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Suzuki - $(\mathcal{Z}_\psi(\alpha,\beta))$ - type rational contractions

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Abstract

In this paper, we obtain a unique common fixed point results by using Suzuki - $(\mathcal{Z}_\psi(\alpha,\beta))$ - type rational contractive mappings in metric spaces. Also we give an example which supports our main theorem.

Keywords: Metric space Suzuki-$\mathcal{Z}_\psi(\alpha,\beta)$-type rational contraction Z- contraction.

1. Introduction

In 2008, the generalization theorem of Banach contraction principle [1], which was introduced by T.Suzuki [3], later this theorem is also referred as Suzuki type contraction. In 2012, Samet et al. [4] introduced the concept of $\alpha-\psi$-contractive and α - admissible mappings and obtained various fixed point theorems for such mappings in complete metric spaces.
Recently, Khojasteh et al. [5] introduced the notion of Simulation function and the notion of Z - contraction with respect to η which generalized the Banach contractions. Following this direction of research, we introduce the notion Suzuki - $\mathcal{Z}_\psi(\alpha,\beta)$ - type rational contractive mapping and establish common fixed point theorems for such mappings in metric spaces.

Throughout this paper, N denotes the set of all nonnegative integers. Further, R represent the real numbers and $R^+ = [0,\infty)$.

2. Preliminaries

Recently, Khojasteh et al. [5] introduced the notion of Simulation function and the notion of Z - contraction with respect to η which generalized the Banach contractions. (see, ([6]- [13])

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Definition 2.1. *[5] Let $\eta : [0,\infty) \to [0,\infty)$ be a mapping, then η is called a simulation function if it satisfies the following condtions:*

(η_1) $\eta(0,0) = 0$,

(η_2) $\eta(t,s) < s - t$ *for all* $t, s > 0$,

(η_3) *if* $\{t_n\}$,$\{s_n\}$ *are the sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *then* $\lim_{n\to\infty} sup\eta(t_n, s_n) < 0$. *We denote the set of all simulation function by* Z.

Joonaghany et al. [6] proposed a new notion, the ψ-simulation function, and with the help of it, the $\mathcal{Z}_\psi$-contraction in the setting of the standard metric space. The notion of the $\mathcal{Z}_\psi$-contraction covers several distinct types of contraction, including the Z-contraction that was defined in [5]

$$\Psi = \{\psi : R^+ \to R^+ | \psi \text{ is continuous and nondecreasing, and } \psi(r) = 0 \Leftrightarrow r = 0\}$$

Definition 2.2. *[6] We say that $\zeta : R^+ \times R^+ \to R$ is a ψ-simulation function, if there exists $\psi \in \Psi$ such that:*

(ζ_1) $\zeta(t,s) < \psi(s) - \psi(t)$ *for all* $t, s > 0$,

(ζ_2) *if* $\{t_n\}$,$\{s_n\}$ *are the sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *then* $\lim_{n\to\infty} sup\zeta(t_n, s_n) < 0$.

Let $\mathcal{Z}_\psi$ be the set of all ψ- simulation functions. Note that if we take ψ as an identity mapping, then "ψ-simulation" becomes "simulation function" in the sencse of [5]

Example 2.3. *[6] Let $\psi \in \Psi$*

(*i*) $\zeta_1(t,s) = k\psi(s) - \psi(t)$ *for all* $t, s \in [0,\infty,$ *where* $k \in [0,1)$.

(*ii*) $\zeta_2(t,s) = \phi(\psi(s)) - \psi(t)$ *for all* $t, s \in [0,\infty,$ *where* $\phi : [0,+\infty) \to [0,+\infty)$ *so that* $\phi(0) = 0$ *and for each* $s > 0$, $\phi(s) < s$

$$\lim_{t\to s} sup\, \phi(t) < s$$

(*iii*) $\zeta_3(t,s) = \psi(s) - \phi(s) - \psi(t)$ *for all* $t, s \in [0,\infty)$,
where $\phi : [0,+\infty) \to [0,+\infty)$ *is a mapping such that, for each* $s > 0$,

$$\lim_{t\to s} inf\, \phi(t) > 0.$$

It is clear that $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{Z}_\psi$.

Remark 2.4. *Each simulation function forms a ψ- simulation function. The contrary of the statement is false [6].*

Lemma 2.5. *(See e.g., [2]) Let (X, d) be a metric space, and let $\{\rho_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

If $\{\rho_{2n}\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ and $d(\rho_{2m_k}, \rho_{2n_k}) \geq \epsilon$ and

(*i*) $\lim_{n\to\infty} d(\rho_{2m_k}, \rho_{2n_k}) = \epsilon$

(ii) $\lim_{n\to\infty} d(\rho_{2m_k-1}, \rho_{2n_{k+1}}) = \epsilon$

(iii) $\lim_{n\to\infty} d(\rho_{2m_k}, \rho_{2n_k+1}) = \epsilon$

(iv) $\lim_{n\to\infty} d(\rho_{2m_{k-1}}, \rho_{2n_k}) = \epsilon.$

In 2012, Samet et al. [4] introduced the class of α - admissible mappings.

Definition 2.6. *[4] A mapping $\mathcal{F} : X \to X$ is called α- admissible if for all $\rho, \sigma \in X$ we have*

$$\alpha(\rho, \sigma) \geq 1 \Rightarrow \alpha(\mathcal{F}\rho, \mathcal{F}\sigma) \geq 1,$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

Definition 2.7. *Let X be a nonempty set, $\mathcal{F}, \mathcal{G} : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$. The two mappings $(\mathcal{F}, \mathcal{G})$ is called a pair of (α, β) - admissible mappings, if*

$$\alpha(\rho, \sigma) \geq 1 \;\; and \;\; \beta(\rho, \sigma) \geq 1$$

implies

$$\alpha(\mathcal{F}\rho, \mathcal{G}\sigma) \geq 1 \;\; and \;\; \beta(\mathcal{G}\rho, \mathcal{F}\sigma) \geq 1 \; and \; \beta(\mathcal{F}\rho, \mathcal{G}\sigma) \geq 1 \;\; and \;\; \alpha(\mathcal{G}\rho, \mathcal{F}\sigma) \geq 1 \;\; for\ all \;\; \rho, \sigma \in X.$$

Motivated by the above results, we introduce the notion of Suzuki- $(\mathcal{Z}_\psi(\alpha, \beta))$ - type rational contraction and prove some common fixed point results in metric spaces. Also we give an example which supports our main theorem.

3. Main Results

We begin with the following notion:

Definition 3.1. *Let (X, d) be a metric space. Let $\mathcal{F}, \mathcal{G} : X \to X$ be two mappings. we say that the pair $(\mathcal{F}, \mathcal{G})$ is Suzuki - $\mathcal{Z}_{(\psi)}(\alpha, \beta)$ - type rational contraction if for all $\rho, \sigma \in X$ and $L \geq 0$*

$\frac{1}{2}\min\{d(\rho, \mathcal{F}\rho), d(\sigma, \mathcal{G}\sigma)\} \leq d(\rho, \sigma)$ *implies*

$$\zeta(\alpha(\rho, \mathcal{F}\rho)\beta(\sigma, \mathcal{G}\sigma)d(\mathcal{F}\rho, \mathcal{G}\sigma), M(\rho, \sigma)) \geq 0 \tag{1}$$

where $\zeta \in \mathcal{Z}_\psi$

$$M(\rho, \sigma) = \max\left\{d(\rho, \sigma), \frac{d(\rho,\mathcal{F}\rho)[1+d(\sigma,\mathcal{G}\sigma)]}{1+d(\rho,\sigma)}, \frac{d(\sigma,\mathcal{G}\sigma)[1+d(\rho,\mathcal{F}\rho)]}{1+d(\rho,\sigma)}, \frac{d(\sigma,\mathcal{G}\sigma)d(\rho,\mathcal{F}\rho)}{d(\rho,\sigma)}\right\} + L\min\{d(\rho, \mathcal{F}\rho), d(\sigma, \mathcal{F}\rho)\}$$

Theorem 3.2. *Let (X, d) be a complete metric space, and let $\mathcal{F}, \mathcal{G} : X \to X$ be two mappings and $\alpha, \beta : X \times X \to [0, \infty)$. Suppose that the following conditions are satisfied:*

(i) $(\mathcal{F}, \mathcal{G})$ is pair of (α, β)- admissible mappings;

(ii) there exists $\rho_0 \in X$ such that $\alpha(\rho_0, \mathcal{F}\rho_0) \geq 1$ and $\beta(\rho_0, \mathcal{G}\rho_0) \geq 1$;

(iii) the pair $(\mathcal{F}, \mathcal{G})$ is Suzuki- $\mathcal{Z}_{(\psi)}(\alpha, \beta)$ - type rational contraction;

(iv) either, $\mathcal{F}$ and $\mathcal{G}$ are continuous or
for every sequence $\{\rho_n\}$ in X such that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ and
$\beta(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in N_0$ and $\rho_n \to x$, we have $\alpha(\rho, \mathcal{F}\rho) \geq 1$ and $\beta(\rho, \mathcal{G}\rho) \geq 1$.

Then $\mathcal{F}$ and $\mathcal{G}$ have a unique common fixed point in X.

Proof. By assumption there exists $\rho_0 \in X$ such that $\alpha(\rho_0, \mathcal{F}\rho_0) \geq 1$. Define the sequence $\{\rho_n\}$ in X by letting $\rho_1 \in X$ such that $\rho_1 = \mathcal{F}\rho_0, \rho_2 = \mathcal{G}\rho_1, \rho_3 = \mathcal{F}\rho_2$,
$\rho_4 = \mathcal{G}\rho_3$,
continuing this process we get $\mathcal{F}\rho_n = \rho_{n+1}$, $\mathcal{G}\rho_{n+1} = \rho_{n+2}$ where $n \geq 0$.
Since $(\mathcal{F}, \mathcal{G})$ is a pair of $(\alpha, \beta) - admissible$, so
$\alpha(\rho_0, \mathcal{F}\mathcal{G}\rho_0) = \alpha(\rho_0, \rho_1) \geq 1$, $\alpha(\mathcal{F}\rho_0, \mathcal{G}\rho_1) = \alpha(\rho_1, \rho_2) \geq 1$ and $\alpha(\mathcal{G}\rho_1, \mathcal{F}\rho_2) = \alpha(\rho_2, \rho_3) \geq 1$
continuing this manner, we obtain

$$\alpha(\rho_n, \rho_{n+1}) \geq 1 \quad for\ all \quad n \geq 0.$$

Similarly, we can get

$$\beta(\rho_n, \rho_{n+1}) \geq 1 \quad for\ all \quad n \geq 0.$$

If $\rho_n = \rho_{n+1}$ for some $n \in N$, then $u = \rho_n$ is a common fixed point for $\mathcal{F}$ or $\mathcal{G}$.
Consequently, we suppose that $\rho_n \neq \rho_{n+1}$ for all $n \in N$.
Since $\frac{1}{2}\min\{d(\rho_{2n}, \mathcal{F}\rho_{2n}), d(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})\} \leq d(\rho_{2n}, \rho_{2n+1})$
from 1, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\rho_{2n}, \mathcal{F}\rho_{2n})\beta(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})d(\mathcal{F}\rho_{2n}, \mathcal{G}\rho_{2n+1}), M(\rho_{2n}, \rho_{2n+1})) \\ 0 &< \psi(M(\rho_{2n}, \rho_{2n+1})) - \psi(\alpha(\rho_{2n}, \mathcal{F}\rho_{2n})\beta(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})d(\rho_{2n+1}, \rho_{2n+2})), \end{aligned}$$

so

$$\psi(M(\rho_{2n}, \rho_{2n+1})) > \psi(\alpha(\rho_{2n}, \mathcal{F}\rho_{2n})\beta(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})d(\rho_{2n+1}, \rho_{2n+2})).$$

Since ψ is strictly increasing,

$$M(\rho_{2n}, \rho_{2n+1}) > \alpha(\rho_{2n}, \mathcal{F}\rho_{2n})\beta(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})d(\rho_{2n+1}, \rho_{2n+2}), \tag{2}$$

on the other hand,

$$\begin{aligned} &M(\rho_{2n}, \rho_{2n+1}) \\ &= \max\left\{ \begin{array}{c} d(\rho_{2n}, \rho_{2n+1}), \frac{d(\rho_{2n}, \mathcal{F}\rho_{2n})[1+d(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})]}{1+d(\rho_{2n}, \rho_{2n+1})}, \\ \frac{d(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})[1+d(\rho_{2n}, \mathcal{F}\rho_{2n})]}{1+d(\rho_{2n}, \rho_{2n+1})}, \frac{d(\rho_{2n+1}, \mathcal{G}\rho_{2n+1})d(\rho_{2n}, \mathcal{F}\rho_{2n})}{d(\rho_{2n}, \rho_{2n+1})} \\ +L\min\{d(\rho_{2n}, \mathcal{F}\rho_{2n}), d(\rho_{2n+1}, \mathcal{F}\rho_{2n})\} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(\rho_{2n}, \rho_{2n+1}), \frac{d(\rho_{2n}, \rho_{2n+1})[1+d(\rho_{2n+1}, \rho_{2n+2})]}{1+d(\rho_{2n}, \rho_{2n+1})}, \\ \frac{d(\rho_{2n+1}, \rho_{2n+2})[1+d(\rho_{2n}, \rho_{2n+1})]}{1+d(\rho_{2n}, \rho_{2n+1})}, \frac{d(\rho_{2n+1}, \rho_{2n+2})d(\rho_{2n}, \rho_{2n+1})}{d(\rho_{2n}, \rho_{2n+1})} \\ +L\min\{d(\rho_{2n}, \rho_{2n+1}), d(\rho_{2n+1}, \rho_{2n+1})\} \end{array} \right\} \\ &= \max\left\{ d(\rho_{2n}, \rho_{2n+1}), \frac{d(\rho_{2n}, \rho_{2n+1})[1+d(\rho_{2n+1}, \rho_{2n+2})]}{1+d(\rho_{2n}, \rho_{2n+1})}, d(\rho_{2n+1}, \rho_{2n+2}) \right\} \end{aligned}$$

for refining the inequality above, we shall consider the following Cases:
Case(i): If $M(\rho_{2n}, \rho_{2n+1}) = d(\rho_{2n+1}, \rho_{2n+2})$, then by 2 we have
$d(\rho_{2n+1}, \rho_{2n+2}) > d(\rho_{2n+1}, \rho_{2n+2})$, which is a contradiction.
Case(ii): If $M(\rho_{2n}, \rho_{2n+1}) = d(\rho_{2n}, \rho_{2n+1})$, then the inequality 2 turns into the inequality

$$d(\rho_{2n+1}, \rho_{2n+2}) < d(\rho_{2n}, \rho_{2n+1}). \tag{3}$$

Case(iii): Suppose that

$$M(\rho_{2n}, \rho_{2n+1}) = \frac{d(\rho_{2n}, \rho_{2n+1})[1+d(\rho_{2n+1}, \rho_{2n+2})]}{1+d(\rho_{2n}, \rho_{2n+1})}.$$

This yields

$$\max\{d(\rho_{2n}, \rho_{2n+1}), d(\rho_{2n+1}, \rho_{2n+2})\} \leq \frac{d(\rho_{2n}, \rho_{2n+1})[1 + d(\rho_{2n+1}, \rho_{2n+2})]}{1 + d(\rho_{2n}, \rho_{2n+1})}. \tag{4}$$

We shall illustrate that this case is not possible. For this reason, we consider the following subcases:
$Case(iii)_a$: Suppose $\max\{d(\rho_{2n},\rho_{2n+1}),d(\rho_{2n+1},\rho_{2n+2})\}=d(\rho_{2n+1},\rho_{2n+2})$, that is,

$$d(\rho_{2n},\rho_{2n+1})\leq d(\rho_{2n+1},\rho_{2n+2}) \tag{5}$$

on the other hand, from 4, we have

$$d(\rho_{2n},\rho_{2n+1})\leq\frac{d(\rho_{2n},\rho_{2n+1})[1+d(\rho_{2n+1},\rho_{2n+2})]}{1+d(\rho_{2n},\rho_{2n+1})}. \tag{6}$$

By a simple conclusion, we have, from the inequality above, that $d(\rho_{2n+1},\rho_{2n+2})<d(\rho_{2n},\rho_{2n+1})$, which contradicts the assumption 5.
$Case(iii)_b$: Assume that

$$\max\{d(\rho_{2n},\rho_{2n+1}),d(\rho_{2n+1},\rho_{2n+2})\}=d(\rho_{2n},\rho_{2n+1}),$$

that is,

$$d(\rho_{2n+1},\rho_{2n+2})<d(\rho_{2n},\rho_{2n+1}). \tag{7}$$

Furthermore, from 6, we have

$$d(\rho_{2n},\rho_{2n+1})\leq\frac{d(\rho_{2n},\rho_{2n+1})[1+d(\rho_{2n+1},\rho_{2n+2})]}{1+d(\rho_{2n},\rho_{2n+1})}. \tag{8}$$

A simple evaluation implies, from the inequality above, that

$$d(\rho_{2n},\rho_{2n+1})<d(\rho_{2n+1},\rho_{2n+2})$$

which contradicts the assumption 7. Hence, Case(iii) does not occur. Hence,

$$d(\rho_{2n+1},\rho_{2n+2})<d(\rho_{2n},\rho_{2n+1}).$$

Hence, we deduce that the sequence $\{d(\rho_n,\rho_{n+1})\}$ is nonnegative and nonincreasing.
Consequently, there exists $r\geq 0$ such that $\lim_{n\to\infty}d(\rho_n,\rho_{n+1})=r$.
We claim that $r=0$. Suppose, on the contrary, that $r>0$.

$$\lim_{n\to\infty}d(\rho_n,\rho_{n+1})=\lim_{n\to\infty}M(\rho_n,\rho_{n+1})=r. \tag{9}$$

For each $n\geq 0$ we have $\frac{1}{2}\min\{d(\rho_{2n},\mathcal{F}\rho_{2n}),d(\rho_{2n+1},\mathcal{G}\rho_{2n+1})\}\leq d(\rho_{2n},\rho_{2n+1})$
from 1, we have

$$\zeta(\alpha(\rho_{2n},\mathcal{F}\rho_{2n})\beta(\rho_{2n+1},\mathcal{G}\rho_{2n+1})d(\mathcal{F}\rho_{2n},\mathcal{G}\rho_{2n+1}),M(\rho_{2n},\rho_{2n+1}))\geq 0$$

so

$$\lim_{n\to\infty}sup\,\zeta(\alpha(\rho_{2n},\mathcal{F}\rho_{2n})\beta(\rho_{2n+1},\mathcal{G}\rho_{2n+1})d(\mathcal{F}\rho_{2n},\mathcal{G}\rho_{2n+1}),M(\rho_{2n},\rho_{2n+1}))\geq 0. \tag{10}$$

Therefore, from (ζ_2)

$$\lim_{n\to\infty}sup\,\zeta(\alpha(\rho_{2n},\mathcal{F}\rho_{2n})\beta(\rho_{2n+1},\mathcal{G}\rho_{2n+1})d(\mathcal{F}\rho_{2n},\mathcal{G}\rho_{2n+1}),M(\rho_{2n},\rho_{2n+1}))<0,$$

which contradicts 10. So the claim is proved, that is,

$$\lim_{n\to\infty}d(\rho_n,\rho_{n+1})=\lim_{n\to\infty}M(\rho_n,\rho_{n+1})=0. \tag{11}$$

Now, we will show that $\{\rho_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{\rho_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon_0 > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$ and $d(\rho_{2m_k}, \rho_{2n_k}) \geq \epsilon_0$

and

(i) $\lim_{n\to\infty} d(\rho_{2m_k}, \rho_{2n_k}) = \epsilon_0$

(ii) $\lim_{n\to\infty} d(\rho_{2m_k-1}, \rho_{2n_{k+1}}) = \epsilon_0$

(iii) $\lim_{n\to\infty} d(\rho_{2m_k}, \rho_{2n_k+1}) = \epsilon_0$

(iv) $\lim_{n\to\infty} d(\rho_{2m_{k-1}}, \rho_{2n_k}) = \epsilon_0$.

Therefore, from the definition of $M(\rho, \sigma)$ we have

$$\begin{aligned}
&\lim_{n\to\infty} M(\rho_{2n_k}, \rho_{2m_k-1}) \\
&= \lim_{n\to\infty} \max\left\{\begin{array}{c} d(\rho_{2n_k}, \rho_{2m_{k-1}}), \frac{d(\rho_{2n_k},\rho_{2n_{k+1}})[1+d(\rho_{2m_{k-1}},\rho_{2m_{k+1}})]}{1+d(\rho_{2n_k},\rho_{2m_{k-1}})}, \\ \frac{d(\rho_{2m_{k-1}},\rho_{2m_{k+1}})[1+d(\rho_{2n_k},\rho_{2n_{k+1}})]}{1+d(\rho_{2n_k},\rho_{2m_{k-1}})}, \frac{d(\rho_{2m_{k-1}},\rho_{2m_{k+1}})d(\rho_{2n_k},\rho_{2n_{k+1}})}{d(\rho_{2n_k},\rho_{2m_{k-1}})} \end{array}\right\} \\
&\quad + L\min\{d(\rho_{2n_k}, \rho_{2n_{k+1}}), d(\rho_{2m_k-1}, \rho_{2n_{k+1}})\} \\
&= \max\{0, \epsilon_0\} = \epsilon_0
\end{aligned}$$

so

$$\lim_{k\to\infty} d(\rho_{2m_k}, \rho_{2n_k+1}) = \lim_{k\to\infty} M(\rho_{2n_k}, \rho_{2m_k-1}) = \epsilon_0 > 0.$$

Hence, ζ_2 implies that

$$\lim_{k\to\infty} d(\rho_{2m_k}, \rho_{2n_k+1}) = \lim_{k\to\infty} M(\rho_{2n_k}, \rho_{2m_k-1}) = \epsilon_0 > 0. \tag{12}$$

On the other hand, we claim that for sufficiently large $k \in N$, if $n_k > m_k > k$, then

$$\frac{1}{2}\min\{d(\mathcal{F}\rho_{n_k}, \rho_{n_k}), d(\rho_{m_k-1}, \mathcal{G}\rho_{m_k-1})\} > d(\rho_{n_k}, \rho_{m_k-1}) \tag{13}$$

on letting as $k \to \infty$ in 13, we get that $\epsilon_0 \leq 0$, contradiction. Therefore

$$\frac{1}{2}\min\{d(\mathcal{F}\rho_{n_k}, \rho_{n_k}), d(\rho_{m_k-1}, \mathcal{G}\rho_{m_k-1})\} \leq d(\rho_{n_k}, \rho_{m_k-1})$$

and from 1, we have

$$0 \leq \quad \zeta(\alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(\rho_{2m_k-1}, \mathcal{G}\rho_{2m_k-1})d(\mathcal{F}\rho_{2n_k}, \mathcal{G}\rho_{2m_k-1}), M(\rho_{2n_k}, \rho_{2m_k-1}))$$

Hence

$$\lim_{k\to\infty} sup\, \zeta(\alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(\rho_{2m_k-1}, \mathcal{G}\rho_{2m_k-1})d(\mathcal{F}\rho_{2n_k}, \mathcal{G}\rho_{2m_k-1}), M(\rho_{2n_k}, \rho_{2m_k-1})) \geq 0$$

which contradicts 12. This contradiction proves that $\{\rho_n\}$ is a Cauchy sequence and, since X is complete, there exists $u \in X$ such that $\{\rho_n\} \to u$ as $n \to \infty$.

We claim that u is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are continuous, we deduce that

$$u = \lim_{n\to\infty} \rho_{2n+1} = \lim_{n\to\infty} \mathcal{F}\rho_{2n} = \mathcal{F}(\lim_{n\to\infty} \rho_{2n}) = \mathcal{F}u$$

and

$$u = \lim_{n\to\infty} \rho_{2n+2} = \lim_{n\to\infty} \mathcal{G}\rho_{2n+1} = \mathcal{G}(\lim_{n\to\infty} \rho_{2n+1}) = \mathcal{G}u.$$

Therefore $\mathcal{F}u = \mathcal{G}u = u$, that is, u is a common fixed point of $\mathcal{F}$ and $\mathcal{G}$.
Since from (iv), we have
for every sequence $\{\rho_n\}$ in X such that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ and $\beta(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in N_0$ and $\rho_n \to u$ as $n \to \infty$, this implies $\rho_{2n_k+1} \to u$ and $\rho_{2n_k+2} \to u$ as $k \to \infty$.
Now we show that $\mathcal{F}u = \mathcal{G}u = u$.
Suppose $u \neq \mathcal{G}u$.
Now we claim that, for each $n \geq 1$, atleast one of the following assertions holds.

$$\frac{1}{2}d(\rho_{n_k-1}, \rho_{n_k}) \leq d(\rho_{n_k-1}, u)$$

or

$$\frac{1}{2}d(\rho_{n_k}, \rho_{n_k+1}) \leq d(\rho_{n_k}, u).$$

On contrary suppose that

$$\frac{1}{2}d(\rho_{n_k-1}, \rho_{n_k}) > d(\rho_{n_k-1}, u)$$

and

$$\frac{1}{2}d(\rho_{n_k}, \rho_{n_k+1}) > d(\rho_{n_k}, u).$$

For some $n \geq 1$. Then we have

$$\begin{aligned} d(\rho_{n_k-1}, \rho_{n_k}) &\leq d(\rho_{n_k-1}, u) + d(u, \rho_{n_k}) \\ &< \tfrac{1}{2}[d(\rho_{n_k-1}, \rho_{n_k}) + d(\rho_{n_k}, \rho_{n_k+1})] \\ &\leq d(\rho_{n_k-1}, \rho_{n_k}), \end{aligned}$$

which is a contradiction and so the claim holds.
From 1 we have $\frac{1}{2}\min\{d(\rho_{2n_k}, \mathcal{F}\rho_{2n_k}), d(u, \mathcal{G}u)\} \leq d(\rho_{2n_k}, u)$ implies

$$\begin{aligned} 0 &\leq \zeta(\alpha(\rho_{2n_k}, \mathcal{G}\rho_{2n_k})\beta(u, \mathcal{G}u)d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u), M(\rho_{2n_k}, u)) \\ &< \psi(M(\rho_{2n_k}, u)) - \psi(\alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(u, \mathcal{G}u)d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u)) \end{aligned}$$

$$\psi(M(\rho_{2n_k}, u)) > \psi(\alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(u, \mathcal{G}u)d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u)).$$

Since ψ is strictly increasing,

$$\alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(u, \mathcal{G}u)d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u) < M(\rho_{2n_k}, u) \tag{14}$$

on the other hand,

$$\begin{aligned} &M(\rho_{2n_k}, u) \\ &\quad = \max\left\{d(\rho_{2n_k}, u), \tfrac{d(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})[1+d(u,\mathcal{G}u)]}{1+d(\rho_{2n_k}, u)}, \tfrac{d(u,\mathcal{G}u)[1+d(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})]}{1+d(\rho_{2n_k}, u)}, \tfrac{d(u,\mathcal{G}u)d(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})}{d(\rho_{2n_k}, u)}\right\} \\ &\qquad + L\min\{d(\rho_{2n_k}, \mathcal{F}\rho_{2n_k}), d(u, \mathcal{F}\rho_{2n_k}). \end{aligned}$$

Taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} M(\rho_{2n_k}, u) = d(u, \mathcal{G}u).$$

Since, from 14, we have

$$\begin{aligned} d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u) &\leq \alpha(\rho_{2n_k}, \mathcal{F}\rho_{2n_k})\beta(u, \mathcal{G}u)d(\mathcal{F}\rho_{2n_k}, \mathcal{G}u) \\ &< M(\rho_{2n_k}, u). \end{aligned} \tag{15}$$

Taking limit $k \to \infty$, in 15 gives $d(u, \mathcal{G}u) < d(u, \mathcal{G}u)$, a contradiction. Hence $u = \mathcal{G}u$. Similarly, we can find that $u = \mathcal{F}u$. Hence, the pair $(\mathcal{F}, \mathcal{G})$ has a common fixed point $u = \mathcal{F}u = \mathcal{G}u$.

We claim $\mathcal{F}$ and $\mathcal{G}$ have a unique common fixed points $u, v \in X$. Therefore $\mathcal{F}u = \mathcal{G}u = u$, $\mathcal{F}v = \mathcal{G}v = v$ and $d(u, v) > 0$.
Therefore

$$\frac{1}{2}\min\{d(u, \mathcal{F}u), d(v, \mathcal{G}v)\} = \frac{1}{2}\min\{0, 0\} = 0 < d(u, v),$$

from 1 we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(u, \mathcal{F}u)\beta(v, \mathcal{G}v)d(\mathcal{F}u, gv), M(u, v)) \\ &< \psi(M(u, v)) - \psi(\alpha(u, \mathcal{F}u)\beta(v, \mathcal{G}v)d(u, v)), \end{aligned}$$

Since ψ is strictly increasing,

$$d(u, v) < \alpha(u, \mathcal{F}u)\beta(v, \mathcal{G}v)d(u, v) < M(u, v) \tag{16}$$

on the other hand,

$$\begin{aligned} &M(u, v) \\ &\quad = \max\left\{d(u, v), \tfrac{d(u, \mathcal{F}u)[1 + d(v, \mathcal{G}v)]}{1 + d(u, v)}, \tfrac{d(v, \mathcal{G}v)[1 + d(u, \mathcal{F}u)]}{1 + d(u, v)}, \tfrac{d(v, \mathcal{G}v)d(u, \mathcal{F}u)}{d(u, v)}\right\} \\ &\quad + L\min\{d(u, \mathcal{F}u), d(v, \mathcal{F}u) \\ &\quad = \max\left\{d(u, v), \tfrac{d(u, u)[1 + d(v, v)]}{1 + d(u, v)}, \tfrac{d(v, v)[1 + d(u, u)]}{1 + d(u, v)}, \tfrac{d(v, v)d(u, u)}{d(u, v)}\right\} \\ &\quad = d(u, v) > 0. \end{aligned}$$

Therefore, from 15, we have

$$d(u, v) < \alpha(u, \mathcal{F}u)\beta(v, v\mathcal{G})d(u, v) < M(u, v) = d(u, v)$$

a contradiction. Hence $\mathcal{F}$ and $\mathcal{G}$ have a unique common fixed point. □

Corollary 3.3. *Let (X, d) be a complete metric space, and let $\mathcal{F} : X \to X$ be a mapping and $\alpha, \beta : X \times X \to [0, \infty)$. Suppose that the following conditions are satisfied:*

(i) if for all $\rho, \sigma \in X$

$\frac{1}{2}\min\{d(\rho, \mathcal{F}\rho), d(\sigma, \mathcal{F}\sigma)\} \leq d(\rho, \sigma)$ *implies*

$$\zeta(\alpha(\rho, \mathcal{F}\rho)\beta(\sigma, \mathcal{F}\sigma)d(\mathcal{F}\rho, \mathcal{F}\sigma), M(\rho, \sigma)) \geq 0 \tag{17}$$

where $\zeta \in \mathcal{Z}_\psi$

$$\begin{aligned} &M(\rho, \sigma) \\ &\quad = \max\left\{d(\rho, \sigma), \tfrac{d(\rho, \mathcal{F}\rho)[1 + d(\sigma, \mathcal{F}\sigma)]}{1 + d(\rho, \sigma)}, \tfrac{d(\sigma, \mathcal{F}\sigma)[1 + d(\rho, \mathcal{F}\rho)]}{1 + d(\rho, \sigma)}, \tfrac{d(\sigma, \mathcal{F}\sigma)d(\rho, \mathcal{F}\rho)}{d(\rho, \sigma)}\right\} \\ &\quad + L\min\{d(\rho, \mathcal{F}\rho), d(\sigma, \mathcal{F}\rho)\} \end{aligned}$$

(ii) $\mathcal{F}$ is (α, β) admissible mapping;

(iii) there exists $\rho_0 \in X$ such that $\alpha(\rho_0, \mathcal{F}\rho_0) \geq 1$;

*(iv) either, $\mathcal{F}$ is continuous or
for every sequence $\{\rho_n\}$ in X such that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ and
$\beta(\rho_n, \rho_{n+1}) \geq 1$ for all $n \in N_0$ and $\rho_n \to x$, we have $\alpha(\rho, \mathcal{F}\rho) \geq 1$ and $\beta(\rho, \mathcal{F}\rho) \geq 1$.*

Then $\mathcal{F}$ has a unique fixed point in X.

Example 3.4. *Let $X = [0, \infty)$, and let $d : X \times X \to [0, \infty)$ be defined by*

$$d(\rho,\sigma) = \begin{cases} \max\{\rho,\sigma\} & if\ \rho \neq \sigma, \\ 0 & \rho = \sigma. \end{cases}$$

*We define $\mathcal{F}, \mathcal{G} : X \to X$ by $\mathcal{F}(\rho) = \frac{\rho}{2}$ and $\mathcal{G}(\rho) = \frac{\rho}{3}$ for all $\rho \in X$.
Clearly (X, d) is complete and $\mathcal{F}$ and $\mathcal{G}$ are continuous self- mappings on X and $\alpha, \beta : X \times X \to [0, \infty)$ are two mappings defined by*

$$\alpha(\rho,\sigma) = \begin{cases} 1 & if\ \rho,\sigma \in [0,1], \\ 0 & otherwise \end{cases}$$

and

$$\beta(\rho,\sigma) = \begin{cases} 1 & if\ \rho,\sigma \in [0,1], \\ 0 & otherwise \end{cases}$$

We now define $\zeta : [0, \infty) \times [0, \infty) \to [0, \infty)$ by $\zeta(t, s) = \frac{1}{2}\psi(s) - \psi(t)$, for all $s, t \in [0, \infty)$ and $\psi(t) = \frac{t}{2}$ Now

$$\frac{1}{2} \min\{d(\rho, \mathcal{F}\rho), d(\sigma, g\sigma)\} \leq d(\rho, \sigma)$$

implies

$$\begin{aligned} \zeta(\alpha(\rho,\mathcal{F}\rho)\beta(\sigma,\sigma)d(\mathcal{F}\rho,\mathcal{F}\sigma), M(\rho,\sigma)) &= \tfrac{1}{2}\psi(M(\rho,\sigma)) - \psi(\alpha(\rho,\mathcal{F}\rho)\beta(\sigma,\mathcal{G}\sigma)d(\mathcal{F}\rho,\mathcal{G}\sigma)) \\ &= \tfrac{1}{2}\psi(M(\rho,\sigma)) - \psi(d(\mathcal{F}\rho,\mathcal{G}\sigma)) \\ &< \tfrac{1}{4}M(\rho,\sigma) - \tfrac{1}{2}d(\mathcal{F}\rho,\mathcal{G}\sigma) \geq 0, \end{aligned}$$

where

$$\begin{aligned} &M(\rho,\sigma) \\ &\quad = \max\left\{d(\rho,\sigma), \tfrac{d(\rho,\mathcal{F}\rho)[1+d(\sigma,\mathcal{G}\sigma)]}{1+d(\rho,\sigma)}, \tfrac{d(\sigma,\mathcal{G}\sigma)[1+d(\rho,\mathcal{F}\rho)]}{1+d(\rho,\sigma)}, \tfrac{d(\sigma,\mathcal{G}\sigma)d(\rho,\mathcal{F}\rho)}{d(\rho,\sigma)}\right\} \\ &\quad +L\min\{d(\rho,\mathcal{F}\rho), d(\sigma,\mathcal{F}\rho)\}. \end{aligned}$$

Hence for $\rho, \sigma \in [0, 1]$ and $L \geq 0$ the pair $(\mathcal{F}, \mathcal{G})$ is a Suzuki - $\mathcal{Z}_{\psi(\alpha,\beta)}$ - type rational contraction. In either case $\alpha(\rho, \sigma) = 0$ and $\beta(\rho, \sigma) = 0$ then pair $(\mathcal{F}, \mathcal{G})$ is a Suzuki - $\mathcal{Z}_{\psi(\alpha,\beta)}$ - type rational contraction. Thus all the assumptions of Theorem 3.2 are satisfied and $\mathcal{F}$ and $\mathcal{G}$ have a common fixed point in X.

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