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The modified simple equation method for solving some nonlinear evolution equations

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Özet

Bu makalede Modifiye Edilmiş Basit Denklem Metodu (MSEM) uygulamalı bilimlerde önemli olan lineer olmayan bazı evolüsyon denklemlerine uygulandı. MSEM metodu iki önemli evolüsyon denklemine yani konveksiyon terimli Fisher ve konveksiyon terimli Fitzhugh–Nagumo denklemlerine uygulanmıştır.

Anahtar Kelimeler: Modifiye edilmiş basit denklem metodu, Tam çözümler.

Abstract

In this paper we applied modified simple equation method (MSEM) for solving some nonlinear evolution equations which are very important in applied sciences. The MSEM is implemented on two very important evolution equations namely the Fisher equation with convection term and the Fitzhugh–Nagumo equation with convection term.

Key Words: Modified simple equation method, Exact solutions.

1.Introduction

The Modified Simple Equation Method have been introduced by Ja'afar Mohamad Jawad, Marko D. Petkovic and Anjan Biswas in 2010 [1, 2].

This paper outlines the application of modified simple equation method (MSEM) for solving the Fisher equation with convection term and the Fitzhugh–Nagumo equation with convection term.

2.Description of the method

We consider a nonlinear evolution equation:

$$F(u,\ u_1,\ u_x,\ u_{xy},\ \ldots) = 0 \qquad (1)$$

where F is a polynomial in u and its partial derivatives.

Step 1. Using the wave transformation

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$$u = u(z),\ z = x - t, \tag{2}$$

we have from (1) and (2) the following ODE:

$$P(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where P is a polynomial in u and its total derivatives and $' = \frac{d}{dz}$.

Step 2. We suppose that Eq. (3) has the formal solution:

$$u(z) = \sum_{k=0}^{N} A_k \left(\frac{\psi'}{\psi}\right)^k, \tag{4}$$

where A_k are arbitrary constants to be determined such that $A_N \neq 0$ while $\psi(z)$ is an unknown function to be determined later.

Step 3. We determine the positive integer N in (4) by balancing the highest order derivatives and the nonlinear terms in Eq. (3).

Step 4. We substitute (4) into (3), we calculate all the necessary derivatives $u', u'', \ldots$ and then we account the function $\psi(z)$. As a result of this substitution, we get a polynomial of $\frac{\psi'}{\psi}$ and its derivatives. In this polynomial, we equate with zero all the coefficients of it. This operation yields a system of equations which can be solved to find A_k and $\psi(z)$. Consequently, we can get the exact solution of Eq. (1).

3. Numerical applications

In this section, we apply MSEM for solving the Fisher equation with convection term and the Fitzhugh–Nagumo equation with convection term.

3.1. The Fisher equation with convection

Consider the Fisher equation with convection term

$$u_t + kuu_x - u_{xx} - u(1-u) = 0 \tag{5}$$

which describes the propagation of nerve pulses [3, 4]. Using the traveling wave $u = u(z),\ z = x - t$ to reduce Eq. (5) to the following ODE:

$$-u' + kuu' - u'' - u(1-u) = 0. \tag{6}$$

Balancing uu' with u'' yields $N = 1$. Consequently, we look for solutions of Eq. (6) in the form

$$u(z) = A_0 + A_1\left(\frac{\psi'}{\psi}\right), \tag{7}$$

where A_0 and A_1 are constants to be determined such that $A_1 \neq 0$, while $\psi(z)$ is an unknown function to be determined. Since

$$u = u(z),\ z = x - t, \tag{2}$$

we have from (1) and (2) the following ODE:

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$$u' = A_1\left(\frac{\psi''}{\psi}-\left(\frac{\psi'}{\psi}\right)^2\right), \tag{8}$$

$$u'' = A_1\left(\frac{\psi'''}{\psi}-3\frac{\psi''\psi'}{\psi^2}+2\left(\frac{\psi'}{\psi}\right)^3\right), \tag{9}$$

then the following expression holds

$$-A_1\left(\frac{\psi''}{\psi}-\left(\frac{\psi'}{\psi}\right)^2\right)+k\left(A_0+A_1\left(\frac{\psi'}{\psi}\right)\right)A_1\left(\frac{\psi''}{\psi}-\left(\frac{\psi'}{\psi}\right)^2\right)-A_1\left(\frac{\psi'''}{\psi}-3\frac{\psi''\psi'}{\psi^2}+2\left(\frac{\psi'}{\psi}\right)^3\right)$$
$$-\left(A_0+A_1\left(\frac{\psi'}{\psi}\right)\right)\left(1-A_0-A_1\left(\frac{\psi'}{\psi}\right)\right)=0. \tag{10}$$

Equating expressions at $\psi^0, \psi^{-1}, \psi^{-2}$ and ψ^{-3} to zero we have the following equations:

$$-A_0 + A_0^2 = 0, \tag{11}$$
$$(1-kA_0)\psi''-\psi'''+(2A_0-1)\psi'=0, \tag{12}$$
$$(kA_1+3)\psi''+(-kA_0+A_1+1)\psi'=0, \tag{13}$$
$$A_1=\frac{-2}{k}. \tag{14}$$

Eq. (11) directly implies

$$A_0 = 0\ ,\ A_0 = 1\ .$$

Case 1: $A_0 = 0.$

Eqs.(12) and (13) becomes

$$\psi'''+\psi''+\psi'=0, \tag{15}$$
$$(kA_1+3)\psi''+(A_1+1)\psi'=0. \tag{16}$$

By substituting Eq. (16) into (15) we get

$$(kA_1+3)\psi'''+((k-1)A_1+2)\psi'=0. \tag{17}$$

Solution of Eq. (17) is given by

$$\psi(z)=c_0+c_1e^{az}+c_2e^{-az}, \tag{18}$$

where c_0, c_1 and c_2 are free arbitrary parameters and

$$a=\pm\sqrt{\frac{(1-k)A_1-2}{kA_1+3}}. \tag{19}$$

Substituting Eq.(18) for $\psi(z)$ into Eq. (7) for $u(x,t)$ we have exact solution in the form:

$$u(x,t)=A_1a\frac{c_1e^{a(x-t)}+c_2e^{-a(x-t)}}{c_0+c_1e^{a(x-t)}+c_2e^{-a(x-t)}}. \tag{20}$$

Case II: $A_0 = 1$.

Eqs.(12) and (13) becomes

$$(1-k)\psi'' - \psi''' + \psi' = 0, \tag{21}$$

$$(k+3)\psi'' + (-k + A_1 + 1)\psi' = 0. \tag{22}$$

By substituting Eq. (22) into (21) we get

$$(k+3)\psi''' + \left(k^2 - k + (1-k)A_1 + 4\right)\psi' = 0. \tag{23}$$

Solution of Eq. (23) is given by

$$\psi(z) = c_0 + c_1 e^{bz} + c_2 e^{-bz}, \tag{24}$$

where c_0, c_1 and c_2 are free arbitrary parameters and

$$b = \pm\sqrt{\frac{k^2 - k + (1-k)A_1 + 4}{k+3}}. \tag{25}$$

Substituting Eq.(24) for $\psi(z)$ into Eq. (7) for $u(x,t)$ we have exact solution in the form:

$$u(x,t) = 1 + A_1 b \frac{c_1 e^{b(x-t)} + c_2 e^{-b(x-t)}}{c_0 + c_1 e^{b(x-t)} + c_2 e^{-b(x-t)}}. \tag{26}$$

3.2. The Fitzhugh–Nagumo equation with convection

Consider the Fitzhugh–Nagumo equation with convection term

$$u_t + kuu_x - u_{xx} - u(1-u)(a-u) = 0, \tag{27}$$

where k and a are constants. The FHN equation, which shows up in the study of electrical pulses in nerve membranes, is a well-studied mathematical model in neurobiology [3, 4].
Using the traveling wave $u = u(z)$, z $\;x-t$ to reduce Eq. (27) to the following ODE:

$$-u' + kuu' - u'' - u(1-u)(a-u) = 0. \tag{28}$$

Substituting Eqs.(7)-(9) into Eq. (28) and equating coefficients of $\psi^0, \psi^{-1}, \psi^{-2}$ and ψ^{-3} to zero, we respectively obtain

$$-aA_0 + (a+1)A_0^2 - A_0^3 = 0, \tag{29}$$

$$-\psi''' + (kA_0 - 1)\psi'' + \left(2(a+1)A_0 - a - 3A_0^2\right)\psi' = 0, \tag{30}$$

$$(kA_1 + 3)\psi'' + \left(1 - kA_0 - 3A_0A_1 + (a+1)A_1\right)\psi' = 0, \tag{31}$$

and

$$A_1 = \frac{-k \mp \sqrt{k^2 - 8}}{2}. \tag{32}$$

Eq. (29) directly implies $A_0 = 0,\ A_0 \;\; 1$ and $A_0 \;\; a$.

Case 1: $A_0 = 0$.

Eqs.(30) and (31) becomes

$$\psi''' + \psi'' + \psi' = 0, \tag{33}$$
$$(kA_1 + 3)\psi'' + ((a+1)A_1 + 1)\psi' = 0. \tag{34}$$

By substituting Eq. (34) into (33) we get

$$(kA_1 + 3)\psi''' + (kaA_1 + 3a - aA_1 - A_1)\psi' = 0. \tag{35}$$

Solution of Eq. (35) is given by

$$\psi(z) = c_0 + c_1 e^{\alpha z} + c_2 e^{-\alpha z}, \tag{36}$$

where c_0, c_1 and c_2 are free arbitrary parameters and

$$\alpha = \pm\sqrt{\frac{A_1(1+a-ak) - 3a + 1}{kA_1 + 3}}. \tag{37}$$

Substituting Eq.(36) for $\psi(z)$ into Eq. (7) for $u(x,t)$ we have exact solution in the form:

$$u(x,t) = A_1 a \frac{c_1 e^{\alpha(x-t)} + c_2 e^{-\alpha(x-t)}}{c_0 + c_1 e^{\alpha(x-t)} + c_2 e^{-\alpha(x-t)}}. \tag{38}$$

Case 2: $A_0 = 1$.

Eqs.(30) and (31) becomes

$$-\psi''' + (k-1)\psi'' + (a-1)\psi' = 0, \tag{39}$$
$$(k+3)\psi'' + (1 - k + A_1(a-2))\psi' = 0. \tag{40}$$

By substituting Eq. (40) into (39) we get

$$(k+3)\psi''' + (-k^2 + 3k - ak - 3a + 2 + A_1(2-a)(1-k))\psi' = 0. \tag{41}$$

Solution of Eq. (41) is given by

$$\psi(z) = c_0 + c_1 e^{\beta z} + c_2 e^{-\beta z}, \tag{42}$$

where c_0, c_1 and c_2 are free arbitrary parameters and

$$\beta = \pm\sqrt{\frac{k^2 - 3k + ak + 3a - 2 + (1-k)(2-a)A_1}{k+3}}. \tag{43}$$

Substituting Eq.(42) for $\psi(z)$ into Eq. (7) for $u(x,t)$ we have exact solution in the form:

$$u(x,t)=1+A_1\beta\frac{c_1e^{\beta(x-t)}+c_2e^{-\beta(x-t)}}{c_0+c_1e^{\beta(x-t)}+c_2e^{-\beta(x-t)}}. \tag{44}$$

Case 3: $A_0=a.$

Eqs.(30) and (31) becomes

$$\psi'''+(1-ak)\psi''-(a-a^2)\psi'=0, \tag{45}$$

$$(kA_1+3)\psi''+(A_1(1-2a)+1-ak)\psi'=0. \tag{46}$$

By substituting Eq. (46) into (45) we get

$$(kA_1+3)\psi'''+(A_1(2a-ak^2-1)+2ak+3a^2-a^2k^2-3a-1)\psi'=0. \tag{47}$$

Solution of Eq. (47) is given by

$$\psi(z)=c_0+c_1e^{\gamma z}+c_2e^{-\gamma z}, \tag{48}$$

where c_0, c_1 and c_2 are free arbitrary parameters and

$$\gamma=\pm\sqrt{\frac{A_1(1+a-ak)-3a+1}{kA_1+3}}. \tag{49}$$

Substituting Eq.(48) for $\psi(z)$ into Eq. (7) for $u(x,t)$ we have exact solution in the form:

$$u(x,t)=a+A_1\gamma\frac{c_1e^{\gamma(x-t)}+c_2e^{-\gamma(x-t)}}{c_0+c_1e^{\gamma(x-t)}+c_2e^{-\gamma(x-t)}}. \tag{50}$$

4. Conclusions

In this paper we use a direct approach for finding the exact solutions of equation (5) and (27). MSEM has been successfully used to obtain exact solitary wave solutions for the Fisher equation with convection term and the Fitzhugh–Nagumo equation with convection term. Calculations in the MSEM are simple and straightforward.

5.References

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