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Difference Series Spaces and Matrix Transformations

G. Canan HAZAR GÜLEÇ[1*]

[1] Department of Mathematics, Pamukkale University, Denizli, Turkey
G. Canan HAZAR GÜLEÇ ORCID No: 0000-0002-8825-5555

**Sorumlu yazar: gchazar@pau.edu.tr*

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Abstract: This paper deals with new series space $|C_\alpha|_p(\nabla)$ introduced by using Cesàro means and difference operator. It is shown that this newly defined space $|C_\alpha|_p(\nabla)$ is a BK- space and has Schauder basis. Furthermore, the α, β, and γ-duals of $|C_\alpha|_p(\nabla)$ are computed and the characterizations of classes of matrix mappings from $|C_\alpha|_p(\nabla)$ to $X = \{\ell_\infty, c, c_0\}$ are also given.

Fark Seri Uzayları ve Matris Dönüşümleri

Anahtar Kelimeler
Fark dizi uzayları, α-β ve γ-dualleri, Matris operatörleri, BK uzayları

Öz: Bu çalışmada, Cesàro ortalaması ve fark operatörü kullanılarak yeni bir $|C_\alpha|_p(\nabla)$ seri uzayı tanımlanmıştır. Bu yeni $|C_\alpha|_p(\nabla)$ uzayının bir BK- uzayı olduğu ve Schauder bazına sahip olduğu gösterilmiştir. Ayrıca, $|C_\alpha|_p(\nabla)$ uzayının α, β, and γ- dualleri hesaplanmış ve $|C_\alpha|_p(\nabla)$ uzayından $X = \{\ell_\infty, c, c_0\}$ uzayına matris dönüşümleri karakterize edilmiştir.

1. INTRODUCTION

Recently, there has been a lot of intrest in studies on the sequence spaces. In the literature, the basic concept is to generate new sequence spaces by means of the matrix domain of triangles (see, [1-17]). Besides this, several authors have studied difference sequence spaces using some newly defined infinite matrices. Also, they have studied some topological properties of them, and they have given the inclusion relations and some characterizations of related matrix transformations.

Throughout this study, ω, ℓ_∞, c, and c_0 will be spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively. Also, by bs, cs and ℓ_p $(1 \leq p < \infty)$, we denote the spaces of all bounded, convergent and p -absolutely convergent series, respectively. A Banach sequence space X is called a BK-space provided each of the maps $P_n : X \to \mathbb{C}$ defined by $P_n(x) = x_n$ $(n \geq 0)$ is continuous, where $\mathbb{C}$ denotes the complex field.

Let U and V be two sequence spaces and $T = (t_{nk})$ be an infinite matrix of complex number. The matrix domain U_T is defined as

$$U_T = \{u \in \omega : Tu \in U\}. \tag{1}$$

Define the set $M(U,V)$ as

$$M(U,V) = \{a = (a_k) \in \omega : au = (a_k u_k) \in V \text{ for all } u = (u_k) \in U\}. \tag{2}$$

By the notation (2), the α, β, and γ-duals of the space U are defined by

$$U^\alpha = M(U,\ell_1), U^\beta = M(U,cs) \text{ and } U^\gamma = M(U,bs),$$

respectively.

Also, T defines a matrix mapping from U into V, if, for every $u=(u_k)\in U$, the sequence $Tu=\big(T_n(u)\big)$, the T-transform of u, exists and is in V, where

$$T_n(u)=\sum_{k=0}^{\infty} t_{nk}u_k$$

for $n\geq 0$. (U,V) denotes the class of all such matrices that maps U into V. Thus, $T\in(U,V)$ if and only if $T_n=(t_{nk})_{k=0}^{\infty}\in U^{\beta}$ for each n and $Tu\in V$ for all $u\in U$.

Throughout this study, q shows the conjugate of p, i.e., $1/p+1/q=1$.

2. DIFFERENCE SERIES SPACES AND CESÀRO MEANS

The notion of difference sequence space has been introduced by Kızmaz [18] as follows.

$$X(\Delta)=\{x=(x_k)\in\omega\ :\Delta x\in X\}$$

for $X\in c_0,c,\ell_\infty$, where $\Delta x=(\Delta x_k)=(x_k-x_{k+1})$ for all $k\in\mathbb{N}$. After, Sarıgöl [14] has defined the sequence space

$$X(\Delta_q)=\{x=(x_k)\ :\Delta_q x=\big(k^q(x_k-x_{k+1})\big)\in X,\ q<1\}.$$

Later on, some new sequence spaces are defined by using the difference operator. For example, several authors including Çolak and Et [3], Orhan [19], Polat and Altay [20], Aydın and Başar [1], Başar and Altay [2], Demiriz and Çakan [4] and others have introduced and studied new sequence spaces by considering difference operators. In this section, following [1-4, 6-11, 14-16], we introduce the difference series space $|C_\alpha|_p(\nabla)$ by using Cesàro means and difference operator and we prove that this space linearly isomorphic to space ℓ_p, and also construct its bases.

Let Σx_v be an infinite series with nth partial sums (s_n), then the nth Cesàro mean (C,α) of order α $(\alpha>-1)$ of the sequence (s_n) is defined by

$$u_n^{\alpha}=\frac{1}{E_n^{\alpha}}\sum_{v=0}^{n}E_{n-v}^{\alpha-1}s_v,$$

where $E_0^{\alpha}=1$, $E_n^{\alpha}=\binom{\alpha+n}{n}$, $E_{-n}^{\alpha}=0$, $n\geq 1$. The series Σx_n is said to be summable $|C,\alpha|_p, p\geq 1$, if (see [21])

$$\sum_{n=1}^{\infty}n^{p-1}|u_n^{\alpha}-u_{n-1}^{\alpha}|^p<\infty.$$

Using the method $|C,\alpha|_p$, the absolute Cesàro series space $|C_\alpha|_p$ has been defined by Sarıgöl in [16]. For any given sequence $x=(x_k)\in|C_\alpha|_p$, $H^{(p)}$-transform of x is in ℓ_p, where the matrix $H^{(p)}=\big(h_{nk}^{p}\big)$ is defined by

$$h_{nk}^{p}=\begin{cases}\dfrac{E_{n-k}^{\alpha-1}k}{n^{1/p}E_n^{\alpha}}, & 1\leq k\leq n\\ 0, & k>n.\end{cases}$$

The main purpose of this study is to define further generalization of the absolute Cesàro series space $|C_\alpha|_p(\nabla)$ using difference operator by

$$|C_\alpha|_p(\nabla)=\{x=(x_k)\in\omega:(\nabla x_k)\in|C_\alpha|_p\}$$

where $\nabla x_k=x_k-x_{k-1}$ for each $k\in\mathbb{N}$.

We first define the difference space $|C_\alpha|_p(\nabla)$ by

$$|C_\alpha|_p(\nabla)=\left\{x=(x_v)\in\omega\ :\sum_{n=1}^{\infty}\left|\frac{1}{n^{1/p}E_n^{\alpha}}\sum_{v=1}^{n}E_{n-v}^{\alpha-1}v\nabla x_v\right|^p<\infty\right\}.$$

Let us define the sequence $y=(y_n)$ as the $H^{(p)}(\nabla)$ transform of the sequence $x=(x_k)$, that is,

$$y_n=\frac{1}{n^{1/p}E_n^{\alpha}}\sum_{v=1}^{n}E_{n-v}^{\alpha-1}v\nabla x_v \tag{3}$$

for each $n\in\mathbb{N}$.

Then the difference space $|C_\alpha|_p(\nabla)$ can be redefined by all sequences whose $H^{(p)}(\nabla)$ transform is in ℓ_p. This leads us together with (1) to the fact that

$$|C_\alpha|_p(\nabla)=\big(\ell_p\big)_{H^{(p)}(\nabla)}. \tag{4}$$

Now, we begin with following theorems which are required in the study.

Theorem 2.1. The difference space $|C_\alpha|_p(\nabla)$ is a BK-space with the norm $\|x\|_{|C_\alpha|_p(\nabla)}=\big\|H^{(p)}(\nabla)(x)\big\|_{\ell_p}$, that is

$$\|x\|_{|C_\alpha|_p(\nabla)}=\left(\sum_{n=1}^{\infty}\left|H_n^{(p)}(\nabla)(x)\right|^p\right)^{1/p}.$$

Proof. It is known that ℓ_p is a BK space according to usual p-norm, (4) holds and the matrix $H^{(p)}(\nabla)$ is a triangle. So, we deduce from Theorem 4.3.2 in [22] that space $|C_\alpha|_p(\nabla)$ is a BK-space with the given norm. This concludes the proof.

Theorem 2.2. The difference space $|C_\alpha|_p(\nabla)$ is linearly isomorphic to the space ℓ_p for $p\geq 1$, that is, $|C_\alpha|_p(\nabla)\cong\ell_p$.

Proof. We should show the existence of a linear bijection between the spaces $|C_\alpha|_p(\nabla)$ and ℓ_p. Consider the transformation $H^{(p)}(\nabla):|C_\alpha|_p(\nabla)\to\ell_p$ such that $H^{(p)}(\nabla)(x)=y$ defined by (3). The linearity of $H^{(p)}(\nabla)$ is clear and also it is seen that $x=\theta$ whenever $H^{(p)}(\nabla)(x)=\theta$. So, $H^{(p)}(\nabla)$ is injective.
Furthermore, let $y\in\ell_p$ and we define a sequence $x=(x_n)$ by

$$x_n=\sum_{j=1}^{n}\sum_{r=j}^{n}\frac{E_{r-j}^{-\alpha-1}E_j^\alpha}{r}j^{1/p}y_j \tag{5}$$

and so

$$\|x\|_{|C_\alpha|_p(\nabla)}=\left\|H^{(p)}(\nabla)(x)\right\|_{\ell_p}=\left(\sum_{n=1}^{\infty}\left|H_n^{(p)}(\nabla)(x)\right|^p\right)^{\frac{1}{p}}$$
$$=\left(\sum_{n=1}^{\infty}\left|\frac{1}{n^{\frac{1}{p}}E_n^\alpha}\sum_{v=1}^{n}E_{n-v}^{\alpha-1}v\nabla x_v\right|^p\right)^{\frac{1}{p}}$$
$$=\|y\|_{\ell_p}.$$

Therefore, $H^{(p)}(\nabla)$ is norm preserving and $x\in|C_\alpha|_p(\nabla)$ for all $y\in\ell_p$, namely, $H^{(p)}(\nabla)$ is surjective. Consequently, $H^{(p)}(\nabla)$ is a linear bijection, this leads the fact that $|C_\alpha|_p(\nabla)\cong\ell_p$, which concludes the proof.

Now, we determine the Schauder basis of the space $|C_\alpha|_p(\nabla)$.

A sequence (b_n) is called a Schauder basis (or briefly basis) of a normed sequence space X, if for each $x\in X$, there exists a unique sequence (α_n) of scalars such that

$$\lim_{m\to\infty}\left\|x-\sum_{k=0}^{m}\alpha_k b_k\right\|_X=0$$

and in this case, we write $x=\sum_{k=0}^{\infty}\alpha_k b_k$.

Since $|C_\alpha|_p(\nabla)\cong\ell_p$, the Schauder basis of the new space $|C_\alpha|_p(\nabla)$ is the inverse image of the basis $\left(e^{(k)}\right)_{k=0}^{\infty}$ of the space ℓ_p, where $e^{(n)}$ $(n=0,1,\dots)$ is the sequence with $e_n^{(n)}=1$, $e_v^{(n)}=0(v\neq n)$ for all $n\geq 0$.

So, we have the following theorem without proof.

Theorem 2.3. Let $\alpha_k=\left(H^{(p)}(\nabla)(x)\right)_k$, for all $k\in\mathbb{N}$. Define the sequence $\tau^{(j)}=\left(\tau_n^{(j)}\right)$ as

$$\tau_n^{(j)}=\begin{cases}j^{1/p}\sum_{r=j}^{n}\frac{E_{r-j}^{-\alpha-1}E_j^\alpha}{r}, 1\leq j\leq n\\ 0, j>n.\end{cases}$$

The sequence $\tau^{(j)}$ is a basis for the space $|C_\alpha|_p(\nabla)$ and any $x\in|C_\alpha|_p(\nabla)$ has a unique representation of the form

$$x=\sum_{j=1}^{\infty}\alpha_j\tau^{(j)}.$$

3. DUAL SPACES AND MATRIX TRANSFORMATIONS

We devote the last section of the paper to determine the α, β and γ -duals of spaces $|C_\alpha|_p(\nabla)$ and to give characterizations of certain matrix classes concerning the spaces $|C_\alpha|_p(\nabla)$.

We continue with quoting following lemmas due to Stieglitz and Tietz [23], Sarıgöl [24] and Maddox [25] for our main results.

Lemma 3.1 [23]. The following statements hold:

a-) $T=(t_{nk})\in(\ell_1,c)$ if and only if

$$\lim_{n\to\infty}t_{nk} \text{ exists for each } k\in\mathbb{N} \tag{6}$$

and

$$\sup_{n,k}|t_{nk}|<\infty. \tag{7}$$

b-) Let $1<p<\infty$. Then, $T=(t_{nk})\in\left(\ell_p,c\right)$ if and only if (6) holds, and

$$\sup_n\sum_{k=0}^{\infty}|t_{nk}|^q<\infty. \tag{8}$$

c-) $T=(t_{nk})\in(\ell_1,\ell_\infty)$ if and only if (7) holds.

d-) Let $1<p<\infty$. Then, $T=(t_{nk})\in\left(\ell_p,\ell_\infty\right)\Leftrightarrow$ (8) holds.

e-) $T=(t_{nk})\in(\ell_1,c_0)\Leftrightarrow$ (7) holds, and

$$\lim_{n\to\infty}t_{nk}=0, \qquad \text{for each } k\in\mathbb{N}. \tag{9}$$

f-) Let $1<p<\infty$. Then, $T=(t_{nk})\in\left(\ell_p,c_0\right)\Leftrightarrow$ (8) and (9) hold.

Lemma 3.2 [24]. Let $1<p<\infty$. Then, $T=(t_{nk})\in\left(\ell_p,\ell_1\right)$ if and only if

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}|t_{nk}|\right)^q<\infty.$$

Lemma 3.3 [25]. Let $1\leq p<\infty$. Then, $T=(t_{nk})\in\left(\ell_1,\ell_p\right)$ if and only if

$$\sup_k \sum_{n=0}^{\infty} |t_{nk}|^p < \infty.$$

We now give details about duals of the spaces $|C_\alpha|_p(\nabla)$.

Theorem 3.4. Let define the sets Λ_1 and Λ_2 as follows.

$$\Lambda_1 = \left\{ a = (a_n) \in \omega : \sum_{j=1}^{\infty} \left(\sum_{n=j}^{\infty} \left| \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} \right|^q \right) < \infty \right\}$$

and

$$\Lambda_2 = \left\{ a = (a_n) \in \omega : \sup_j \sum_{n=j}^{\infty} \left| \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j \right| < \infty \right\}.$$

Then, the α-dual of the spaces $|C_\alpha|_p(\nabla)$ for $p > 1$ and $|C_\alpha|_1(\nabla)$ are given by

$$\{|C_\alpha|_p(\nabla)\}^{\alpha} = \Lambda_1$$

and

$$\{|C_\alpha|_1(\nabla)\}^{\alpha} = \Lambda_2,$$

respectively.

Proof. Let $a = (a_n) \in w$ and $p > 1$. Then, we write

$$a_n x_n = a_n \sum_{j=1}^{n} \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} y_j = \sum_{j=1}^{n} \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} y_j = (F^p y)_n,$$

where the matrix $F^p = (f_{nj}^p)$ is defined via the sequence $a = (a_n)$ by

$$f_{nj}^p = \begin{cases} \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p}, 1 \le j \le n \\ 0, j > n. \end{cases}$$

Therefore, we deduce that $ax = (a_n x_n) \in \ell_1$ whenever $x \in |C_\alpha|_p(\nabla)$ if and only if $F^p y \in \ell_1$ whenever $y \in \ell_p$, which implies that $a \in \{|C_\alpha|_p(\nabla)\}^{\alpha}$ if and only if $F^p \in (\ell_p, \ell_1)$ by Lemma 3.2, we obtain $a \in \{|C_\alpha|_p(\nabla)\}^{\alpha}$ if and only if

$$\sum_{j=1}^{\infty} \left(\sum_{n=j}^{\infty} \left| \sum_{r=j}^{n} \frac{a_n E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} \right|^q \right) < \infty.$$

Thus, we have $\{|C_\alpha|_p(\nabla)\}^{\alpha} = \Lambda_1$.

Using Lemma 3.3 instead of Lemma 3.2, the proof can be completed in a similar way.

Theorem 3.5. Let define the sets Λ_3, Λ_4 and Λ_5 by

$$\Lambda_3 = \left\{ a = (a_n) \in \omega : \sup_m \sum_{j=1}^{m} \left| \sum_{n=j}^{m} a_n \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} \right|^q < \infty \right\}, \quad (10)$$

$$\Lambda_4 = \left\{ a = (a_n) \in \omega : \sup_{m,j} \left| \sum_{n=j}^{m} a_n \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j \right| < \infty \right\}, \quad (11)$$

and

$$\Lambda_5 = \left\{ a = (a_n) \in \omega : \lim_{m\to\infty} \sum_{n=j}^{m} a_n \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1}}{r} \; exists \; for \; each \; j \in \mathbb{N} \right\},$$

respectively. Then, the β-dual of the spaces $|C_\alpha|_p(\nabla)$ for $p > 1$ and $|C_\alpha|_1(\nabla)$ are given by

$$\{|C_\alpha|_p(\nabla)\}^{\beta} = \Lambda_3 \cap \Lambda_5$$

and

$$\{|C_\alpha|_1(\nabla)\}^{\beta} = \Lambda_4 \cap \Lambda_5$$

respectively.

Proof. Let $a = (a_n) \in w$ and $p > 1$. Then, we consider the following equation.

$$\sum_{n=1}^{m} a_n x_n = \sum_{n=1}^{m} a_n \sum_{j=1}^{n} \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} y_j = \sum_{j=1}^{m} j^{1/p} E_j^{\alpha} \sum_{n=j}^{m} a_n \sum_{r=j}^{n} \frac{E_{r-j}^{-\alpha-1}}{r} y_j = \sum_{j=1}^{m} b_{mj} y_j = (By)_m,$$

where the matrix $B=(b_{mj})$ is defined via the sequence $a=(a_n)$ by

$$b_{mj}=\begin{cases} j^{1/p}E_j^{\alpha}\sum_{n=j}^{m}a_n\sum_{r=j}^{n}\frac{E_{r-j}^{-\alpha-1}}{r}, 1\le j\le m,\\ 0, j>m.\end{cases}$$

Therefore, we deduce that $ax=(a_nx_n)\in cs$ whenever $x\in|C_\alpha|_p(\nabla)$ if and only if $By\in c$ whenever $y\in\ell_p$, which implies that $a\in\{|C_\alpha|_p(\nabla)\}^{\beta}$ if and only if $B\in(\ell_p,c)$, by part b-) of Lemma 3.1, we obtain that $a\in\{|C_\alpha|_p(\nabla)\}^{\beta}$ if and only if

$$\sup_m\sum_{j=1}^{m}\left|j^{1/p}E_j^{\alpha}\sum_{n=j}^{m}a_n\sum_{r=j}^{n}\frac{E_{r-j}^{-\alpha-1}}{r}\right|^q<\infty$$

and

$$\lim_{m\to\infty}\sum_{n=j}^{m}a_n\sum_{r=j}^{n}\frac{E_{r-j}^{-\alpha-1}}{r}\, exists\ for\ each\ j\in\mathbb{N}.$$

Thus, we have $\{|C_\alpha|_p(\nabla)\}^{\beta}=\Lambda_3\cap\Lambda_5$.

Using part a-) instead of part b-) of Lemma 3.1, the proof can be completed in a similar way.

Since the proof is similar to the previous one, we give following theorem without proof.

Theorem 3.6. Let define the sets Λ_3 and Λ_4 by (10) and (11), respectively. The γ-dual of the spaces $|C_\alpha|_p(\nabla)$ for $p>1$ and $|C_\alpha|_1(\nabla)$ are given by

$$\{|C_\alpha|_p(\nabla)\}^{\gamma}=\Lambda_3$$

and

$$\{|C_\alpha|_1(\nabla)\}^{\gamma}=\Lambda_4,$$

respectively.

Now, we characterize matrix transformations from $|C_\alpha|_p(\nabla)$ to ℓ_∞, c, c_0. Let us define the matrix $B^{(p)}=\left(b_{nj}^{(p)}\right)$ via an infinite matrix $T=(t_{nk})$ by

$$b_{nj}^{(p)}=\sum_{k=j}^{\infty}t_{nk}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}E_j^{\alpha}}{r}j^{1/p}. \tag{12}$$

We may begin with characterization of matrix classes $(|C_\alpha|_1(\nabla),X)$, where $X=\{\ell_\infty,c,c_0\}$.

Theorem 3.7. Consider the matrix $B^{(p)}=\left(b_{nk}^{(p)}\right)$ as in (12) with $p=1$. Then,

i-) $T=(t_{nk})\in(|C_\alpha|_1(\nabla),\ell_\infty)$ if and only if

$$\lim_{m\to\infty}\sum_{k=j}^{m}t_{nk}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}}{r}\, exists\ for\ all\ n,j\in\mathbb{N}, \tag{13}$$

$$\sup_{m,j}\left|\sum_{k=j}^{m}t_{nk}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}E_j^{\alpha}}{r}j\right|<\infty,\quad for\ each\ n\in\mathbb{N}, \tag{14}$$

$$\sup_{n,k}\left|b_{nk}^{(1)}\right|<\infty. \tag{15}$$

ii-) $T=(t_{nk})\in(|C_\alpha|_1(\nabla),c)$ if and only if (13), (14), (15) hold and

$$\lim_{n\to\infty}b_{nk}^{(1)}\, exists\ for\ each\ k\in\mathbb{N}.$$

iii-) $T=(t_{nk})\in(|C_\alpha|_1(\nabla),c_0)$ if and only if (13), (14), (15) hold and

$$\lim_{n\to\infty}b_{nk}^{(1)}=0, for\ each\ k\in\mathbb{N}.$$

Proof. i-) $T=(t_{nk})\in(|C_\alpha|_1(\nabla),\ell_\infty)$ iff Tx exists and is in ℓ_∞ for all $x\in|C_\alpha|_1(\nabla)$. Then $(t_{nk})_{k=1}^{\infty}\in\left(|C_\alpha|_1(\nabla)\right)^{\beta}$ and so the conditions (13) and (14) hold. Moreover, the series $\Sigma_k t_{nk}x_k$ converges uniformly in n and so

$$\lim_{n\to\infty}T_n(x)=\sum_{k=0}^{\infty}\lim_{n\to\infty}t_{nk}x_k. \tag{16}$$

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To prove necessity and sufficiency of (15), let $x\in|C_\alpha|_1(\nabla)$ be given and consider the operator $H^{(1)}(\nabla):|C_\alpha|_1(\nabla)\to\ell_1$ defined by (3) with $p=1$. Further, $x\in|C_\alpha|_1(\nabla)$ iff $y=H^{(1)}(\nabla)(x)\in\ell_1$, and also by (5), let us consider the equality

$$\begin{aligned}\sum_{k=1}^{m}t_{nk}x_k&=\sum_{k=1}^{m}t_{nk}\sum_{j=1}^{k}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}E_j^{\alpha}}{r}jy_j\\&=\sum_{j=1}^{m}\sum_{k=j}^{m}t_{nk}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}E_j^{\alpha}}{r}jy_j\\&=\sum_{j=1}^{m}\psi_{mj}^{(n)}y_j,\end{aligned} \tag{17}$$

where

$$\psi_{mj}^{(n)}=\begin{cases}\sum_{k=j}^{m}t_{nk}\sum_{r=j}^{k}\frac{E_{r-j}^{-\alpha-1}E_j^{\alpha}}{r}j, 1\le j\le m\\ 0, \qquad j>m.\end{cases}$$

Then, since $y\in\ell_1$ and $\Psi^{(n)}=\left(\psi_{mj}^{(n)}\right)\in(\ell_1,c)$, $\Psi^{(n)}$ exists and so the series $\sum_j\psi_{mj}^{(n)}y_j$ converges uniformly for every $n \in \mathbb{N}$. Hence, by (16), this yields us under the assumption that as $m \to \infty$ in (17),

$$T_n(x) = \sum_{j=1}^{\infty} \left(\lim_{m\to\infty} \psi_{mj}^{(n)} \right) y_j = \sum_{j=1}^{\infty} b_{nj}^{(1)} y_j = B_n^{(1)}(y),$$

where $b_{nj}^{(1)} = \lim_{m\to\infty} \psi_{mj}^{(n)}$. This means that $Tx \in \ell_\infty$ whenever $x \in |C_\alpha|_1(\nabla)$ if and only if $B^{(1)} y \in \ell_\infty$ whenever $y \in \ell_1$. Therefore, it follows from part c-) of Lemma 3.1 that $B^{(1)} \in (\ell_1, \ell_\infty)$ iff (15) is satisfied, and this step completes the proof of the part i-).

Since ii-) and iii-) are proved easily as in i-) using parts a-), e-) instead of part c-) of Lemma 3.1, so we omit the detail.

Now, we prove the following result on matrix transformations.

Theorem 3.8. Let $1 < p < \infty$ and define the matrix $B^{(p)} = \left(b_{nk}^{(p)} \right)$ as in (12). Then,

i-) $T = (t_{nk}) \in \left(|C_\alpha|_p(\nabla), \ell_\infty \right)$ if and only if (13) holds, and

$$\sup_m \sum_{j=1}^{m} \left| \sum_{k=j}^{m} t_{nk} \sum_{r=j}^{k} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} \right|^q < \infty, \quad for\ all\ n \geq 1, \tag{18}$$

$$\sup_n \sum_{k=1}^{\infty} \left| b_{nk}^{(p)} \right|^q < \infty. \tag{19}$$

ii-) $T = (t_{nk}) \in \left(|C_\alpha|_p(\nabla), c \right)$ if and only if (13), (18), (19) hold, and

$$\lim_{n\to\infty} b_{nk}^{(p)} exists\ for\ each\ k \in \mathbb{N}.$$

iii-) $T = (t_{nk}) \in \left(|C_\alpha|_p(\nabla), c_0 \right)$ if and only if (13), (18), (19) hold, and

$$\lim_{n\to\infty} b_{nk}^{(p)} = 0, for\ each\ k \in \mathbb{N}.$$

Proof. i-) Given $T = (t_{nk}) \in \left(|C_\alpha|_p(\nabla), \ell_\infty \right)$. Then, equivalently, Tx exists and is in ℓ_∞ for all $x \in |C_\alpha|_p(\nabla)$. Then $(t_{nk})_{k=1}^{\infty} \in \left(|C_\alpha|_p(\nabla) \right)^\beta$ and so the conditions (13) and (18) hold. Moreover, the series $\Sigma_k t_{nk} x_k$ converges uniformly in n and so (16) holds.

To prove necessity and sufficiency of (19), consider the operator $H^{(p)}(\nabla) : |C_\alpha|_p(\nabla) \to \ell_p$ defined by (3) and let $x \in |C_\alpha|_p(\nabla)$ be given. Then $x \in |C_\alpha|_p(\nabla)$ iff $y = H^{(p)}(\nabla)(x) \in \ell_p$. Let us now consider the following equality derived by using the relation (5),

$$\begin{aligned} \sum_{k=1}^{m} t_{nk} x_k &= \sum_{k=1}^{m} t_{nk} \sum_{j=1}^{k} \sum_{r=j}^{k} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} y_j \\ &= \sum_{j=1}^{m} \sum_{k=j}^{m} t_{nk} \sum_{r=j}^{k} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p} y_j \\ &= \sum_{j=1}^{m} \tilde{\psi}_{mj}^{(n)} y_j, \end{aligned} \tag{20}$$

where

$$\tilde{\psi}_{mj}^{(n)} = \begin{cases} \sum_{k=j}^{m} t_{nk} \sum_{r=j}^{k} \frac{E_{r-j}^{-\alpha-1} E_j^{\alpha}}{r} j^{1/p}, & 1 \leq j \leq m \\ 0, & j > m. \end{cases}$$

Then, since $y \in \ell_p$ and $\tilde{\Psi}^{(n)} = \left(\tilde{\psi}_{mj}^{(n)} \right) \in \left(\ell_p, c \right)$, $\tilde{\Psi}^{(n)}$ exists and so the series $\sum_j \tilde{\psi}_{mj}^{(n)} y_j$ converges uniformly for every $n \in \mathbb{N}$. Therefore, if we pass to the limit in (20) as $m \to \infty$, then we obtain by (16) that

$$T_n(x) = \sum_{j=1}^{\infty} \left(\lim_{m\to\infty} \tilde{\psi}_{mj}^{(n)} \right) y_j = \sum_{j=1}^{\infty} b_{nj}^{(p)} y_j = B_n^{(p)}(y),$$

where $b_{nj}^{(p)} = \lim_{m\to\infty} \tilde{\psi}_{mj}^{(n)}$, $n \geq 1$. Thus, we deduce that $Tx \in \ell_\infty$ whenever $x \in |C_\alpha|_p(\nabla)$ if and only if $B^{(p)} y \in \ell_\infty$ whenever $y \in \ell_p$, which implies that $B^{(p)} \in \left(\ell_p, \ell_\infty \right)$, and so it follows from part d-) of Lemma 3.1 that $B^{(p)} \in \left(\ell_p, \ell_\infty \right)$ iff (19) is satisfied. This completes the proof of part i-) of the theorem.

Since parts ii-) and iii-) can be proved by using the similar way of that used in the proof of part i-) taking account of parts b-) and f-) instead of part d-) of Lemma 3.1, respectively; we leave the details to the reader.

REFERENCES

[1] Aydın C, Başar F. Some new difference sequence spaces. Appl. Math. Comput. 2004;157(3):677–693.

[2] Başar F, Altay B. On the space of sequences of p-bounded variation and related matrix mappings. Ukrainian Math. J. 2003;55(1):136–147.

[3] Çolak R, Et M. On some generalized difference sequence spaces and related matrix transformations. Hokkaido Math. J. 1997;26(3):483–492.

[4] Demiriz S, Çakan C. Some topological and geometrical properties of a new difference sequence space. Abstract and Applied Analysis. 2011; Volume 2011, Article ID 213878: 14 pages, doi:10.1155/2011/213878.

[5] Demiriz S, Çakan C. Some new paranormed sequence spaces and α −core of a sequence. Pure and Applied Mathematics Letters. 2016; Volume 2016: 32-45.

[6] Duyar O, Demiriz S, Özdemir O. On some new generalized difference sequence spaces of nonabsolute type. Journal of Mathematics. 2014;

Volume 2014, Article ID 876813: 13 pages, http://dx.doi.org/10.1155/2014/876813.
[7] Ellidokuzoğlu HB, Demiriz S. Euler-Riesz Difference Sequence Spaces. Turk. J. Math. Comput. Sci. 2017;7:63-72.
[8] Hazar GC, Sarıgöl MA. On absolute Nörlund spaces and matrix operators. Acta Math. Sin. (Engl. Ser.). 2018;34(5):812-826.
[9] Hazar Güleç GC, Sarıgöl MA. Compact and Matrix Operators on the Space $|C,-1|_k$. J. Comput. Anal. Appl. 2018;25(6):1014-1024.
[10] Hazar Güleç GC. Compact Matrix Operators on Absolute Cesàro Spaces. Numer. Func. Anal. Opt. 2020;41(1):1-15.
[11] Hazar Güleç GC. Characterization of some classes of compact and matrix operators on the sequence spaces of Cesàro means. Operator and Matrices. 2019;13(3):809-822.
[12] İlkhan M, Kara EE. A new Banach space defined by Euler totient matrix operator. Operator Matrices. 2019;13(2):527-544.
[13] İlkhan M, Demiriz S, Kara EE. A new paranormed sequence space defined by Euler totient matrix. Karaelmas Sci. Eng. J. 2019; 9(2).
[14] Sarıgöl MA. On difference sequence spaces. J. Karadeniz Tech. Univ. Fac. Arts Sci. Ser. Math.-Phys. 1987;10:63-71.
[15] Sarıgöl MA. Matrix transformations on fields of absolute weighted mean summability. Studia Sci. Math. Hungar. 2011;48(3):331-341.
[16] Sarıgöl MA. Spaces of series summable by absolute Cesàro and matrix operators. Comm. Math Appl. 2016;7(1):11-22.
[17] Sezer SA, Çanak İ. On a Tauberian theorem for the weighted mean method of summability. Kuwait J. Sci. 2015;42:1-9.
[18] Kizmaz H. On certain sequence spaces. Canad. Math. Bull. 1981;24(2):169–176.
[19] Orhan C. Matrix transformations on Cesàro difference sequence spaces. Comm. Fac. Sci. Univ. Ankara Ser. A1 1984;33(1):1–8.
[20] Polat H, Altay B. On some new Euler difference sequence spaces. Southeast Asian Bull. Math. 2006;30:209–220.
[21] Flett TM. On an extension of absolute summability and some theorems of Littlewood and Paley. Proc. London Math. Soc. 1957;7:113-141.
[22] Wilansky A. Summability Through Functional Analysis. North-Holland Mathematical Studies. vol. 85. Elsevier Science Publisher; 1984.
[23] Stieglitz M, Tietz H. Matrixtransformationen von folgenraumen eine ergebnisüberischt. Math Z. 1977;154:1-16.
[24] Sarıgöl MA. Extension of Mazhar's theorem on summability factors. Kuwait J. Sci. 2015;42(3):28-35.
[25] Maddox IJ. Elements of functional analysis. Cambridge University Press. London, New York; 1970.