Advances in the Theory of Nonlinear Analysis and its Applications **5** (2021) No. 2, 193–202.
https://doi.org/10.31197/atnaa.755309
Available online at www.atnaa.org
Research Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Fractional Derivatives and Expansion Formulae of Incomplete H and $\overline{H}$-Functions

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Abstract

In this paper, we investigate the fractional derivatives and expansion formulae of incomplete H and $\overline{H}$-functions for one variable. Further, we also obtain results for repeated fractional order derivatives and some special cases are also discussed. Various other analogues results are also established. The results obtained here are very much helpful for the further research and useful in the study of applied problems of sciences, engineering and technology.

Keywords: Fractional calculus operators, Incomplete Gamma functions, Incomplete H-functions, Mellin-Barnes type contour
2010 MSC: 33B20, 33E20, 26A33.

1. Introduction and Preliminaries

The fractional derivative from Oldham and Spanier [12] (see also; Ross [17]) of complex order α of a function $f(t)$ is defined by

$$ {}_{\beta}\mathfrak{D}_t^{\alpha}\left[f(t)\right] = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_{\beta}^{t} (t-x)^{-\alpha-1} f(x)dx, & \Re(\alpha) < 0, \\ \frac{d^m}{dx^m} \cdot {}_{\beta}\mathfrak{D}_t^{\alpha-m}\left\{f(t)\right\}, & 0 \leq \Re(\alpha) < m, \end{cases} \tag{1}$$

where $m \in I^+$(Positive integer). For our convenience, we use $\mathfrak{D}_t^{\alpha} \equiv {}_0\mathfrak{D}_t^{\alpha}$.

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Received June 20, 2020; Accepted: February 21, 2021; Online: February 24, 2021.

The equation (1) will reduces to Riemman-Liouville representation and Weyl sense for differintegral of arbitrary order if $\beta \to 0$ and $\beta \to \infty$ respectively.
In our present paper, we use the Leibniz rule for the derivative of arbitrary order of a product (see; Ross and Northover [16]) which is the generalization of the n^{th} order derivative.

$$ {}_{\beta}\mathfrak{D}_t^{\alpha}\left\{f(t)g(t)\right\} = \sum_{m=0}^{\infty} \left(\begin{array}{c} \alpha \\ m \end{array} \right) \cdot {}_{\beta}\mathfrak{D}_t^{m}\left\{f(t)\right\} \cdot \left\{ {}_{\beta}\mathfrak{D}_t^{\alpha-m} g(t)\right\}, \quad \forall \alpha. \tag{2} $$

Leibniz's rule helps to predict any interesting transformations, summations, generating functions and extensions referring to the various special functions (including q-functions) of one and often more variables, see papers [1, 3, 10, 11, 13, 23] and references therein.
For the study of incomplete generalized hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, the following pair of Mellin-Barnes type contour integral representation is defined by Srivastava et al. [19] as:

$$ \begin{aligned} {}_p\gamma_q \left[\begin{array}{c} (u_1, x), u_2, ..., u_p; \\ v_1, v_2, ..., v_q; \end{array} z \right] &= \frac{\prod_{j=1}^{q}\Gamma(v_j)}{\prod_{j=1}^{p}\Gamma(u_j)} \sum_{k=0}^{\infty} \frac{\gamma(u_1+k, x)\prod_{j=2}^{p}\Gamma(u_j+k)}{\prod_{j=1}^{q}\Gamma(v_j+k)} \frac{z^k}{k!} \\ &= \frac{1}{2\pi i} \frac{\prod_{j=1}^{q}\Gamma(v_j)}{\prod_{j=1}^{p}\Gamma(u_j)} \int_{\mathfrak{L}} \frac{\gamma(u_1+s, x)\prod_{j=2}^{p}\Gamma(u_j+s)}{\prod_{j=1}^{q}\Gamma(v_j+s)} \Gamma(-s)(-z)^s ds, \end{aligned} \tag{3} $$

and

$$ \begin{aligned} {}_p\Gamma_q \left[\begin{array}{c} (u_1, x), u_2, ..., u_p; \\ v_1, v_2, ..., v_q; \end{array} z \right] &= \frac{\prod_{j=1}^{q}\Gamma(v_j)}{\prod_{j=1}^{p}\Gamma(u_j)} \sum_{k=0}^{\infty} \frac{\Gamma(u_1+k, x)\prod_{j=2}^{p}\Gamma(u_j+k)}{\prod_{j=1}^{q}\Gamma(v_j+k)} \frac{z^k}{k!} \\ &= \frac{1}{2\pi i} \frac{\prod_{j=1}^{q}\Gamma(v_j)}{\prod_{j=1}^{p}\Gamma(u_j)} \int_{\mathfrak{L}} \frac{\Gamma(u_1+s, x)\prod_{j=2}^{p}\Gamma(u_j+s)}{\prod_{j=1}^{q}\Gamma(v_j+s)} \Gamma(-s)(-z)^s ds, \end{aligned} \tag{4} $$

where ($|\arg(-z)| < \pi$) and $\mathfrak{L}$ is the Mellin-Barnes type contour which starts from $\tau - i\infty$ and terminate at $\tau + i\infty$ ($\tau \in \Re$) with indentations that each set poles are separate to each other in the integrand in each case.
Recently, in this sequel of (3) and (4), Srivastava et al. [20] introduce and investigate the corresponding pairs of incomplete H-functions. Further, they also define their various properties. The incomplete H-functions $\gamma_{p,q}^{m,n}$ and $\Gamma_{p,q}^{m,n}$ are defined as follow:

$$ \begin{aligned} \gamma_{p,q}^{m,n}(z) &= \gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (u_1, U_1, x), (u_j, U_j)_{2,p} \\ (v_j, V_j)_{1,q} \end{array} \right. \right] \\ &= \gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (u_1, U_1, x), (u_2, U_2), \cdots, (u_p, U_p) \\ (v_1, V_1), (v_2, V_2), \cdots, (v_q, V_q) \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathfrak{L}} g(s, x) z^{-s} ds, \end{aligned} \tag{5} $$

where

$$ g(s, x) = \frac{\gamma(1-u_1-U_1 s, x)\prod_{j=1}^{m}\Gamma(v_j+V_j s)\prod_{j=2}^{n}\Gamma(1-u_j-U_j s)}{\prod_{j=m+1}^{q}\Gamma(1-v_j-V_j s)\prod_{j=n+1}^{p}\Gamma(u_j+U_j s)}, \tag{6} $$

and

$$ \begin{aligned} \Gamma_{p,q}^{m,n}(z) &= \Gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (u_1, U_1, x), (u_j, U_j)_{2,p} \\ (v_j, V_j)_{1,q} \end{array} \right. \right] \\ &= \Gamma_{p,q}^{m,n}\left[z \left| \begin{array}{c} (u_1, U_1, x), (u_2, U_2), \cdots, (u_p, U_p) \\ (v_1, V_1), (v_2, V_2), \cdots, (v_q, V_q) \end{array} \right. \right] \\ &= \frac{1}{2\pi i} \int_{\mathfrak{L}} G(s, x) z^{-s} ds, \end{aligned} \tag{7} $$

where

$$G(s,x)=\frac{\Gamma(1-u_1-U_1s,x)\prod_{j=1}^{m}\Gamma(v_j+V_js)\prod_{j=2}^{n}\Gamma(1-u_j-U_js)}{\prod_{j=m+1}^{q}\Gamma(1-v_j-V_js)\prod_{j=n+1}^{p}\Gamma(u_j+U_js)}. \tag{8}$$

These functions are exists for all $x \geq 0$ for the same conditions as (see, for details [6, 8, 9, 18]). Also, this decomposition formula holds:

$$\gamma_{p,q}^{m,n}(z)+\Gamma_{p,q}^{m,n}(z)=H_{p,q}^{m,n}(z).$$

Further, we study the $\overline{H}$- function which is defined by Buschman and Srivastava [2] in the form Mellin-Barnes type contour integral of type $\mathfrak{L}_0$. Since due to the wide applications of $\overline{H}$- function for fractional-order differential equation, Srivastava et al. [20] defined the following set of incomplete $\overline{H}$-functions in term of $\overline{\gamma}_{p,q}^{m,n}$ and $\overline{\Gamma}_{p,q}^{m,n}$ as follows:

$$\begin{aligned}\overline{\gamma}_{p,q}^{m,n}(z)&=\overline{\gamma}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(u_j,U_j;\gamma_j)_{2,n},(u_j,U_j)_{n+1,p}\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q}\end{matrix}\right]\\&=\overline{\gamma}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(u_2,U_2;\gamma_2),\cdots,(u_n,U_n;\gamma_n),\\(v_1,V_1),\cdots,(v_m,V_m),\end{matrix}\right.\\&\qquad\left.\begin{matrix}(u_{n+1},U_{n+1}),\cdots,(u_p,U_p)\\(v_{m+1},V_{m+1};\delta_{m+1}),\cdots,(v_q,V_q;\delta_q)\end{matrix}\right]\\&=\frac{1}{2\pi i}\int_{\mathfrak{L}}\overline{g}(s,x)z^{-s}ds,\end{aligned} \tag{9}$$

where

$$\overline{g}(s,x)=\frac{[\gamma(1-u_1-U_1s,x)]^{\gamma_1}\prod_{j=1}^{m}\Gamma(v_j+V_js)\prod_{j=2}^{n}[\Gamma(1-u_j-U_js)]^{\gamma_j}}{\prod_{j=m+1}^{q}[\Gamma(1-v_j-V_js)]^{\delta_j}\prod_{j=n+1}^{p}\Gamma(u_j+U_js)}, \tag{10}$$

and

$$\begin{aligned}\overline{\Gamma}_{p,q}^{m,n}(z)&=\overline{\Gamma}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(u_j,U_j;\gamma_j)_{2,n},(u_j,U_j)_{n+1,p}\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q}\end{matrix}\right]\\&=\overline{\Gamma}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(u_2,U_2;\gamma_2),\cdots,(u_n,U_n;\gamma_n),\\(v_1,V_1),\cdots,(v_m,V_m),\end{matrix}\right.\\&\qquad\left.\begin{matrix}(u_{n+1},U_{n+1}),\cdots,(u_p,U_p)\\(v_{m+1},V_{m+1};\delta_{m+1}),\cdots,(v_q,V_q;\delta_q)\end{matrix}\right]\\&=\frac{1}{2\pi i}\int_{L}\overline{G}(s,x)z^{-s}ds,\end{aligned} \tag{11}$$

where

$$\overline{G}(s,x)=\frac{[\Gamma(1-u_1-U_1s,x)]^{\gamma_1}\prod_{j=1}^{m}\Gamma(v_j+V_js)\prod_{j=2}^{n}[\Gamma(1-u_j-U_js)]^{\gamma_j}}{\prod_{j=m+1}^{q}[\Gamma(1-v_j-V_js)]^{\delta_j}\prod_{j=n+1}^{p}\Gamma(u_j+U_js)}. \tag{12}$$

Also fractional calculus have several applications in modelling real-world problems, interested may refer [5, 7, 14, 15, 21, 22].

2. Main results

In this section, we derive the expansion formulae for incomplete H and $\overline{H}$-functions for one variable with the help of (Leibniz rule for arbitrary order) fractional order derivative (see also [4]), and also find the results for repeated fractional order derivatives. Our results are shown in the below in form of Theorems.

Theorem 2.1. *For $\mu > 0, \lambda > 0$, then*

$$
\begin{aligned}
&{}_{\beta}\mathfrak{D}_{t}^{\alpha}\left\{(et+f)^{\eta}(gt+h)^{\xi}\,.\,\Gamma_{p,q}^{m,n}\left[z\,(et+f)^{\mu}\,(gt+h)^{\lambda}\right]\right\}\\
&= A\sum_{P,Q=0}^{\infty}\phi_{P,Q}\left(\frac{t-\beta}{t+f/e}\right)^{P}\left(\frac{t-\beta}{t+h/g}\right)^{Q}\\
&\cdot\,\Gamma_{p+2,q+2}^{m,n+2}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\;\middle|\;\begin{matrix}(u_1,U_1,x)\,,(-\eta,\mu)\,,(-\xi,\lambda)\,,(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\,,(P-\eta,\mu)\,,(Q-\xi,\lambda)\end{matrix}\right].
\end{aligned}
\tag{13}
$$

Theorem 2.2. *For $\mu_i > 0, \lambda_i > 0$, then*

$$
\begin{aligned}
&{}_{\beta_1}\mathfrak{D}_{t_1}^{\alpha_1}\cdots{}_{\beta_r}\mathfrak{D}_{t_r}^{\alpha_r}\left[\left\{\prod_{i=1}^{R}(e_it_i+f_i)^{\eta_i}\,(g_it_i+h_i)^{\xi_i}\right\}\right.\\
&\left.\,.\left\{\Gamma_{p,q}^{m,n}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}\,(g_it_i+h_i)^{\lambda_i}\right]\right\}\right]\\
&=\prod_{i=1}^{R}(A_i)\sum_{P_i,Q_i=0}^{\infty}\dots\sum_{P_R,Q_R=0}^{\infty}\prod_{i=1}^{R}\left\{\phi_{P_i,Q_i}\left(\frac{t_i-\beta_i}{t_i+f_i/e_i}\right)^{P_i}\left(\frac{t_i-\beta_i}{t_i+h_i/g_i}\right)^{Q_i}\right\}\\
&\cdot\,\Gamma_{p+2R,q+2R}^{m,n+2R}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\;\right|\\
&\qquad\left.\begin{matrix}(u_1,U_1,x)\,,(-\eta_R,\mu_R)\,,(-\xi_R,\lambda_R)\,,(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\,,(P_R-\eta_R,\mu_R)\,,(Q_R-\xi_R,\lambda_R)\end{matrix}\right].
\end{aligned}
\tag{14}
$$

Theorem 2.3. *For $\mu > 0, \lambda > 0$, then*

$$
\begin{aligned}
&{}_{\beta}\mathfrak{D}_{t}^{\alpha}\left\{(et+f)^{\eta}(gt+h)^{\xi}\,.\,\overline{\Gamma}_{p,q}^{m,n}\left[z\,(et+f)^{\mu}\,(gt+h)^{\lambda}\right]\right\}\\
&= A\sum_{P,Q=0}^{\infty}\phi_{P,Q}\left(\frac{t-\beta}{t+f/e}\right)^{P}\left(\frac{t-\beta}{t+h/g}\right)^{Q}\\
&\cdot\overline{\Gamma}_{p+2,q+2}^{m,n+2}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\;\middle|\;\begin{matrix}(u_1,U_1;\gamma_1:x)\,,(-\eta,\mu;1)\,,\\(v_j,V_j)_{1,m}\,,(v_j,V_j;\delta_j)_{m+1,q}\,,\end{matrix}\right.\\
&\qquad\left.\begin{matrix}(-\xi,\lambda;1)\,,\cdots,(u_n,U_n;\gamma_n)\,(u_j,U_j)_{n+1,p}\\(P-\eta,\mu;1)\,,(Q-\xi,\lambda;1)\end{matrix}\right].
\end{aligned}
\tag{15}
$$

Theorem 2.4. *For $\mu_i > 0, \lambda_i > 0$, then*

$$
\begin{aligned}
&{}_{\beta_1}\mathfrak{D}_{t_1}^{\alpha_1}\cdots{}_{\beta_r}\mathfrak{D}_{t_r}^{\alpha_r}\left[\left\{\prod_{i=1}^{R}(e_it_i+f_i)^{\eta_i}\,(g_it_i+h_i)^{\xi_i}\right\}\right.\\
&\left.\,.\left\{\overline{\Gamma}_{p,q}^{m,n}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}\,(g_it_i+h_i)^{\lambda_i}\right]\right\}\right]\\
&=\prod_{i=1}^{R}(A_i)\sum_{P_i,Q_i=0}^{\infty}\cdots\sum_{P_R,Q_R=0}^{\infty}\prod_{i=1}^{R}\left\{\phi_{P_i,Q_i}\left(\frac{t_i-\beta_i}{t_i+f_i/e_i}\right)^{P_i}\left(\frac{t_i-\beta_i}{t_i+h_i/g_i}\right)^{Q_i}\right\}
\end{aligned}
$$

$$\cdot\overline{\Gamma}^{m,n+2R}_{p+2R,q+2R}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\,\middle|\, \begin{matrix}(u_1,U_1;\gamma_1:x),(-\eta_R,\mu_R;1),\\ (v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{matrix}\right.$$

$$\left.\begin{matrix}(-\xi_R,\lambda_R;1),\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\ (P_R-\eta_R,\mu_R;1),(Q_R-\xi_R,\lambda_R;1)\end{matrix}\right]. \tag{16}$$

In this manner, the following results can also be defined for $\gamma^{m,n}_{p,q}$ and $\overline{\gamma}^{m,n}_{p,q}$ as follows:

Theorem 2.5. *For $\mu>0,\lambda>0$, then*

$$_{\beta}\mathfrak{D}^{\alpha}_{t}\left\{(et+f)^{\eta}(gt+h)^{\xi}\,.\gamma^{m,n}_{p,q}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\right]\right\}$$

$$=A\sum_{P,Q=0}^{\infty}\phi_{P,Q}\left(\frac{t-\beta}{t+f/e}\right)^{P}\left(\frac{t-\beta}{t+h/g}\right)^{Q}$$

$$\cdot\,\gamma^{m,n+2}_{p+2,q+2}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\,\middle|\,\begin{matrix}(u_1,U_1,x),(-\eta,\mu),(-\xi,\lambda),(u_j,U_j)_{2,p}\\ (v_j,V_j)_{1,q},(P-\eta,\mu),(Q-\xi,\lambda)\end{matrix}\right]. \tag{17}$$

Theorem 2.6. *For $\mu_i>0,\lambda_i>0$, then*

$$_{\beta_1}\mathfrak{D}^{\alpha_1}_{t_1}\cdots{}_{\beta_r}\mathfrak{D}^{\alpha_r}_{t_r}\left[\left\{\prod_{i=1}^{R}(e_it_i+f_i)^{\eta_i}(g_it_i+h_i)^{\xi_i}\right\}\right.$$

$$\left.\cdot\left\{\gamma^{m,n}_{p,q}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\right]\right\}\right]$$

$$=\prod_{i=1}^{R}(A_i)\sum_{P_i,Q_i=0}^{\infty}\cdots\sum_{P_R,Q_R=0}^{\infty}\prod_{i=1}^{R}\left\{\phi_{P_i,Q_i}\left(\frac{t_i-\beta_i}{t_i+f_i/e_i}\right)^{P_i}\left(\frac{t_i-\beta_i}{t_i+h_i/g_i}\right)^{Q_i}\right\}$$

$$\cdot\gamma^{m,n+2R}_{p+2R,q+2R}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\,\right|$$

$$\left.\begin{matrix}(u_1,U_1,x),(-\eta_R,\mu_R),(-\xi_R,\lambda_R),(u_j,U_j)_{2,p}\\ (v_j,V_j)_{1,q},(P_R-\eta_R,\mu_R),(Q_R-\xi_R,\lambda_R)\end{matrix}\right]. \tag{18}$$

Theorem 2.7. *For $\mu>0,\lambda>0$, then*

$$_{\beta}\mathfrak{D}^{\alpha}_{t}\left\{(et+f)^{\eta}(gt+h)^{\xi}\,.\overline{\gamma}^{m,n}_{p,q}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\right]\right\}$$

$$=A\sum_{P,Q=0}^{\infty}\phi_{P,Q}\left(\frac{t-\beta}{t+f/e}\right)^{P}\left(\frac{t-\beta}{t+h/g}\right)^{Q}$$

$$\cdot\overline{\gamma}^{m,n+2}_{p+2,q+2}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(-\eta,\mu;1),\\ (v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{matrix}\right.$$

$$\left.\begin{matrix}(-\xi,\lambda;1),\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\ (P-\eta,\mu;1),(Q-\xi,\lambda;1)\end{matrix}\right]. \tag{19}$$

Theorem 2.8. *For $\mu_i > 0, \lambda_i > 0$, then*

$$
{}_{\beta_1}\mathfrak{D}_{t_1}^{\alpha_1} \cdots {}_{\beta_r}\mathfrak{D}_{t_r}^{\alpha_r}\left[\left\{\prod_{i=1}^{R}(e_it_i+f_i)^{\eta_i}(g_it_i+h_i)^{\xi_i}\right\}\right.
$$

$$
\left.\cdot\left\{\overline{\gamma}_{p,q}^{m,n}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\right]\right\}\right]
$$

$$
=\prod_{i=1}^{R}(A_i)\sum_{P_i,Q_i=0}^{\infty}\cdots\sum_{P_R,Q_R=0}^{\infty}\prod_{i=1}^{R}\left\{\phi_{P_i,Q_i}\left(\frac{t_i-\beta_i}{t_i+f_i/e_i}\right)^{P_i}\left(\frac{t_i-\beta_i}{t_i+h_i/g_i}\right)^{Q_i}\right\}
$$

$$
\cdot\overline{\gamma}_{p+2R,q+2R}^{m,n+2R}\left[z\prod_{i=1}^{R}(e_it_i+f_i)^{\mu_i}(g_it_i+h_i)^{\lambda_i}\,\middle|\,\begin{matrix}(u_1,U_1;\gamma_1:x),(-\eta_R,\mu_R;1),\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{matrix}\right.
$$

$$
\left.\begin{matrix}(-\xi_R,\lambda_R;1),\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\(P_R-\eta_R,\mu_R;1),(Q_R-\xi_R,\lambda_R;1)\end{matrix}\right]. \tag{20}
$$

In our above results, we consider

$$
A=\frac{(et+f)^{\eta}(gt+h)^{\xi}}{(t-\beta)^{\alpha}};\quad \phi_{P,Q}=\begin{pmatrix}\alpha\\P\end{pmatrix}\begin{pmatrix}\alpha-P\\Q\end{pmatrix}\cdot\frac{1}{\Gamma(1+P+Q-\alpha)},
$$

and

$$
A_i=\frac{(e_it_i+f_i)^{\eta_i}(g_it_i+h_i)^{\xi_i}}{(t_i-\beta_i)^{\alpha_i}};\ \phi_{P_i,Q_i}=\begin{pmatrix}\alpha_i\\P_i\end{pmatrix}\begin{pmatrix}\alpha_i-P_i\\Q_i\end{pmatrix}
$$

$$
\cdot\frac{1}{\Gamma(1+P_i+Q_i-\alpha_i)}
$$

.

Proof. To prove equation (13), let L.H.S. be denoted by I_1, and then, we have

$$
I_1={}_{\beta}\mathfrak{D}_t^{\alpha}\left\{(et+f)^{\eta}(gt+h)^{\xi}.\Gamma_{p,q}^{m,n}\left[z(et+f)^{\mu}(gt+h)^{\lambda}\right]\right\},
$$

now by making use of equation (7), we have

$$
I_1=\frac{1}{2\pi i}\int_{\mathfrak{L}}G(s,x)\,z^{-s}{}_{\beta}\mathfrak{D}_t^{\alpha}\left\{(et+f)^{\eta-\mu s}(gt+h)^{\xi-\lambda s}\right\}ds,
$$

now to solve above fractional derivative, we use Leibniz rule for differentiation of arbitrary order, for this we let $f(t)=(et+f)^{\eta-\mu s}(gt+h)^{\xi-\lambda s}$ and $g(t)=1$, then equation (2) gives

$$
I_1=\frac{1}{2\pi i}\int_{\mathfrak{L}}G(s,x)\,z^{-s}\sum_{P=0}^{\infty}\begin{pmatrix}\alpha\\P\end{pmatrix}\frac{d^P}{dt^P}\left\{(et+f)^{\eta-\mu s}(gt+h)^{\xi-\lambda s}\right\}
$$

$$
\cdot\left\{{}_{\beta}D_t^{\alpha-P}t^0\right\}ds
$$

$$
=\frac{1}{2\pi i}\int_{\mathfrak{L}}G(s,x)\,A.z^{-s}\sum_{P,Q=0}^{\infty}\phi_{P,Q}\frac{\Gamma(1+\eta-\mu s)\Gamma(1+\xi-\lambda s)}{\Gamma(1+\eta-\mu s-P)\Gamma(1+\xi-\lambda s-Q)}
$$

$$
.\left(\frac{t-\beta}{t+f/e}\right)^{P}\left(\frac{t-\beta}{t+h/g}\right)^{Q}ds,
$$

now making the use of equation (7), which gives the desired result. □

In this similar way, we can also obtain the above results (14)-(20), details are omitted here.

3. Special cases

In this section, we establish some special cases and various analogous results by substituting particular values, as if we put $e = g = 1, h = \beta = 0$ in equation (13) and (15), then the fractional derivative of double series representation reduced to single series representation, then for $\mu > 0, \lambda > 0$, we have

Corollary 3.1.

$$\mathfrak{D}_t^\alpha \left\{ t^\xi (t+f)^\eta . \Gamma_{p,q}^{m,n} \left[zt^\lambda (t+f)^\mu \right] \right\} = t^{\xi-\alpha}(t+f)^\eta \sum_{P=0}^{\infty} \begin{pmatrix} \alpha \\ P \end{pmatrix} \left(\frac{t}{t+f} \right)^P$$

$$\cdot \Gamma_{p+2,q+2}^{m,n+2} \left[zt^\lambda (t+f)^\mu \middle| \begin{array}{c} (u_1, U_1, x), (-\eta, \mu), (-\xi, \lambda), (u_j, U_j)_{2,p} \\ (v_j, V_j)_{1,q}, (P-\eta, \mu), (\alpha - P - \xi, \lambda) \end{array} \right] \tag{21}$$

and

Corollary 3.2.

$$\mathfrak{D}_t^\alpha \left\{ t^\xi (t+f)^\eta . \overline{\Gamma}_{p,q}^{m,n} \left[zt^\lambda (t+f)^\mu \right] \right\}$$

$$= t^{\xi-\alpha}(t+f)^\eta \sum_{P=0}^{\infty} \begin{pmatrix} \alpha \\ P \end{pmatrix} \left(\frac{t}{t+f} \right)^P \cdot \overline{\Gamma}_{p+2,q+2}^{m,n+2} \left[zt^\lambda (t+f)^\mu \right|$$

$$\left. \begin{array}{c} (u_1, U_1; \gamma_1 : x), (-\eta, \mu; 1), (-\xi, \lambda; 1), \cdots, (u_n, U_n; \gamma_n), (u_j, U_j)_{n+1,p} \\ (v_j, V_j)_{1,m}, (v_j, V_j; \delta_j)_{m+1,q}, (P-\eta, \mu; 1), (\alpha - P - \xi, \lambda; 1) \end{array} \right]. \tag{22}$$

Proof. To prove the result (21), we start from the L.H.S. by making use of equation (7), we have

$$\mathfrak{D}_t^\alpha \left\{ t^\xi (t+f)^\eta . \Gamma_{p,q}^{m,n} \left[zt^\lambda (t+f)^\mu \right] \right\}$$

$$= \frac{1}{2\pi i} \int_{\mathfrak{L}} G(s,x)\, z^{-s} \mathfrak{D}_t^\alpha \left\{ (t+f)^{\eta-\mu s}\, t^{\xi - \lambda s} \right\} ds,$$

now, we use Leibniz rule for differentiation of arbitrary order, we assume $f(t) = (t+f)^{\eta-\mu s}$ and $g(t) = t^{\xi-\lambda s}$, then equation (2) gives

$$\mathfrak{D}_t^\alpha \left\{ t^\xi (t+f)^\eta . \Gamma_{p,q}^{m,n} \left[zt^\lambda (t+f)^\mu \right] \right\}$$

$$= \frac{1}{2\pi i} \int_{\mathfrak{L}} G(s,x)\, z^{-s} \sum_{P=0}^{\infty} \begin{pmatrix} \alpha \\ P \end{pmatrix} \frac{d^P}{dt^P} \left\{ (t+f)^{\eta-\mu s} \right\} \left\{ D_t^{\alpha-P} (t^{\xi-\lambda s}) \right\} ds$$

$$= \frac{1}{2\pi i} \int_{\mathfrak{L}} G(s,x)\, z^{-s} \sum_{P=0}^{\infty} \begin{pmatrix} \alpha \\ P \end{pmatrix} \frac{\Gamma(\eta - \mu s + 1)}{\Gamma(\eta - \mu s - P + 1)} (t+f)^{\eta - \mu s - P}$$

$$\cdot \frac{\Gamma(\xi - \lambda s + 1)}{\Gamma(\xi - \lambda s - \alpha + P + 1)} t^{\xi - \lambda s - \alpha + P},$$

rearranging the terms and making the use of equation (7), which gives the desired result (21). Similarly we can prove the result (22). □

In this manner, we can also establish the results for $\gamma_{p,q}^{m,n}$ and $\overline{\gamma}_{p,q}^{m,n}$ as below:
For $\mu > 0, \lambda > 0$, then results holds true:

Corollary 3.3.

$$\mathfrak{D}_t^{\alpha}\left\{t^{\xi}\left(t+f\right)^{\eta}.\gamma_{p,q}^{m,n}\left[zt^{\lambda}\left(t+f\right)^{\mu}\right]\right\}=t^{\xi-\alpha}(t+f)^{\eta}\sum_{P=0}^{\infty}\left(\begin{array}{c}\alpha\\P\end{array}\right)\left(\frac{t}{t+f}\right)^{P}$$

$$.\gamma_{p+2,q+2}^{m,n+2}\left[zt^{\lambda}(t+f)^{\mu}\left|\begin{array}{c}(u_1,U_1,x),(-\eta,\mu),(-\xi,\lambda),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q},(P-\eta,\mu),(\alpha-P-\xi,\lambda)\end{array}\right.\right] \tag{23}$$

and

Corollary 3.4.

$$\mathfrak{D}_t^{\alpha}\left\{t^{\xi}\left(t+f\right)^{\eta}.\overline{\gamma}_{p,q}^{m,n}\left[zt^{\lambda}\left(t+f\right)^{\mu}\right]\right\}=t^{\xi-\alpha}(t+f)^{\eta}\sum_{P=0}^{\infty}\left(\begin{array}{c}\alpha\\P\end{array}\right)\left(\frac{t}{t+f}\right)^{P}$$

$$.\overline{\gamma}_{p+2,q+2}^{m,n+2}\left[zt^{\lambda}(t+f)^{\mu}\left|\begin{array}{c}(u_1,U_1;\gamma_1:x),(-\eta,\mu;1),(-\xi,\lambda;1),\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{array}\right.\right.$$

$$\left.\begin{array}{c}\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\(P-\eta,\mu;1),(\alpha-P-\xi,\lambda;1)\end{array}\right]. \tag{24}$$

In particular, if we put $f=0$ in the L.H.S of equation (21), we obtain

Corollary 3.5.

$$\mathfrak{D}_t^{\alpha}\left\{t^{\eta+\xi}.\,\Gamma_{p,q}^{m,n}\left[zt^{\mu+\lambda}\right]\right\}$$

$$=t^{\eta+\xi-\alpha}\cdot\Gamma_{p+1,q+1}^{m,n+1}\left[zt^{\mu+\lambda}\left|\begin{array}{c}(u_1,U_1,x),(-\eta-\xi,\mu+\lambda),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q},(-\eta-\xi+\alpha,\mu+\lambda)\end{array}\right.\right]. \tag{25}$$

Proof. To prove (25), we start from the L.H.S. and then by making use of equation (7), we have

$$\mathfrak{D}_t^{\alpha}\left\{t^{\eta+\xi}.\,\Gamma_{p,q}^{m,n}\left[zt^{\mu+\lambda}\right]\right\}=\frac{1}{2\pi i}\int_{\mathfrak{L}}G(s,x)z^{-s}\mathfrak{D}_t^{\alpha}\left\{t^{(\eta+\xi)-s(\mu+\lambda)}\right\}ds$$

$$=\frac{1}{2\pi i}\int_{L}G(s,x)z^{-s}\frac{\Gamma\left(1+\eta+\xi-s(\mu+\lambda)\right)}{\Gamma\left(1+\eta+\xi-\alpha-s(\mu+\lambda)\right)}t^{\eta+\xi-\alpha-s(\mu+\lambda)}ds,$$

by making the use of equation (7), which gives (25). □

Now, if we compare equation (25) with equation (21) with particular case $f=0$, then we have
For $\mu>0,\lambda>0$, then the results holds true:

Corollary 3.6.

$$\Gamma_{p+1,q+1}^{m,n+1}\left[zt^{\mu+\lambda}\left|\begin{array}{c}(u_1,U_1,x),(-\eta-\xi,\mu+\lambda),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q},(-\eta-\xi+\alpha,\mu+\lambda)\end{array}\right.\right]$$

$$=\sum_{P=0}^{\infty}\left(\begin{array}{c}\alpha\\P\end{array}\right)\cdot\Gamma_{p+2,q+2}^{m,n+2}\left[zt^{\mu+\lambda}\left|\begin{array}{c}(u_1,U_1,x),(-\eta,\mu),(-\xi,\lambda),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q},(P-\eta,\mu),(\alpha-P-\xi,\lambda)\end{array}\right.\right]. \tag{26}$$

Similarly from equation (22), we obtain the following result:

Corollary 3.7.

$$\overline{\Gamma}_{p+1,q+1}^{m,n+1}\left[zt^{\mu+\lambda}\left|\begin{array}{c}(u_1,U_1;\gamma_1:x),(-\eta-\xi,\mu+\lambda;1),\cdots,(u_n,U_n;\gamma_n),\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{array}\right.\begin{array}{c}(u_j,U_j)_{n+1,p}\\(-\eta-\xi+\alpha,\mu+\lambda;1)\end{array}\right]$$

$$=\sum_{P=0}^{\infty}\binom{\alpha}{P}\cdot\overline{\Gamma}_{p+2,q+2}^{m,n+2}\left[zt^{\mu+\lambda}\left|\begin{array}{c}(u_1,U_1,\gamma_1:x),(-\eta,\mu;1),(-\xi,\lambda;1),\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{array}\right.\begin{array}{c}\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\(P-\eta,\mu;1),(\alpha-P-\xi,\lambda;1)\end{array}\right]. \tag{27}$$

Similarly, we can also find the results for repeated fractional derivative from equation (14) and (16) as follows: For $\mu_i>0,\lambda_i>0$, then

Corollary 3.8.

$$\Gamma_{p+R,q+R}^{m,n+R}\left[z\prod_{i=1}^{R}(t_i^{\mu_i+\lambda_i})\left|\begin{array}{c}(u_1,U_1,x),(-\eta_R-\xi_R,\mu_R+\lambda_R),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q},(\alpha_R-\eta_R-\xi_R,\mu_R+\lambda_R)\end{array}\right.\right]$$

$$=\sum_{P_1,\ldots,P_R=0}^{\infty}\prod_{i=1}^{R}(\phi_{P_i,Q_i})$$

$$\cdot\Gamma_{p+2R,q+2R}^{m,n+2R}\left[z\prod_{i=1}^{R}(t_i^{\mu_i+\lambda_i})\left|\begin{array}{c}(u_1,U_1,x),(-\eta_R,\mu_R),(-\xi_R,\lambda_R),\\(v_j,V_j)_{1,q},(P_R-\eta_R,\mu_R),\end{array}\right.\begin{array}{c}(u_j,U_j)_{2,p}\\(\alpha_R-P_R-\xi+R,\lambda_R)\end{array}\right]. \tag{28}$$

and

Corollary 3.9.

$$\overline{\Gamma}_{p+R,q+R}^{m,n+R}\left[z\prod_{i=1}^{R}(t_i^{\mu_i+\lambda_i})\left|\begin{array}{c}(u_1,U_1;\gamma_1:x),((-\eta_R-\xi_R,\mu_R+\lambda_R;1)),\\(v_j,V_j)_{1,m},(v_j,V_j;\delta_j)_{m+1,q},\end{array}\right.\begin{array}{c}\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\((\alpha_R-\eta_R-\xi_R,\mu_R+\lambda_R;1))\end{array}\right].$$

$$=\sum_{P_1,\cdots,P_R=0}^{\infty}\prod_{i=1}^{R}(\phi_{P_i,Q_i})\,\overline{\Gamma}_{p+2R,q+2R}^{m,n+2R}\left[z\prod_{i=1}^{R}(t_i^{\mu_i+\lambda_i})\left|\begin{array}{c}(u_1,U_1;\gamma_1:x),\\(v_j,V_j)_{1,m},(v_j,)\end{array}\right.\begin{array}{c}(-\eta_R,\mu_R;1),(-\xi_R,\lambda_R;1),\cdots,(u_n,U_n;\gamma_n),(u_j,U_j)_{n+1,p}\\(V_j;\delta_j)_{m+1,q},(P_R-\eta_R,\mu_R;1),(\alpha_R-P_R-\xi+R,\lambda_R;1)\end{array}\right]. \tag{29}$$

Similar manner, we can also defined the results for $\gamma_{p,q}^{m,n}$ and $\overline{\gamma}_{p,q}^{m,n}$ of above equations (26), (27), (28) and (29).

4. Conclusions

In this present investigation, we obtained the fractional derivatives and expansion formulae of incomplete H and $\overline{H}$- functions for one variable. We also found from the results for recurrent and repeated fractional order derivatives and discussed about their some special cases. Further, various other analogues results are also established. The results obtained here are very much helpful for the further research and also useful in the study of applied problems of sciences, engineering and technology.

Acknowledgements

The authors express their sincere thanks to the referees for their careful reading and suggestions that helped to improve this paper.

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