SDÜ FEN EDEBİYAT FAKÜLTESİ
FEN DERGİSİ (E-DERGİ). 2007, 2 (1), 111-116

ON NULL CURVES ON SURFACES AND NULL VECTORS IN LORENTZ SPACE

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Received: 30 March 2007, Accepted: 22 April 2007

Abstract: In this work, we compare the Darboux frame and the Frenet frame of a null curve lying on a spacelike surface in the three-dimensional Lorentz space, and we show that the normal curvature of the curve is a constant. Then we study the inner product of null vectors, to have results in terms of angles.

Keywords: Null curves, Null vectors, Lorentz space, Normal curvature, Darboux frame.

Mathematics Subject Classifications (2000): 53C50, 53A04, 53A05.

LORENTZ UZAYINDA YÜZEYLER ÜZERİNDEKİ NULL EĞRİLER VE NULL VEKTÖRLER ÜZERİNE

Özet: Bu çalışmada, üç boyutlu Lorentz uzayında bir spacelike yüzey üzerinde yatan bir null eğrinin Darboux çatısı ile Frenet çatısı karşılaştırıldı ve normal eğriliğin sabit olduğu gösterildi. Daha sonra da null vektörlerin iç çarpımları incelendi ve açılar cinsinden sonuçlar elde edildi.

Anahtar Kelimeler: Null eğriler, Null vektörler, Lorentz uzayı, Normal eğrilik, Darboux çatısı.

1. INTRODUCTION

There are some difficulties in studying geometry of a null curve since the arc-length parameter cannot be used, and normalizing the tangent is not possible. In this manner the pseudo-arc parameter, that is used to normalize the acceleration vector, was introduced (BONNOR 1969), and by this way the canonical frame, the Cartan frame, was obtained.

Differential geometry of null curves in Lorentz spaces is well-known (BONNOR 1969, FERRANDEZ et al. 2001), and now we will discuss null curves on hypersurfaces: There is no null curve on a timelike hypersurface (a hypersurface with a timelike normal) of a Lorentz space, since any tangent space of the hypersurface is Euclidean. When the surface is null the only null curves on it are the null rulings (BONNOR 1972). Last of all, there may be different types of null curves on a spacelike hypersurface, for instance a Lorentz cylinder, as its tangent planes are Lorentzian.

In this paper, we consider null curves lying on a spacelike surface in the three-dimensional Lorentz space. Section 2 consists of a comparison of the Darboux frame and the Frenet frame. We show that the only null curves lying on a totally umbilical spacelike surface are null straight lines (Proposition 2.1), and after taking the Cartan frame of a null curve into account the normal curvature is obtained to be ± 1 (Eq. 6), in case the surface is not totally umbilical. This constraint gives rise to some simplicities, such as, the Darboux frame and the Frenet frame are related up to a quantity, the geodesic curvature (Eq. 11).

Section three is devoted to have relations for inner products and cross products of null vectors in Lorentz spaces by using an orthogonal projection onto Euclidean spaces. So we get an inequality similar to the Schwarz inequality (Eq. 13).

2. NULL CURVES LYING ON SPACELIKE SURFACES IN A THREE DIMENSIONAL LORENTZ SPACE

Let L^3 denote the vector space $\mathbf{R}_1^3$ with its standard Lorentz metric of signature $(-,+,+)$. Consider a regular and orientable spacelike surface $M \subset L^3$ and a null curve α locally parametrized by $\alpha : I \to M$. The Darboux frame (SPIVAK 1979) at α(t) is the orthonormal basis $\{T,U,V\}$ of L^3, where T is the tangent of α, U is the unit normal of M, and V is the unique vector obtained by

$$V = \frac{1}{<X,T>}\{X - \frac{<X,X>}{2<X,T>}T\},\ X \in T_{\alpha(t)}M\ ,\ <X,T> \neq 0\,,$$

which appears in (DUGGAL & BEJANCU 1996). Then the normalization conditions

$$<T,T>=<V,V>=<T,U>=<V,U>=0, <T,V>=<U,U>=1 \tag{1}$$

are satisfied, and then the change of the moving frame is

$$\begin{bmatrix} T' \\ V' \\ U' \end{bmatrix} = \begin{bmatrix} k_g & 0 & k_n \\ 0 & -k_g & \tau_g \\ -\tau_g & -k_n & 0 \end{bmatrix} \begin{bmatrix} T \\ V \\ U \end{bmatrix}, \tag{2}$$

where ($'$) denotes the covariant derivative with respect to T. These functions; k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Next, from (2) we have

$$k_n = <ST,T> \tag{3}$$

and

$$\tau_g = <ST,V> \tag{4}$$

where S is the shape operator of M. Accordingly, this result immediately follows.

Proposition 2.1. If M is totally umbilical, then the only null curves lying on it are null straight lines of L^3.

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Proof: Suppose M is totally umbilical, then its shape operator is a scalar. Hence from (3), $k_n = 0$ is derived, and inserting this into (2) leads to

$$T^{'} = k_g T \ ,$$

which implies

$$T(t) = \exp(\int_{t_0}^{t} k_g du)a,$$

where a is a constant null vector. Then, if a suitable parameter is chosen, α has the form

$$\alpha(s) = at + b$$

where b is a constant vector, as well. This concludes the proof.

Since the normal curvature of a null curve on a totally umbilical surface is identically zero, henceforth we will omit this case.

As it is known the null curves except the null straight lines are called Cartan curves and their Frenet equations with respect to the pseudo-arc parameter, in matrix form, are

$$\begin{bmatrix} T^{'} \\ N^{'} \\ W^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -\kappa \\ \kappa & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ W \end{bmatrix}, \qquad (5)$$

where $\{T, N, W\}$ is the Cartan frame which is also pseudo-orthonormal (FERRANDEZ et al. 2001). After comparing (2) and (5)

$$k_n = \pm 1 \qquad (6)$$

is derived. Without loss of generality we may assume $k_n = 1$, then (2) reads

$$\begin{bmatrix} T^{'} \\ V^{'} \\ U^{'} \end{bmatrix} = \begin{bmatrix} k_g & 0 & 1 \\ 0 & -k_g & \tau_g \\ -\tau_g & -1 & 0 \end{bmatrix} \begin{bmatrix} T \\ V \\ U \end{bmatrix}, \qquad (7)$$

and so

$$< W, U >= 1, \ < W, V >= k_g \qquad (8)$$

are valid. Next an easy computation after derivations provides

$$< N, U >= -k_g, \ < N, V >= -k_g^2 / 2, \qquad (9)$$

and

$$-\kappa = -k_g^2 - 2k_g^{'} + 2\tau. \qquad (10)$$

Thus combining (1), (8) and (9) together yields, in matrix form,

$$\begin{bmatrix} T \\ N \\ W \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_g^2/2 & 1 & -k_g \\ k_g & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ V \\ U \end{bmatrix}, \qquad (11)$$

Observe that the frames coincide when $k_g = 0$, and τ_g never vanishes by (4).

We give an example to illustrate these results.

Example 2.2. Let's take into account the cylinder $C = \{(x_0, x_1, x_2) : -x_0^2 + x_1^2 = 1\}$. Thus it may be written in parametric coordinates as

$$\varphi(t, \lambda) = (\sinh t, \cosh t, \lambda)$$

and so its unit normal is

$$U = (-\sinh t, -\cosh t, 0)$$

and its shape operator is

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is obvious that the null helix on C, parametrized by,

$$\alpha(t) = (\sinh t, \cosh t, t)$$

is a geodesic of C and $\tau_g = -1/2$ for it. But an another null curve

$$\beta(t) = (t, \sqrt{1+t^2}, \sinh t),$$

is not a geodesic, since

$$k_g = \cosh t,$$

and we compute $\tau_g = -1/2$ for this curve, as well.

Now we state a theorem that uses the method of the Euler theorem (SPIVAK 1979).

Theorem 2.3. Let M be a spacelike surface on in L^3 and let X_1 and X_2 are two principal directions on M, X_1 being timelike and X_2 spacelike, corresponding the principal curvatures k_1 and k_2, respectively, and

$$< T, X_1 >= a.$$

Then

$$a^2 k_1 - a^2 k_2 = \eta,$$

where $\eta = 1$ if the frames have the same orientation and $\eta = -1$, otherwise.

Proof. By the hypothesis we can write

$$T = aX_1 + bX_2,$$

and since T is null we have $a = \pm b$. Since $k_n = \pm 1$, the result follows.

3. PRODUCTS OF NULL VECTORS

Let's consider the null vectors of the four-dimensional Lorentz space L^4 with the metric of signature $(-,+,+,+)$. If a null vector L is given in the standard coordinates by

$$L = (x_0, x_1, x_2, x_3),$$

then it is obvious that

$$|x_0| = \sqrt{x_1^2 + x_2^2 + x_3^2} \neq 0.$$

Next write

$$L = L_t + L_s,$$

where

$$L_t = (x_0, 0,0,0), \quad L_s = (0, x_1, x_2, x_3),$$

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and call L_t and L_s the time part and the spatial part of L, respectively. Therefore the null vector

$$\overline{L} = \frac{L}{\|L_s\|}$$

has a unit spatial part. These types of null vectors may be called pseudo-unit null vectors and these null vectors determine a part of the null cone uniquely.

Since the induced metric on a set of spatial parts of null vectors is Euclidean, it is easy to check for two null vectors L and N that

$$< L, N >= \|L_s\|\|N_s\|(\varepsilon + \cos\theta)\,, \tag{12}$$

where θ is the Euclidean angle between L_s and N_s, and $\varepsilon = -1$ if L and N are both future-pointing or past-pointing and $\varepsilon = 1$, otherwise. If the pseudo-null vectors such that $\varepsilon = -1$ are taken then (12) reduces to

$$< \overline{L}, \overline{N} >= -1 + \cos\theta = -\sin^2\frac{\theta}{2}\,, \tag{13}$$

which gives the relation

$$\left| < \overline{L}, \overline{N} > +1 \right| \leq 1\,,$$

reminding the Schwarz inequality.

A similar argument could be done for Lorentz cross-product defined by

$$X \wedge Y = (x_2 y_1 - x_1 y_2, x_2 y_0 - x_0 y_2, x_0 y_1 - x_1 y_0)\,,$$

where $X = (x_0, x_1, x_2)$ and $Y = (y_0, y_1, y_2)$. As it is known (CHOI 1995)

$$\|L \wedge N\| = \left| < L, N > \right|,$$

where L and N are two arbitrary null vectors of L^3.

As an application of pseudo-unit null vectors we set up a Frenet frame of a null curve, locally parametrized by $\alpha : I \to L^4$. If

$$\alpha^{'} = L\,,$$

then we may choose a pseudo-unit null vector

$$\overline{L} = \frac{L}{\|L_s\|}\,,$$

so that

$$\overline{L}^{'} = k_1 W_1\,,$$

where W_1 is a unit spacelike vector. Next we may take an another pseudo-unit null vector such that

$$< \overline{L}, \overline{N} >= -1\,,$$

which results

$$< \overline{N}, W_1 >= 0$$

from (13). Lastly, if a unit spacelike vector W_2 is chosen in order to be orthogonal to the pseudo-orthonormal system $\{\overline{L}, \overline{N}, W_1\}$, then the Frenet equations with respect to the frame $\{\overline{L}, \overline{N}, W_1, W_2\}$ will be

$$\begin{aligned} \alpha' &= L\,, \\ \overline{L}' &= k_1 W_1\,, \\ W_1' &= k_2 \overline{L} + k_1 \overline{N}\,, \\ \overline{N}' &= k_2 W_1 + k_3 W_2\,, \\ W_2' &= k_3 \overline{L}\,. \end{aligned}$$

Note that this frame is not the canonical frame since the number of the corresponding curvatures is not of minimum number. For another way of constructing this frame, please see (DUGGAL & BEJANCU 1996).

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