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Faster Convergent Modified Lindstedt-Poincare Solution of Nonlinear Oscillators

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Abstract

The modified Lindstedt-Poincare method has been extended to obtain a faster convergent solution of nonlinear oscillators. First of all a classical type Lindstedt-Poincare solution has been determined and then a conversion formula has been used to find the desired solution. The solution has been compared and justified by corresponding numerical solution.

1. Introduction

Poincare [19] developed different methods to solve differential equations. Poincare and Lindstedt developed Lindstedt-Poincare method [1,2]. The Lindstedt-Poincare method [1,2] was originally developed for handling a weak nonlinear oscillator

$$\ddot{x}+\omega_0^2 x+\varepsilon f(x,\dot{x},\ddot{x})=0, \tag{1.1}$$

where ε is a small parameter, ω_0 is a constant, over dots denote differentiation with respect to t and $x(0)=a_0$, $\dot{x}(0)=0$ are the given initial conditions. Then Krylov-Bogoliubov's [3] and multiple time scale [1] methods were presented to investigate Eq. (1.1). The classical perturbation methods agree with numerical solutions (e. g. Runge-Kutta 4^{th} order method [19], finite elements method [5], etc.) when ε is very close to zero.

Several authors [4]- [6], [16] extended the Lindstedt-Poincare method to solve stronger nonlinear problems. Jones [4] presented an approximate technique by introducing a new parameter, $\alpha(\varepsilon)$ rather than the small parameter, ε. Such approximate solution is valid even for large value of ε. Burton [5] presented a modified version of the Lindstedt-Poincare method. Cheung et al. [6] further modified this method. However, all the approximate solutions obtained by approaches of [4]- [6] are effective for Duffing oscillator with cubical nonlinearity. The aim of this article is to present a new form of the modified Lindstedt-Poincare method of Cheung et al. [6] based on the conversion formula presented by Alam et al. [14] by introducing a parameter k. The solutions obtained for various nonlinear oscillators nicely agree with corresponding numerical solutions and provide better results than other existing solutions.

Besides the classical perturbation methods, many approximate techniques have been presented for solving the stronger nonlinear oscillators. Among them the asymptotic expansions [15,18], the homotopy perturbation [7], harmonic balance [8,9], energy balance [10] and iteration methods [11] are widely used. Singular differential equations are also solved using optimal successive complementary expansion method by F. Say [17].

2. The Lindstedt-Poincare method

Introducing a new variable, $\tau=\omega t$, t can be replaced and Eq. (1.1) is written as

$$\omega^2 x''+\omega_0^2 x+\varepsilon f(x,\omega x',\omega^2 x'')=0. \tag{2.1}$$

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Here ω is known as the frequency of the oscillator and the primes denote differentiation with respect to τ. According to Lindstedt-Poincare method [1,2], x and ω can be expanded in powers of ε as

$$x=\sum_{n=0}^{\infty}x_n\varepsilon^n, \tag{2.2}$$

and

$$\omega^2=\omega_0^2+\sum_{n=1}^{\infty}\omega_n\varepsilon^n. \tag{2.3}$$

Earlier it was chosen that $\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\mathscr{O}(\varepsilon^3)$. But Veronis [12] and Burton [5] and Burton et al. [13] used series Eq. (2.3). In this article we have used the series in Eq. 2.3 for faster convergent solution.
By substituting x and ω into Eq. (2.1) and equating the coefficients of like powers of ε, we obtain the following equations:

$$\omega_0^2x_0''+\omega_0^2x_0=0, \tag{2.4}$$

$$\omega_0^2x_1''+\omega_0^2x_1=-2\omega_0\omega_1x_0''-f(x_0,x_0',x_0''), \tag{2.5}$$

$$\omega_0^2x_2''+\omega_0^2x_2=-2\left(\omega_0\omega_1+\omega_1^2\right)x_0''-2\omega_0\omega_1x_1''-x_1\frac{\partial f(x_0,x_0',x_0'')}{\partial x}$$

$$-\left(\omega_0x_1'+\frac{\omega_1x_0'}{2\omega_0}\right)\frac{\partial f(x_0,x_0',x_0'')}{\partial x'}-\left(\omega_0^2x_1''+\omega_1x_0''\right)\frac{\partial f(x_0,x_0',x_0'')}{\partial x''}. \tag{2.6}$$

The initial conditions are usually replaced by $x_0(0)=a_0$, $x_0'(0)=0$, $x_1(0)=x_1'(0)=x_2(0)=0\cdots$, and x_0,x_1 and ω_1,x_2 and ω_2 etc. are determined sequentially. In this article we only follow the initial conditions of $x_0'(0)=x_1'(0)=\cdots=0$, and

$$a_0=x_0(0)+\varepsilon x_1(0)+\varepsilon^2x_2(0)+\mathscr{O}(\varepsilon^3). \tag{2.7}$$

This assumption was introduced in [9] following [3].

3. Conversion formulae

Recently a conversion formula [14] has been presented to the modified Lindstedt-Poincare solution [6] from its classical version. This conversion formula can be used to obtain a faster convergent solution (concern of this article). Cheung et al. [6] reconsidered Eq. (2.3) to the following form

$$\omega^2=\left(\omega_0^2+\varepsilon\omega_1\right)\left(1+\frac{\varepsilon^2\omega_2}{\omega_0^2+\varepsilon\omega_1}+\frac{\varepsilon^3\omega_3}{\omega_0^2+\varepsilon\omega_1}+\mathscr{O}(\varepsilon^4)\right). \tag{3.1}$$

Then a new parameter α is chosen such as

$$\alpha(\varepsilon)=\frac{\varepsilon\omega_1}{\omega_0^2+\varepsilon\omega_1}. \tag{3.2}$$

Thus Eq. (3.1) can be rewritten in a series of α,

$$\omega^2=\frac{\omega_0^2}{(1-\alpha)}\left(1+\sum_{n=2}^{\infty}\delta_n\alpha^n\right). \tag{3.3}$$

Substituting the value of α from Eq. (3.2) into Eq. (3.3), we obtain a power series of ε,

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\frac{\varepsilon^2\omega_1^2\delta_2}{\omega_0^2}+\frac{\varepsilon^3\omega_1^3(-\delta_2+\delta_3)}{\omega_0^4}+\frac{\varepsilon^4\omega_1^4(\delta_2-2\delta_3+\delta_4)}{\omega_0^6}+\mathscr{O}(\varepsilon^5). \tag{3.4}$$

Now Eq. (2.3) and Eq. (3.4) are identical. Therefore, we obtain

$$\frac{\omega_1^2\delta_2}{\omega_0^2}=\omega_2,\ \frac{\omega_1^3(-\delta_2+\delta_3)}{\omega_0^4}=\omega_3,\ \frac{\omega_1^4(\delta_2-2\delta_3+\delta_4)}{\omega_0^6}=\omega_4,\cdots, \tag{3.5}$$

or,

$$\delta_2 = \frac{\omega_2\omega_0^2}{\omega_1^2},\ \delta_3 = \frac{\omega_0^2\omega_1\omega_2+\omega_0^4\omega_3}{\omega_1^3},\ \delta_4 = \frac{\omega_0^2\omega_1^2\omega_2+2\omega_0^4\omega_1\omega_3+\omega_0^6\omega_4}{\omega_1^4},\cdots. \tag{3.6}$$

The above relations measures the unknown coefficients $\delta_2,\delta_3,\cdots$ etc., where $\omega_0,\omega_1,\omega_2,\cdots$ etc. are calculated by classical Lindstedt-Poincare method [1,2]. Thus we can convert the frequency obtained by classical Lindstedt-Poincare method [1,2] to its modified form presented by Cheung et al. [6]. On the other hand transformation Eq. (3.2) makes Eq. (2.2) to the form

$$x = x_0+\alpha\tilde{x_1}+\alpha^2\tilde{x_2}+\mathcal{O}(\alpha^3). \tag{3.7}$$

The unknown coefficients $\tilde{x_1},\tilde{x_2},\cdots$ etc. still to be determined. We can substitute the value of α from Eq. (3.2) into Eq. (3.7) and obtain a series of ε,

$$x = x_0+\frac{\varepsilon\omega_1\tilde{x_1}}{\omega_0^2}+\frac{\varepsilon^2\omega_1^2(-\tilde{x_1}+\tilde{x_2})}{\omega_0^4}+\frac{\varepsilon^3\omega_1^3(\tilde{x_1}-2\tilde{x_2}+\tilde{x_3})}{\omega_0^6}+\mathcal{O}(\varepsilon^4). \tag{3.8}$$

Clearly that Eq. (2.2) is identical to Eq. (3.8). So, comparing equal powers of ε, we obtain the following algebraic equations:

$$\frac{\omega_1\tilde{x_1}}{\omega_0^2} = x_1,\ \frac{\omega_1^2(-\tilde{x_1}+\tilde{x_2})}{\omega_0^4} = x_2,\ \frac{\omega_1^3(\tilde{x_1}-2\tilde{x_2}+\tilde{x_3})}{\omega_0^6} = x_3,\cdots, \tag{3.9}$$

or,

$$\tilde{x_1} = \frac{\omega_0^2x_1}{\omega_1},\ \tilde{x_2} = \frac{\omega_0^2\omega_1x_1+\omega_0^4x_2}{\omega_1^2},\ \tilde{x_3} = \frac{\omega_0^2\omega_1^2x_1+2\omega_0^4\omega_1x_2+\omega_0^6x_3}{\omega_1^3},\cdots. \tag{3.10}$$

When $x_1,x_2,\cdots$ together with $\omega_0,\omega_1,\omega_2,\cdots$ are known, $\tilde{x_1},\tilde{x_2},\cdots$ are found by Eq. (3.10).

4. Example

Let us consider Duffing oscillator (cubical) $\ddot{x}+x+\varepsilon x^3 = 0$. For this problem, $\omega_0 = 1$ and $f(x,\dot{x},\ddot{x}) = x^3$. Therefore, Eqs. (2.4)-(2.6) becomes

$$x_0''+x_0 = 0, \tag{4.1}$$

$$x_1''+x_1 = -\omega_1x_0''-x_0^3, \tag{4.2}$$

$$x_2''+x_2 = -3x_0^2x_1-x_1''\omega_1-x_0''\omega_2. \tag{4.3}$$

The solution of Eq. (4.1) is

$$x_0 = a\cos\tau. \tag{4.4}$$

Substituting this value of x_0 in Eq. (4.2) and simplifying we obtain

$$x_1''+x_1 = \omega_1a\cos\tau-\frac{3}{4}a^3(3\cos\tau+\cos3\tau). \tag{4.5}$$

It is noted that $x_1,x_2,\cdots$ do not contain the fundamental term to avoid secular terms. Therefore, the coefficient of $\cos\tau$ of Eq. (4.5) vanishes. Thus we obtain

$$\omega_1 = \frac{3a^2}{4}. \tag{4.6}$$

The particular solution of Eq. (4.5) is

$$x_1 = \frac{a^3\cos3\tau}{32}. \tag{4.7}$$

According to Lindstedt-Poincare method, $x_1(0) = x_1'(0) = 0$. Therefore, the solution of Eq. (4.5) becomes

$$x_1 = \frac{a^3(-\cos\tau+\cos3\tau)}{32}. \tag{4.8}$$

It has already been mentioned that we do strictly follow this rule. We may consider

$$x_1 = \frac{a^3(-k\cos\tau+\cos3\tau)}{32}, \tag{4.9}$$

where k is a constant.
Alam et al. [9] was chosen a periodic solution of $\ddot{x}+\omega_0^2 x=\varepsilon f(x)$, $x(0)=a_0$, $\dot{x}(0)=0$, as

$$x=a\cos\varphi+a^3C_3(a)\cos 3\varphi+a^5C_5(a)\cos 5\varphi+\mathscr{O}(a^7),$$

where a and $\dot{\varphi}$ are constants. Alam et al. [9] considered above solution by choosing $k=0$.
$k=1$ is strictly followed by Cheung et al. [6] and various methods of perturbation for solving nonlinear oscillators. Thus the value of k can be considered as parameter. This will give us additional variation to find more accurate solutions of nonlinear oscillators. Determination of higher order solution will increase accuracy of the solution. But choosing k as a parameter we have found faster convergent solutions without finding higher order approximations. By finding a proper value of k, solution can be made more accurate with first few approximations. We have introduced k in the first approximate solution and consequently k appear in the second, third and fourth approximations.
Choosing a suitable value of k, we can find a series of ω which converge faster than that of obtained by Cheung et al. [6] and Alam et al. [14]. Carrying on a similar process, we have solved the higher order equations (e.g., Eq. (4.3),$\cdots$) and obtained the following results:

$$\omega_2=-\frac{3}{128}a^4\left(-1+2k\right),\ \omega_3=\frac{3a^6\left(-19+36k+7k^2\right)}{4096},\ \omega_4=-\frac{3a^8\left(-335+556k+342k^2+30k^3\right)}{131072}, \tag{4.10}$$

and

$$x_2=C_{2,1}\cos\tau+C_{2,3}\cos 3\tau+C_{2,5}\cos 5\tau,$$

$$x_3=C_{3,1}\cos\tau+C_{3,3}\cos 3\tau+C_{3,5}\cos 5\tau+C_{3,7}\cos 7\tau,$$

$$x_4=C_{4,1}\cos\tau+C_{4,3}\cos 3\tau+C_{4,5}\cos 5\tau+C_{4,7}\cos 7\tau+C_{4,9}\cos 9\tau, \tag{4.11}$$

where

$$C_{2,1}=\frac{a^5\left(20k+3k^2\right)}{1024},\ C_{2,3}=\frac{-a^5\left(21+3k\right)}{1024},\ C_{2,5}=\frac{a^5}{1024},\ C_{3,1}=-\frac{a^7k\left(375+160k+12k^2\right)}{32768},$$

$$C_{3,3}=\frac{3a^7\left(139+55k+4k^2\right)}{32768},\ C_{3,5}=-\frac{a^7\left(43+5k\right)}{32768},\ C_{3,7}=\frac{a^7}{32768},\ C_{4,1}=\frac{a^9k\left(6521+5750k+1100k^2+55k^3\right)}{1048576},$$

$$C_{4,3}=-\frac{a^9\left(7797+6144k+1125k^2+55k^3\right)}{1048576},\ C_{4,5}=\frac{a^9\left(1340+401k+25k^2\right)}{1048576},\ C_{4,7}=-\frac{a^9(65+7k)}{1048576},\ C_{4,9}=\frac{a^9}{1048576}. \tag{4.12}$$

Now utilizing the transformation formulae Eq. (3.6) and Eq. (3.10), we obtain respectively

$$\delta_2=\frac{1}{24}\left(1-2k\right),\delta_3=\frac{1}{576}\left(5-12k+7k^2\right),\delta_4=\frac{-1+20k-6k^2-30k^3}{13824}, \tag{4.13}$$

and

$$\tilde{x}_1=\tilde{C}_{1,1}\cos\tau+\tilde{C}_{1,3}\cos 3\tau,$$

$$\tilde{x}_2=\tilde{C}_{2,1}\cos\tau+\tilde{C}_{2,3}\cos 3\tau+\tilde{C}_{2,5}\cos 5\tau,$$

$$\tilde{x}_3=\tilde{C}_{3,1}\cos\tau+\tilde{C}_{3,3}\cos 3\tau+\tilde{C}_{3,5}\cos 5\tau+\tilde{C}_{3,7}\cos 7\tau,$$

$$\tilde{x}_4=\tilde{C}_{4,1}\cos\tau+\tilde{C}_{4,3}\cos 3\tau+\tilde{C}_{4,5}\cos 5\tau+\tilde{C}_{4,7}\cos 7\tau+\tilde{C}_{4,9}\cos 9\tau, \tag{4.14}$$

where

$$\tilde{C}_{1,1}=-\frac{ak}{24},\ \tilde{C}_{1,3}=\frac{a}{24},\ \tilde{C}_{2,1}=\frac{ak\left(-4+3k\right)}{576},\ \tilde{C}_{2,3}=\frac{a(1-k)}{192},\ \tilde{C}_{2,5}=\frac{a}{576},\ \tilde{C}_{3,1}=-\frac{ak\left(-9+16k+12k^2\right)}{13824},$$

$$\tilde{C}_{3,3}=\frac{a\left(-5+7k+4k^2\right)}{4608},\ \tilde{C}_{3,5}=-\frac{5a\left(-1+k\right)}{13824},\ \tilde{C}_{3,7}=\frac{a}{13824},\ \tilde{C}_{4,1}=\frac{ak\left(257-586k+236k^2+55k^3\right)}{331776},$$

$$\tilde{C}_{4,3}=-\frac{a\left(237-552k+261k^2+55k^3\right)}{331776},\ \tilde{C}_{4,5}=\frac{a\left(-28+41k+25k^2\right)}{331776},\ \tilde{C}_{4,7}=-\frac{7a\left(-1+k\right)}{331776},\ \tilde{C}_{4,9}=\frac{a}{331776}. \tag{4.15}$$

For the initial conditions, we obtain

$$\tilde{x}_1(0)=\frac{a(1-k)}{24},\ \tilde{x}_2(0)=\frac{a\left(4-7k+3k^2\right)}{576},\ \tilde{x}_3(0)=\frac{a\left(-9+25k-4k^2-12k^3\right)}{13824},$$

$$\tilde{x}_4(0)=\frac{a\left(-257+843k-822k^2+181k^3+55k^4\right)}{331776}. \tag{4.16}$$

It is clear that $\tilde{x}_1(0)=\tilde{x}_2(0)=\tilde{x}_3(0)=\tilde{x}_4(0)=0$ when $k=1$ and $x(0)=a_0=a$. When $k\neq 1$, we obtain the following nonlinear algebraic equation

$$a_0=a\left(1+\frac{a(1-k)\alpha}{24}+\frac{a\left(4-7k+3k^2\right)\alpha^2}{576}+\frac{a\left(-9+25k-4k^2-12k^3\right)\alpha^3}{13824}+\frac{a\left(-257+843k-822k^2+181k^3+55k^4\right)\alpha^4}{331776}\right), \tag{4.17}$$

where $\alpha=\frac{\frac{3a^2}{4}}{1+\frac{3a^2}{4}}$. In general a_0 is given; so that a would be found solving Eq. (4.17) by an iteration method (numerical). It is noted that the higher order terms of α are small whatever the values of a and ε if we chose a suitable value of k. Therefore it requires one or two iterations to obtain a desired result.

5. Results and discussion

A faster convergent modified Lindstedt-Poincare solution has been determined. The solution is identical to that of Cheung et al. [6] and Alam et al. [14] for $k=1$. When $k=1$, then from Eq. (4.13) we get,

$$\delta_2=-\frac{1}{24},\delta_3=0,\delta_4=-\frac{17}{13824}.$$

The above results are same as obtained by Cheung et al. [6] and Alam et al. [14]. When $k=\frac{5}{7}$, we obtain

$$\delta_2=-\frac{1}{56},\delta_3=0,\delta_4=-\frac{9}{175616}.$$

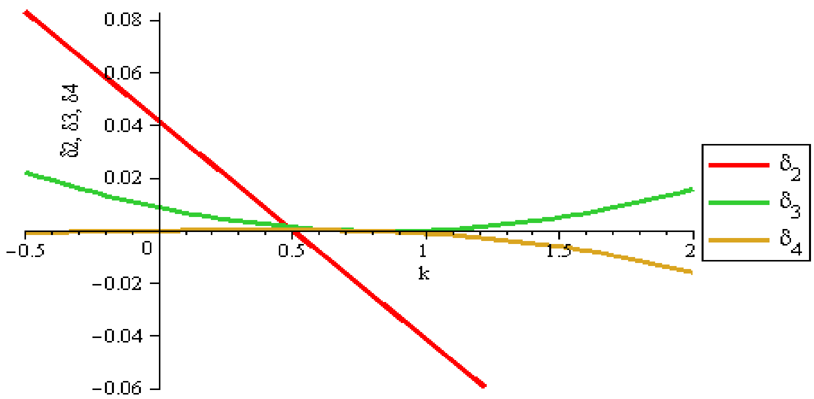

Figure 5.1: Variation of $\delta_2,\delta_3,\delta_4$ with k for duffing oscillator to determine small value of $\delta_2,\delta_3,\delta_4$.

It is clear that the α-series (Eq. (3.3)) converges faster when coefficients $|\delta_i|, i=2,3,\cdots$ etc. become small. We have plotted $\delta_2,\delta_3,\delta_4$ against k in the Fig. 5.1 for Duffing oscillator. We have found that $\delta_2,\delta_3,\delta_4$ all are small in the region $0.4<k<1$. The series (Eq. (3.3)) of frequency for the Duffing oscillator converges faster when $k=\frac{5}{7}$. For several values of a_0, the frequency ω have been calculated for both $k=1$ (Alam et al. [14] and Cheung et al. [6]) and $k=\frac{5}{7}$, and presented in Table 1 together with numerical results obtained by Runge-Kutta 4^{th} order method.
It is hard to say what would be the suitable value of k for other nonlinear oscillators. We have plotted $\delta_2,\delta_3,\delta_4$ against k in the Fig. 5.2 for the quintic oscillator. We find from Fig. 5.2 that $\delta_2,\delta_3,\delta_4$ all are small in the region $0<k<1$. For the cubic Duffing oscillator, we see that δ_3 vanishes for both $k=1$ and $k=\frac{5}{7}$. But for the quintic oscillator (i.e., $\ddot{x}+x+\varepsilon x^5=0$) δ_3 never vanishes. For this oscillator, we have obtained

$$\delta_2=\frac{1}{120}(19-32k),\ \delta_3=\frac{1}{14400}\left(1009-254k+1664k^2\right),$$
$$\delta_4=\frac{1}{1728000}\left(14441-65806k+140552k^2-104448k^3\right).$$

We see from Fig. 5.2 that the values of these coefficients are opposite in sign when $\frac{19}{32}<k$. But all are positive when $k\leq\frac{19}{32}$ and δ_2 vanishes when $k=\frac{19}{32}$. Thus for $k=1$ and $k=\frac{19}{32}$, we have obtained respectively

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{5}{7})$ $Er(\%)$	ω_{nu}
1	1.31778 0.00000	1.31778 0.00000	1.31778
10	8.53390 0.003633	8.53351 0.000937	8.53359
100	84.7309 0.004013	84.7266 0.001062	84.7275
1000	847.248 0.004013	847.205 0.001062	847.214

Table 1: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the Duffing oscillator (where $Er(\%)$ denotes absolute percentage error).

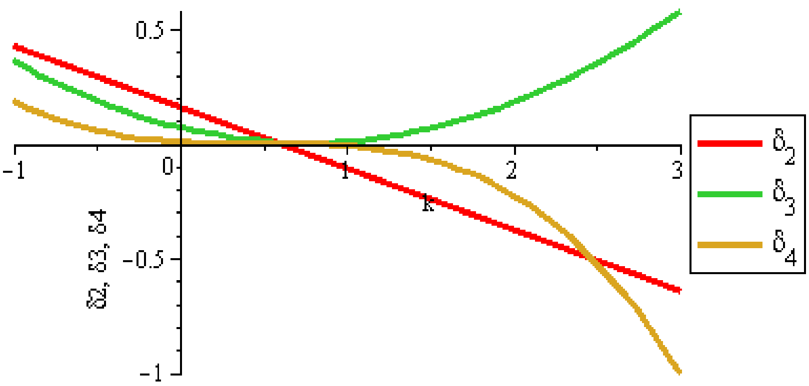

Figure 5.2: Variation of $\delta_2, \delta_3, \delta_4$ with k for quintic oscillator to determine small value of $\delta_2, \delta_3, \delta_4$.

$$\delta_2 = -\frac{13}{120}, \ \delta_3 = \frac{2}{225}, \ \delta_4 = -\frac{5087}{576000},$$

and

$$\delta_2 = 0, \ \delta_3 = \frac{541}{92160}, \ \delta_4 = \frac{391129}{221184000}.$$

Comparing these results, we easily expect that α-series (Eq. (3.3)) converges faster for $k=\frac{19}{32}$. To verify this matter, we have calculated some results choosing $k=1$ (Alam et al. [14] and Cheung et al. [6]) and $k=\frac{19}{32}$ and presented in Table 2 together with corresponding numerical results.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{19}{32})$ $Er(\%)$	ω_{nu}
1	1.26470 0.000791	1.26471 0.000000	1.26471
10	74.6618 0.038961	74.6768 0.018878	74.6909
100	7465.44 0.038831	7466.93 0.018880	7468.34
1000	746531.22 0.040569	746701.04 0.0178304	746834.20

Table 2: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the quintic oscillator (where $Er(\%)$ denotes absolute percentage error).

For the nonlinear oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ we have obtained

$$\delta_2 = \frac{1}{8}(3+2k), \ \delta_3 = \frac{1}{192}\left(63+76k+21k^2\right),$$

$$\delta_4 = \frac{1}{1563}\left(407+668k+426k^2+90k^3\right).$$

Thus for $k=1$ and $k=\frac{2}{5}$, we have obtained respectively

$$\delta_2 = \frac{5}{8}, \ \delta_3 = \frac{5}{6}, \ \delta_4 = \frac{1591}{1536},$$

and

$$\delta_2 = \frac{19}{40},\ \delta_3 = \frac{2419}{4800},\ \delta_4 = \frac{18703}{38400}.$$

We have calculated some results choosing $k=1$ (Alam et al. [14] and Cheung et al. [6]) and $k=\frac{2}{5}$ and presented in Table 3 together with corresponding numerical results. From Fig. 5.3 we see that $\delta_2, \delta_3, \delta_4$ are small near $k=\frac{-3}{2}$ but for $k=\frac{2}{5}$ obtained results are better for larger values of a_0.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=\frac{-3}{2})$ $Er(\%)$	$\omega(k=\frac{2}{5})$ $Er(\%)$	ω_{nu}
0.01	1.00001 0.000000	1.00001 0.000000	1.00001 0.000000	1.00001
0.1	1.00125 0.000000	1.00125 0.000000	1.00125 0.000000	1.00125
1	1.13651 0.023713	1.13682 0.0035187	1.13666 0.0105561	1.13678
10	9.12723 8.07624	10.3405 4.14306	9.95623 0.272934	9.92913
100	93.4396 6.55395	104.866 4.87324	101.947 1.95403	99.9931

Table 3: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ (where $Er(\%)$ denotes absolute percentage error).

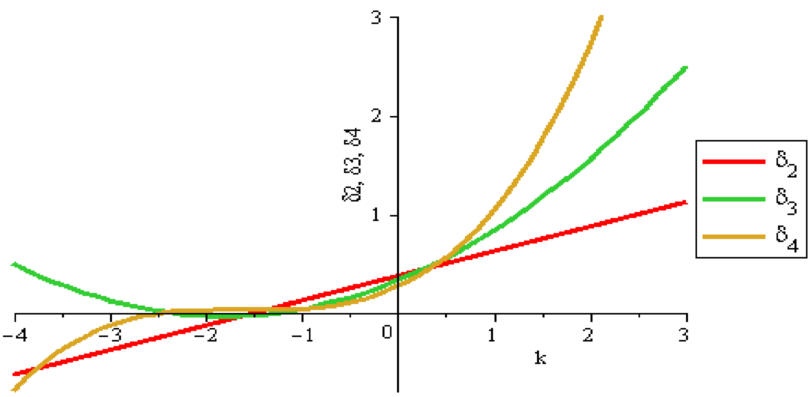

Figure 5.3: Variation of $\delta_2, \delta_3, \delta_4$ with k for the oscillator $\ddot{x}+x+\varepsilon\dot{x}^2x=0$ to determine small value of $\delta_2, \delta_3, \delta_4$.

For the nonlinear oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$, we have obtained

$$\delta_2 = \frac{1}{72}(-11+6k),\ \delta_3 = \frac{1}{1728}\left(-17-36k+21k^2\right),$$
$$\delta_4 = \frac{1}{124416}\left(-3359+2812k-1242k^2+270k^3\right).$$

Thus for $k=1$, we have obtained

$$\delta_2 = \frac{-5}{72},\ \delta_3 = \frac{-1}{54},\ \delta_4 = \frac{1519}{124416},$$

and which are same as obtained in Alam et al. [14].

For different values of the unknown constant k we have calculated some results and presented in Table 4 together with corresponding numerical results and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$). From Table 4 it is clear that frequency of the oscillator depends on the parameter k and comparing various results suitable value of k can be determined. From Fig 5.4 we see the variation of $\delta_2, \delta_3, \delta_4$ with the unknown constant k, shows the region of convergence.

6. Conclusion

The modified Lindstedt-Poincare method of Cheung et al. [6] based on Alam et al. [14] has been presented in a new form introducing an unknown constant, k. All the coefficients related to the solution depend on this constant. When $k=1$, the solution is identical to that of Cheung et al. [6] and Alam et al. [14]. But a better result would be found for a particular value of k. Comparing various results of the unknown coefficients, $|\delta_i(k)|, i=2,3,\cdots$, the suitable value of k can be determined. The method is applied to obtain the approximate solution of Duffing oscillator, quintic oscillator and another two nonlinear equations whose nonlinear response is significant. All the solutions show a good agreement with numerical solutions obtained by Runge-Kutta 4^{th} order method and provide better results than other existing solutions. The results may be useful to the researches in the field of nonlinear mechanics for investigating periodic solution of some higher order nonlinear problems.

a_0	$\omega(k=1)$ $Er(\%)$	$\omega(k=2)$ $Er(\%)$	$\omega(k=3)$ $Er(\%)$	$\omega(k=5)$ $Er(\%)$	ω_{nu}
0.01	0.999963 0.000000	0.999963 0.000000	0.999963 0.000000	0.999963 0.000000	0.999963
0.1	0.996273 0.000000	0.996273 0.000000	0.996273 0.000000	0.996273 0.000000	0.996273
1	0.761518 0.00800967	0.761545 0.00446441	0.761568 0.00144438	0.761712 0.0174637	0.761579
10	0.120712 2.11720	0.121174 1.74258	0.121670 1.34038	0.124195 0.707086	0.123323
100	0.0121717 2.83240	0.0122225 2.42686	0.0122776 1.98699	0.0125643 0.30176	0.0125256
1000	0.00121728 2.87246	0.00122235 2.46792	0.00122788 2.02668	0.00125658 0.263309	0.00125328
10000	0.000121728 2.87479	0.000122235 2.47026	0.000122788 2.02903	0.000125658 0.260909	0.000125331

Table 4: Comparison of the approximate frequencies obtained by present method with the numerical and other existing frequencies (Alam et al. [14] and Cheung et al. [6], $k=1$) for the oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$ (where $Er(\%)$ denotes absolute percentage error).

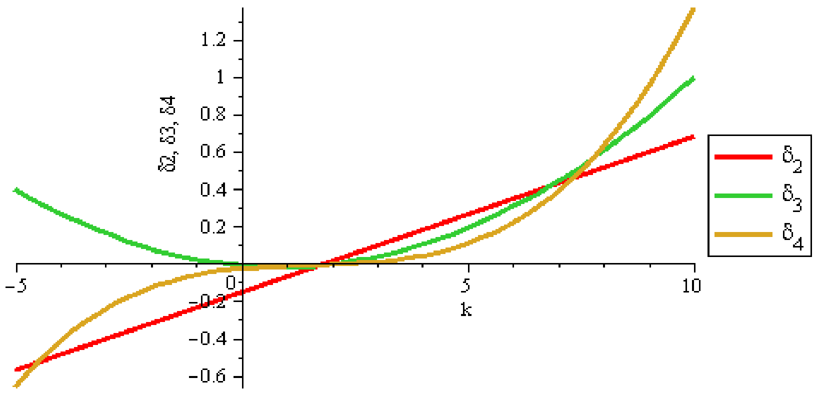

Figure 5.4: Variation of $\delta_2,\delta_3,\delta_4$ with k for the oscillator $\ddot{x}+x+\varepsilon\ddot{x}x^2=0$ to determine small value of $\delta_2,\delta_3,\delta_4$.

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