy continuous function. Since each $K \in \Phi \subseteq \tau_2$ and $f$ is a pythagorean fuzzy continuous function, then $f^{-1}[K] \in \tau_1$.

Concersely, assume that $\Phi$ is a pythagorean fuzzy subbase for $\tau_2$ and $f^{-1}[K] \in \tau_1$ for each $K \in \Phi$. Then for arbitrary a pythagorean fuzzy open set $P \in \tau_2$,

$$\begin{aligned} f^{-1}[P] &= f^{-1}\left[\underset{i_j\in I}{\cup}(K_{i_1} \cap K_{i_2} \cap ... \cap K_{i_n})\right] \\ &= \underset{i_j\in I}{\cup}\left(f^{-1}[K_{i_1}] \cap f^{-1}[K_{i_2}] \cap ... \cap f^{-1}[K_{i_n}]\right) \in \tau_1 \end{aligned}$$

That is, $f$ is a pythagorean fuzzy continuous function. □

**Definition 3.15.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

**a** $f$ is called a pythagorean fuzzy open function if $f[P]$ is a pythagorean fuzzy open set over $Y$ for every pythagorean fuzzy open set $P$ over $X$.

**b** $f$ is called a pythagorean fuzzy closed function if $f[K]$ is a pythagorean fuzzy closed set over $Y$ for every pythagorean fuzzy closed set $K$ over $X$.

**Example 3.16.** Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Consider the following families of pythagorean fuzzy sets $\tau_1 = \{0_X, 1_X, P_1, P_2, P_3, P_4\}$ and $\tau_2 = \{0_Y, 1_Y, S_1, S_2, S_3, S_4\}$ where

$$\begin{aligned} P_1 &= \{\langle x_1, 0.3, 0.5\rangle, \langle x_2, 0.6, 0.2\rangle, \langle x_3, 0.6, 0.5\rangle\}, \\ P_2 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.8, 0.3\rangle, \langle x_3, 0.7, 0.6\rangle\}, \\ P_3 &= \{\langle x_1, 0.6, 0.5\rangle, \langle x_2, 0.8, 0.2\rangle, \langle x_3, 0.7, 0.5\rangle\}, \\ P_4 &= \{\langle x_1, 0.3, 0.5\rangle, \langle x_2, 0.6, 0.3\rangle, \langle x_3, 0.6, 0.6\rangle\}, \end{aligned}$$

$$\begin{aligned}
S_1 &= \{\langle y_1, 0.6, 0.2\rangle, \langle y_2, 0.3, 0.5\rangle, \langle y_3, 0.6, 0.5\rangle\},\\
S_2 &= \{\langle y_1, 0.8, 0.3\rangle, \langle y_2, 0.6, 0.5\rangle, \langle y_3, 0.7, 0.6\rangle\},\\
S_3 &= \{\langle y_1, 0.8, 0.2\rangle, \langle y_2, 0.6, 0.5\rangle, \langle y_3, 0.7, 0.5\rangle\},\\
S_4 &= \{\langle y_1, 0.6, 0.3\rangle, \langle y_2, 0.3, 0.5\rangle, \langle y_3, 0.6, 0.6\rangle\},
\end{aligned}$$

It is clear that $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ are pythagorean fuzzy topological spaces. If pythagorean fuzzy function $f : X \to Y$ is defined as;

$$\begin{aligned}
f(x_1) &= y_2\\
f(x_2) &= y_1\\
f(x_3) &= y_3
\end{aligned}$$

Then $f$ is a pythagorean fuzzy open function. However $f$ is not pythagorean fuzzy closed function on pythagorean fuzzy topological spaces $(X, \tau_1)_p$.

**Theorem 3.17.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then,

**i** $f$ is a pythagorean fuzzy open function if $f[int(P)] \subseteq int(f[P])$ for each pythagorean fuzzy set $P$ over $X$.

**ii** $f$ is a pythagorean fuzzy closed function if $cl(f[P]) \subseteq f[cl(P)]$ for each pythagorean fuzzy set $P$ over $X$.

PROOF. i) Let $f$ be a pythagorean fuzzy open function and $P$ be a $PFS$ over $X$. Then, $int(P)$ is a pythagorean fuzzy open set and $int(P) \subseteq P$. Since $f$ is a pythagorean fuzzy open function, $f[int(P)]$ is a pythagorean fuzzy open set over $Y$ and $f[int(P)] \subseteq f[P]$. Thus, $f[int(P)] \subseteq int(f[P])$ is obtained.

Conversely, suppouse that $P$ is any pythagorean fuzzy open set over $X$. Then $P = int(P)$. From the condition of theorem, we have $f[int(P)] \subseteq int(f[P])$. Then $f[P] = f[int(P)] \subseteq int(f[P]) \subseteq f[P]$. This implies that $f[P] = int(f[P])$. That is, $f$ is a pythagorean fuzzy open function.

ii) Let $f$ be a pythagorean fuzzy closed function and $P$ be a $PFS$ over $X$. Since $f$ is a pythagorean fuzzy closed function then $f[cl(P)]$ is a pythagorean fuzzy closed set over $Y$ and $f[P] \subseteq f[cl(P)]$. Thus, $cl(f[P]) \subseteq f[cl(P)]$ is obtained.

Conversely, assume that $P$ is any pythagorean fuzzy closed set over $X$. Then $P = cl(P)$. From the condition of theorem, we have $cl(f[P]) \subseteq f[cl(P)] = f[P] \subseteq cl(f[P])$. This means that, $cl(f[P]) = f[P]$. That is, $f$ is a pythagorean fuzzy closed function. □

**Definition 3.18.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then $f$ is a called a pythagorean fuzzy homeomerphism, if

**i** $f$ is a bijection,

**ii** $f$ is a pythagorean fuzzy continuous function,

**iii** $f^{-1}$ is a pythagorean fuzzy continuous function.

**Theorem 3.19.** Let $(X, \tau_1)_p$ and $(Y, \tau_2)_p$ be two $PFTS$s and $f : X \to Y$ be a pythagorean fuzzy function. Then the following conditions are equivalent;

**i** $f$ is a pythagorean fuzzy homeomerphism,

**ii** $f$ is a pythagorean fuzzy continuous function and pythagorean fuzzy open function,

**iii** $f$ is a pythagorean fuzzy continuous function and pythagorean fuzzy closed function.

PROOF. The proof can be easily obtained by using the previous theorems on continuity, opennes and closedness are omitted. □

## Conclusion

In this paper, we introduced the concept of pythagorean fuzzy interior, pythagorean fuzzy closure, pythagorean fuzzy boundary, pythagorean fuzzy basic on pythagorean fuzzy topological spaces. We also study pythagorean fuzzy open (closed) function and pythagorean fuzzy homeomorphism. Some basic properties of these concepts are explored. We hope that, the results of this study may help in the investigation of pythagorean fuzzy topological spaces in many researches.

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