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PERIODIC BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract: We prove existence results for the solutions of the periodic boundary value problem concerning the n-th order functional differential equation with impulses effects

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x(\alpha_1(t))\cdots, x(\alpha_m(t))), \text{ a.e. } t \in [0,T], \\ \Delta x^{(i)}(t_k) = I_{i,k}(x(t_k),\cdots, x^{(n-1)}(t_k)), \ k = 1,\cdots, p, \end{cases}$$

and the periodic boundary conditions

$$x^{(i)}(0) = x^{(i)}(T), i = 0,\cdots, n-1.$$

Our method is based upon the coincidence degree theory of Mawhin and some technical inequalities. Examples are presented to illustrate the main results.

Key words. Periodic boundary value problem; solution; n-th order impulsive functional differential equation;

Mathematics Subject Classifications (2000): 34B10, 34B15

1. INTRODUCTION

In this paper, we investigate the periodic boundary value problem (PBVP for short) consisting of the n-th order functional differential equation with impulses effects

$$\begin{cases} x^{(n)}(t)) = f(t, x(t), x(\alpha_1(t))\cdots, x(\alpha_m(t))), \text{ a.e. } t \in [0,T], \\ \Delta x^{(i)}(t_k) = I_{i,k}(x(t_k),\cdots, x^{(n-1)}(t_k)), \ k = 1,\cdots, p, \end{cases} \quad (1)$$

and the periodic boundary conditions

$$x^{(i)}(0) = x^{(i)}(T), i = 0,1,\cdots, n-1, \quad (2)$$

where $T > 0$ is a constant, $f:[0,1]\times R^{m+1} \to R$ is an impulsive Carathedeodory function, $n \ge 2$ an integer, $I_{i,k}$ are continuous functions, $\alpha_i \in C^1([0,T],[0,T])$.

The motivation for this paper is as follows. First, there exist many papers concerning with the solvability of the PBVPs for first order ordinary or functional differential equations, see (NIETO 2002, FRANCO 1998, NIETO 1997, HE 2002, LADDE 1985, JIANG 2004, HAKL 2003, NIETO 2002, CABADA 1994, NIETO 1996, PIERSON-GOREZ 1993, VATSALA 1992, LIU 1990). We now discuss briefly several of the appropriate papers on the topic.

The pioneer papers concerning the solvability of PBVP may be (HU 1989) and (BAINOV 1989). In (NIETO 1997), Nieto studied the PBVP

$$\begin{cases} x'(t)+\lambda x(t)=F(t,x(t)), \quad t\in[0,T]\setminus\{t_1,\cdots,t_p\}, \\ x(t_k^+)-x(t_k)=I_k(x(t_k)), \quad k=1,\cdots,p \\ x(0=x(T), \end{cases} \tag{3}$$

where $\lambda \neq 0$, $J=[0,T]$, $0=t_0<t_1<\cdots<t_p<t_{p+1}=T$, by using fixed point theorems and lower and upper solution methods.

In (NIETO 2002) and (NIETO 1996), Nieto considered the following PBVP for first order impulsive differential equation

$$\begin{cases} x'(t)+F(t,x(t))=0, \quad a.e.\, t\in[0,1]\setminus\{t_1,\cdots,t_p\}, \\ x(t_k^+)-x(t_k)=I_k(x(t_k)), \quad k=1,2,\cdots,p \\ x(0)=x(T), \end{cases} \tag{4}$$

where $0=t_0<t_1<\cdots<t_p<t_{p+1}=1$, F is an impulsive Caratheodory function, I_k is continuous. The methods used in these papers are different from those in (NIETO 1997).
In (FRANCO 1998), Franco and Neito studied the first order PBVP

$$\begin{cases} x'(t)=f(t,x(t)), \quad a.e.\, t\in J\setminus\{t_1,\cdots,t_p\}, \\ x(t_k^+)-x(t_k)=I_k(x(t_k)), \quad k=1,2,\cdots,p \\ x(0)=x(T). \end{cases} \tag{5}$$

Using upper and lower solutions method and the monotone technique, they proved (5) has at least one solution under the existence assumptions of lower solution α and upper solution β.
In a recent paper (LI 2006), Li and Shen studied the PBVPs consisting of the functional impulsive differential equations

$$\begin{cases} x'(t)=f(t,x(t),x(\theta(t))), t\in[0,T], t\neq t_k, k=1,\cdots,m, \\ \Delta x(t_k)=I_k(x(t_k)), k=1,\cdots,m, \\ x(0)=x(T), \end{cases} \tag{6}$$

by using upper and lower solution methods and monotone techniques under certain assumptions, but their methods are different from those in (FRANCO 1998).
Second, there exist some papers concerned with the solvability of PBVPs for the second order functional differential equations with or without impulses effects, see (CABADA 2000, JIANG 2004, CHEN 2006, LAKSHMIKANTHAM 1984, GUO 1997, DING 2004, JIANG 2005). We address some of the relative papers.
Jiang, Chu and Zhang (JIANG 2005) studied the PBVP

$$\begin{cases} x''(t)+a(t)x(t)=f(t,x(t)), t\in[0,T], \\ x(0)=x(1), \ x'(0)=x'(1), \end{cases} \tag{7}$$

where f has a repulsive singularity near $x=0$ using nonlinear alternative of Leray-Schauder type and of Krasnoselskii fixed point theorem on compression and expansion in cones.
Kiguradze and Stanek (KIGURADZE 2002) studied the following PBVP

$$\begin{cases} x''(t)=f(t,x(t),x'(t)), t\in[0,T], \\ x(a)=x(b), \ x'(a)=x'(b), \end{cases} \tag{8}$$

their methods are based upon upper and lower solution methods and monotone iterative technique.

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In (DING 2004), Ding, Han and Yan investigated the following PBVP

$$\begin{cases} x''(t) = f(t, x(t), x(\theta(t))), t \in [0, T], \\ x(0) = x(T), \ \ x'(0) = x'(T), \end{cases} \tag{9}$$

the proofs are based upon upper and lower solution methods and comparison principle.
In (CHEN 2006), Chen and Sun studied the existence of positive solutions of the following PBVP of the second order impulsive functional differential equations

$$\begin{cases} -x''(t) = f(t, x(t), x(\theta(t))), t \in [0, T], t \neq t_k, k = 1, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k)), k = 1, \cdots, m, \\ \Delta x'(t_k) = I_k^*(x(t_k)), k = 1, \cdots, m, \\ x(0) = x(T) + k_1, x'(0) = \lambda x'(T) + k_2. \end{cases} \tag{10}$$

Using upper and lower solution methods and monotone iterative technique, they established existence results for above problems. PBVP(10) contains the following PBVP

$$\begin{cases} -x''(t) = f(t, x(t), x(\theta(t))), t \in [0, T], t \neq t_k, k = 1, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k)), k = 1, \cdots, m, \\ \Delta x'(t_k) = I_k^*(x(t_k)), k = 1, \cdots, m, \\ x(0) = x(T), x'(0) = x'(T) \end{cases}$$

as special case.
In papers (GUO 1997, YANG 2007, LIANG 2007, DING 2005, DING 2004, HU 1989), the authors, by developing some new comparison results and using the monotone iterative technique, obtained existence of minimal and maximal solutions of some PBVPs for second-order nonlinear impulsive differential, functional differential or integro-differential equations.
There exist a few papers concerned with the solvability of the PBVPs of higher order functional differential equations, see (KONG 2001, CHU 2006, LIU 2005, CONG 1998, CONG 2000, LI 2007, LIU 2005, CONG 2004). We give a simple list concerned with this topic.
Chu and Zhou (CHU 2006), Kong, Wang and Wang (KONG 2001) studied the existence of solutions of the following PBVP

$$\begin{cases} x'''(t) + \rho^3 x(t) = f(t, x(t)) \ t \in [0, T], \\ x(0) = x(2\pi), \ \ x'(0) = x'(2\pi), \end{cases} \tag{11}$$

Cong and Huang (CONG 1998, CONG 2000, CONG 2004) established the existence results for solutions of the following PBVP

$$\begin{cases} x^{(2n)} + \sum_{j=1}^{n-1} c_j x^{(2j)}(t) = f(t, x(t)), t \in [0, 2\pi], \\ x^{(i)}(0) = x^{(i)}(2\pi), \ \ i = 0, \cdots, 2n-1 \end{cases} \tag{12}$$

and

$$\begin{cases} x^{(2n+1)} + \sum_{j=0}^{n-1} c_j x^{(2j+1)}(t) = f(t, x(t)), t \in [0, 2\pi], \\ x^{(i)}(0) = x^{(i)}(2\pi), \ \ i = 0, \cdots, 2n, \end{cases} \tag{13}$$

respectively. Li, Li and Liang in (LI 2007) studied the following PBVP

$$\begin{cases}(-1)mx^{(2m)}+\sum_{j=1}^{m}(-1)^{m-j}c_j x^{(2(m-j)}(t)=f(t,x(t)),\ t\in[0,1],\\ x^{(i)}(0)=x^{(i)}(1),\ \ i=0,\cdots,2m-1\end{cases} \tag{14}$$

In a recent paper (LIU 2005, LIU 2005), Liu and Ge, different from (CHEN 2006, KONG 2001, CHU 2006, CONG 1998, CONG 2000, LI 2007), studied the following PBVP

$$\begin{cases}x^{(n)}+\sum_{j=0}^{n-1}c_j x^{(j)}(t)=f(t,x(t)),\ t\in[0,2\pi],\\ x^{(i)}(0)=x^{(i)}(2\pi),\ \ i=0,\cdots,n-1\end{cases} \tag{15}$$

and

$$\begin{cases}x^{(n)}=f(t,x(t),\cdots,x^{(n-1)}(t)),\ t\in[0,T],\\ x^{(i)}(0)=x^{(i)}(T),\ \ i=0,\cdots,n-1\end{cases} \tag{16}$$

To the best of our knowledge, the existence of solutions of the PBVPs of the higher-order impulsive functional differential equations has not been well studied till now. Our purpose is to provide sufficient conditions for the existence of solutions of PBVP(1)-(2). This will be done by applying the well known coincidence degree theory and some technical inequalities. The methods used is different from those used in papers (CHEN 2006, GUO 1997, YANG 2007, LIANG 2007, DING 2005, DING 2004, VATSALA 1992, HU 1989) and the text book (BAINOV 1993).

The organization of this paper is as follows. In section 2, we present some preliminary results. The main results concerned with the even order case will be given in section 3, and concerned with the odd order case in section 4 , and the examples to illustrate the main results will be given in section 5.

2. Preliminary Results

To establish sufficient conditions for the existence of at least one solution of PBVP(1)-(2), we, in this section, introduce some notations and an abstract existence theorem by Gaines and Mawhin (GAINES 1977).

Let X and Y be Banach spaces, $L:\text{dom } L(\subset X)\to Y$ be a Fredholm operator of index zero, $P:X\to X$, $Q:Y\to Y$ be projectors such that

$$\text{Im } P=\text{Ker } L,\ \text{Ker } Q=\text{Im } L,\ \ X=\text{Ker } L\oplus \text{Ker } P,\ \ Y=\text{Im } L\oplus \text{Im } Q.$$

It follows that

$$L|_{\text{dom } L\cap \text{Ker } P}:\text{dom } L\cap \text{Ker } P\to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\text{dom } L\cap\overline{\Omega}\neq\emptyset$, the map $N:X\to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\overline{\Omega}\to X$ is compact.

Lemma 2.1 (GAINES 1977). Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:

(i). $Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(\text{dom}L\setminus \text{Ker}L)\cap\partial\Omega]\times(0,1)$;

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(ii). $Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;

(iii). $\deg(\wedge QN|_{KerL}, \Omega \cap KerL, 0) \neq 0$, where $\wedge : KerL \to Y/ImL$ is an isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

Let $u : J = [0,T] \to R$, and $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, for $k = 0,\cdots,p$, define the function $u_k : (t_k, t_{k+1}] \to R$ by $u_k(t) = u(t)$. We will use the following real Banach spaces

$$X = \left\{ \begin{array}{l} u : J \to R, u_k \in C^0(t_k, t_{k+1}], k = 0,\cdots,p, \textit{there exist the limits} \\ \lim\limits_{t \to t_k^+} u(t), \lim\limits_{t \to 0^+} u(t) = u(0) \end{array} \right\}$$

and $Y = X \times R^{np}$ with the norms $\| u \| = \sup_{t\in[0,T]} | u(t) |$ for $u \in X$ and

$$\| y \| = \max\left\{ \| u \|, \max_{1\le k\le np} \{ | x_k | \} \right\}$$

for $y = \{u, x_1, \cdots, x_{np}\} \in Y$.

A function F is an impulsive Carathedeodory function if

* $F(\bullet, u_0, u_1, \cdots, u_m) \in X$ for each $u \in R^{m+1}$;
* $F(t, \bullet, \cdots, \bullet)$ is continuous for a.e. $t \in J$;
* for each $r > 0$ there is $h_r \in L^1(J)$ so that

$$| F(t, u_0, u_1, \cdots, u_m) | \le h_r(t), \; a.e. \; t \in J \setminus \{t_1, \cdots, t_p\}$$

and every u satisfying $\| (u_0, u_1, \cdots, u_m) \| > r$.

By a solution of PBVP(1)-(2) we mean a function $u \in X$ satisfying (1) and (2).

Define the linear operator L and the nonlinear operator N by

$$L : X \cap domL \to Y, \quad Lx(t) = \begin{pmatrix} x^{(n)}(t) \\ \Delta x(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x^{(n-1)}(t_1) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x(t_p) \\ \cdot \\ \cdot \\ \cdot \\ \Delta x^{(n-1)}(t_p) \end{pmatrix} \text{for } x \in domL,$$

where

$$domL = \left\{ \begin{array}{l} x \in X : x_k^{(n)} \in C^0(t_k, t_{k+1}], k = 0, \cdots, p, \text{there exist the limits} \lim\limits_{t \to t_k^-} x^{(i)}(t) = x^{(i)}(t_k), \\ \lim\limits_{t \to t_k^+} x^{(i)}(t), \lim\limits_{t \to 0^+} x^{(i)}(t) = x^{(i)}(0), \lim\limits_{t \to T^-} x^{(i)}(t) = x^{(i)}(T), \\ with x^{(i)}(0) = x^{(i)}(T), i = 0, \cdots, n-1 \end{array} \right\}.$$

$$N : X \to Y, \quad Nx(t) = \begin{pmatrix} f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_m(t))) \\ I_{0,1}(x(t_1), \cdots, x^{(n-1)}(t_1)) \\ . \\ . \\ . \\ I_{n-1,1}(x(t_1), \cdots, x^{(n-1)}(t_1)) \cdot \\ . \\ . \\ . \\ I_{0,p}(x(t_p), \cdots, x^{(n-1)}(t_p)) \\ . \\ . \\ . \\ I_{n-1,p}(x(t_p), \cdots, x^{(n-1)}(t_p)) \end{pmatrix}$$

for $x \in X$.

Lemma 2.2. *The following results hold.*

(i). $\mathrm{KerL} = \{x(t) \equiv c, t \in [0,T], \ c \in R\}$;

(ii). $\mathrm{ImL} = \left\{ (y(t), a_{0,1}, \cdots, a_{0,p}, \cdots, a_{n-1,1}, \cdots, a_{n-1,p}) \in Y, \int_0^T y(u)du + \sum_{k=1}^p a_{n-1,k} = 0 \right\}$;

(iii). L *is a Fredholm operator of index zero;*

(iv). There exist projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{KerL} = \mathrm{ImP}$ *and* $\mathrm{KerQ} = \mathrm{ImL}$. *Furthermore, let* $\Omega \subset X$ *be an open bounded subset with* $\overline{\Omega} \cap \mathrm{domL} \neq \emptyset$, *then* N *is* $L-$*compact on* $\overline{\Omega}$*;*

(v). $x(t)$ is a solution of $\mathrm{PBVP}(1)$ - (2) if and only if x is a solution of the operator equation $Lx = Nx$ in domL.

Proof: The proofs are similar to those of Lemmas in (HE 2002, LADDE 1985, CABADA 2000, LI 2006, JIANG 2004) and are omitted. We list $P : X \to X, Q : Y \to Y$ and the generalized inverse $K_p : \mathrm{ImL} \to \mathrm{domL} \cap \mathrm{ImP}$ and the isomorphism $\wedge : \mathrm{KerL} \to Y/\mathrm{ImL}$.

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$$Px(t) = x(0) \ \ for \ x \in X,$$

$$Q(y(t), a_{0,1}, \cdots, a_{0,p}, \cdots, a_{n-1,1}, \cdots, a_{n-1,p})$$

$$= \left(\frac{1}{T} \int_0^T y(s)ds + \frac{1}{T} \sum_{i=1}^{p} a_{n-1,p}, 0, \cdots, 0 \right)$$

$$for \ (y, a_{0,1}, \cdots, a_{0,p}, \cdots, a_{n-1,1}, \cdots, a_{n-1,p}) \in Y,$$

$$K_p(y(t), a_{0,1}, \cdots, a_{0,p}, \cdots, a_{n-1,1}, \cdots, a_{n-1,p})$$

$$= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds + \sum_{i=0}^{n-2} \frac{a_i}{(i+1)!} t^{i+1} + \sum_{i=1}^{n-1} \int_0^t \frac{(t-s)^i}{i!} \sum_{0<t_k<s} a_{i,k} ds + \sum_{0<t_k<t} a_{0,k}$$

$$for \ (y, a_{0,1}, \cdots, a_{0,p}, \cdots, a_{n-1,1}, \cdots, a_{n-1,p}) \in Y,$$

$$\wedge(c) = (c, 0, \cdots, 0) \ \ for \ c \in R.$$

Suppose the followings which will be used in the main results.

(A_1). There exist impulsive Caratheodory functions $h(t, x_0, x_1, \cdots, x_m)$, $e(t)$ and $g_i(t, x)(i = 0, 1, \cdots, m)$, positive number β and q such that

- f satisfies

$$f(t, x_0, x_1, \cdots, x_m) = e(t) + h(t, x_0, x_1, \cdots, x_m) + \sum_{i=0}^{m} g_i(t, x_i),$$

- h satisfies

$$(-1)^n x_0 h(t, x_0, x_1, \cdots, x_m) \le -\beta |x_0|^{q+1}$$

for all $t \in [0, T]$ and $(x_0, x_1, \cdots, x_m) \in R^{m+1}$,

- g_i satisfies

$$\limsup_{|x| \to \infty, t \in [0,T]} \frac{|g_i(t, x)|}{|x|^q} = r_i, \ \ for \ i = 0, 1, \cdots, m$$

with $r_i \ge 0$ for $i = 0, 1, \cdots, m$;

(A_2). There exist impulsive Caratheodory functions $h(t, x_0, x_1, \cdots, x_m)$, $e(t)$, $g_i(t, x)(i = 0, 1, \cdots, m)$, positive number β and q such that

- f satisfies

$$f(t, x_0, x_1, \cdots, x_m) = e(t) + h(t, x_0, x_1, \cdots, x_m) + \sum_{i=0}^{m} g_i(t, x_i),$$

- h satisfies

$$(-1)^n x_0 h(t, x_0, x_1, \cdots, x_m) \ge \beta |x_0|^{q+1}$$

for all $t \in [0, T]$ and $(x_0, x_1, \cdots, x_m) \in R^{m+1}$,

- g_i satisfies

$$\limsup_{|x| \to \infty, t \in [0,T]} \frac{|g_i(t, x)|}{|x|^q} = r_i, \ \ for \ i = 0, 1, \cdots, m$$

with $r_i \ge 0$ for $i = 0, 1, \cdots, m$;

(A_3). There exists a constant $M_0 > 0$ such that

$$c\left[\frac{1}{T}\int_0^T f(t,c,c,\cdots,c)dt + \frac{1}{T}\sum_{k=1}^{p} I_{n-1,k}(c,0,\cdots,0)\right] > 0$$

for all $|c| > M_0$ or

$$c\left[\frac{1}{T}\int_0^T f(t,c,c,\cdots,c)dt + \frac{1}{T}\sum_{k=1}^{p} I_{n-1,k}(c,0,\cdots,0)\right] < 0$$

for all $|c| > M_0$.

(A_4). There exist positive functions $p_i, r \in X$ so that

$$|f(s,x_0,x_1,\cdots,x_m)|ds \le \sum_{i=0}^{m} p_i(t)|x_i|^q + r(t), \quad t \in [0,T]$$

3. Existence Results for the Even Order Case

The even order case of PBVP(1)-(2) is as follows.

$$\begin{cases} x^{(2n)}(t)) = f(t,x(t),x(\alpha_1(t))\cdots,x(\alpha_m(t))), \ a.e.\, t\in[0,T], \\ \Delta x^{(i)}(t_k) = I_{i,k}(x(t_k),\cdots,x^{(2n-1)}(t_k)), \ k=1,\cdots,p, i=0,\cdots,2n-1, \\ x^{(i)}(0) = x^{(i)}(T), \ i=0,\cdots,2n-1, \end{cases} \tag{17}$$

where $n \ge 1$ is an integer. Suppose

(A_5). For all $(x_0,\cdots,x_{2n-1}) \in R^{2n}$ and $i=1,\cdots,n$, we have

$$(-1)^{i+n}\left(x_{2n-i}I_{i-1,k}(x_0,\cdots,x_{2n-1}) + x_{i-1}I_{2n-i,k}(x_0,\cdots,x_{2n-1}) + I_{i-1,k}(x_0,\cdots,x_{2n-1})I_{2n-i,k}(x_0,\cdots,x_{2n-1})\right) \ge 0.$$

(A_6). For all $(x_0,\cdots,x_{2n-1}) \in R^{2n}$ and $i=1,\cdots,n-1$ we have $x_i(x_i + I_{i,k}(x_0,\cdots,x_{2n-1})) \ge 0$;

(A_7). There exist constants $\alpha_{i,k} \ge 0$ such that $|I_{i,k}(x_0,\cdots,x_{2n-1})| \le \alpha_{i,k}|x_i|$ with $\sum_{k=1}^{p}\alpha_{i,k} < 1$, $i=0,\cdots,n-1$ and $k=1,\cdots,p$;

Theorem 3.1. Suppose (A_1), (A_3), (A_5), (A_6) and (A_7) hold. Then problem (17) has at least one solution if

$$r_0 + \sum_{k=1}^{m} r_k \|\beta_k'\|_\infty^{q/(q+1)} < \beta, \tag{18}$$

where $s = \beta_k(u)$ is the inverse function of $u = \alpha_k(s)$, $k=1,\cdots,m$.

Proof. To apply Lemma 2.1, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Lemma 2.1 hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in domL \setminus KerL, Lx = \lambda Nx \ for\ some\ \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. Suppose $x \in \Omega_1$. Then

$$\begin{cases} x^{(2n)}(t) = \lambda f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_m(t))), t \in [0,T], t \neq t_k, k = 1, \cdots, p, \\ \Delta x^{(i)}(t_k) = \lambda I_{i,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)), k = 1, \cdots, p, i = 0, \cdots, 2n-1, \\ x^{(i)}(0) = x^{(i)}(T), \ \ i = 0, \cdots, 2n-1. \end{cases} \tag{19}$$

Substep 1.1. Prove that there is a constant $M > 0$ such that $\int_0^T |x(s)|^{q+1} ds \leq M$.

Multiplying both sides of the first equation of (19) by $x(t)$, integrating it from 0 to T, we get from (A_1) that

$$\begin{aligned} & \sum_{i=1}^{n} (-1)^{i+1} \left(x^{(2n-i)}(T) x^{(i-1)}(T) - x^{(2n-i)}(0) x^{(i-1)}(0) \right) \\ & + \sum_{i=1}^{n} (-1)^{i} \sum_{k=1}^{p} \left(x^{(2n-i)}(t_k^+) x^{(i-1)}(t_k^+) - x^{(2n-i)}(t_k) x^{(i-1)}(t_k) \right) + (-1)^n \int_0^T [x^{(n)}(s)]^2 ds \\ = \ \ & \lambda \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s))) x(s) ds \\ = \ \ & \lambda \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s))) x(s) ds + \int_0^T g_0(s, x(s)) x(s) ds \right. \\ & \left. + \sum_{i=1}^{n} \int_0^T g_i(s, x(\alpha_i(s)) x(s) ds + \int_0^T e(s) x(s) ds \right). \end{aligned}$$

It follows from (A_5), for $i = 1, \cdots, n$ and $k = 1, \cdots, p$, that

$$\begin{aligned} & (-1)^{i+n} \left[x^{(2n-i)}(t_k^+) x^{(i-1)}(t_k^+) - x^{(2n-i)}(t_k) x^{(i-1)}(t_k) \right] \\ = \ \ & (-1)^{i+n} \left(x^{(2n-i)}(t_k) I_{i-1,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)) + x^{(i-1)}(t_k) I_{2n-i,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)) \right. \\ & \left. + I_{i-1,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)) I_{2n-i,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)) \right) \\ \geq \ \ & 0. \end{aligned}$$

Hence we get

$$\begin{aligned} & (-1)^n \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_n(s))) x(s) ds + \int_0^T g_0(s, x(s)) x(s) ds \right. \\ & \left. + \sum_{i=1}^{n} \int_0^T g_i(s, x(\alpha_i(s)) x(s) ds + \int_0^T e(s) x(s) ds \right) \geq 0. \end{aligned}$$

It follows from (A_1) that

$$\begin{aligned} & \beta \int_0^T |x(s)|^{q+1} ds \\ \leq \ \ & (-1)^n \left(\int_0^T g_0(s, x(s)) x(s) ds + \sum_{i=1}^{n} \int_0^1 g_i(s, x(\alpha_i(s)) x(s) ds + \int_0^T e(s) x(s) ds \right) \\ \leq \ \ & \int_0^T |g_0(s, x(s))| |x(s)| ds + \sum_{i=1}^{n} \int_0^T |g_i(s, x(\alpha_i(s))| |x(s)| ds + \int_0^T |e(s)| |x(s)| ds. \end{aligned}$$

Choose $\varepsilon > 0$ satisfy that

$$(r_0 + \varepsilon) + \sum_{k=1}^{m} (r_k + \varepsilon) \| \beta_k{}' \|_\infty^{q/(q+1)} < \beta. \tag{20}$$

For such $\varepsilon > 0$, there is $\delta > 0$ so that for every $i = 0, 1, \cdots, n$,

$$|g_i(t, x)| < (r_i + \varepsilon) |x|^q \ \textit{uniformly for } t \in [0,T] \textit{ and } |x| > \delta. \tag{21}$$

Let, for $i = 1, \cdots, n$,

$\Delta_{1,i}=\{t:t\in[0,T],|x(\alpha_i(t))|\le\delta\}$, $\Delta_{2,i}=\{t:t\in[0,T],|x(\alpha_i(t))|>\delta\}$, $g_{\delta,i}=\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|$, and $\Delta_1=\{t\in[0,T],|x(t)|\le\delta\}$, $\Delta_2=\{t\in[0,T],|x(t)|>\delta\}$. Then we get

$$
\begin{aligned}
\beta\int_0^T|x(s)|^{q+1}\,ds \le\ & \int_{\Delta_1}|g_0(s,x(s))||x(s)|\,ds+\int_{\Delta_2}|g_0(s,x(s))||x(s)|\,ds\\
&+\sum_{i=1}^{n}\int_{\Delta_{1,i}}|g_i(s,x(\alpha_i(s))||x(s)|\,ds+\sum_{i=1}^{n}\int_{\Delta_{2,i}}|g_i(s,x(\alpha_i(s))||x(s)|\,ds\\
&+\int_0^T|e(s)||x(s)|\,ds\\
\le\ & (r_0+\varepsilon)\int_0^T|x(s)|^{q+1}\,ds+\sum_{k=1}^{m}(r_k+\varepsilon)\int_0^T|x(\alpha_k(s))|^q|x(s)|\,ds\\
&+\int_0^T|e(s)|\,|x(s)|\,ds+\sum_{k=0}^{m}g_{\delta,k}\int_0^T|x(s)ds\\
\le\ & (r_0+\varepsilon)\int_0^T|x(s)|^{q+1}\,ds\\
&+\sum_{k=1}^{m}(r_k+\varepsilon)\left(\int_0^T|x(\alpha_k(s))|^{q+1}\,ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\left(\int_0^T|e(s)|^{(q+1)/q}\,ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\sum_{k=0}^{m}g_{\delta,k}\int_0^T|x(s)|\,ds\\
=\ & (r_0+\varepsilon)\int_0^T|x(s)|^{q+1}\,ds\\
&+\sum_{k=1}^{m}(r_k+\varepsilon)\left|\int_{\alpha_k(0)}^{\alpha_k(T)}|x(u)|^{q+1}|\beta_k{}'(u)|\,du\right|^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\left(\int_0^T|e(s)|^{(q+1)/q}\,ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\sum_{k=0}^{m}g_{\delta,k}T^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
\le\ & (r_0+\varepsilon)\int_0^T|x(s)|^{q+1}\,ds\\
&+\sum_{k=1}^{m}(r_k+\varepsilon)\|\beta_k{}'\|_\infty^{q/(q+1)}\left(\int_0^T|x(u)|^{1+q}|\,du\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\left(\int_0^T|e(s)|^{(q+1)/q}\,ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\sum_{k=0}^{m}g_{\delta,k}T^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
=\ & \left((r_0+\varepsilon)+\sum_{k=1}^{m}(r_k+\varepsilon)\|\beta_k{}'\|_\infty^{q/(q+1)}\right)\int_0^T|x(s)|^{q+1}\,ds\\
&+\left(\int_0^T|e(s)|^{(q+1)/q}\,ds\right)^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}\\
&+\sum_{k=0}^{m}g_{\delta,k}T^{q/(q+1)}\left(\int_0^T|x(s)|^{q+1}\,ds\right)^{1/(q+1)}
\end{aligned}
$$

.

It follows from (20) that there is a constant $M>0$ such that $\int_0^T|x(s)|^{q+1}\,ds\le M$.

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Substep 1.2. Prove that there is a constant $M_1 > 0$ so that $\| x \|_\infty \le M_1$.

It follows from Substep 1.1 that there is $\xi \in [0, T]$ so that $| x(\xi) | \le (M/T)^{1/(q+1)}$. On the other hand, we get

$$\begin{aligned}
\int_0^T [x^{(n)}(s)]^2 ds &= \lambda(-1)^n \int_0^T f(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_m(s)))x(s)ds \\
&\quad + \sum_{i=1}^{n} (-1)^{i+n+1}[x^{(2n-i)}(t_k^+)x^{(i-1)}(t_k^+) - x^{(2n-i)}(t_k)x^{(i-1)}(t_k)] \\
&\le (-1)^n \left(\int_0^T h(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_m(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds \right. \\
&\quad \left. + \sum_{i=1}^{m} \int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T e(s)x(s)ds \right) \\
&\le -\beta \int_0^T | x(s) |^{q+1} ds + \int_0^T | g_0(s, x(s)) \| x(s) | ds + \sum_{i=1}^{m} \int_0^T | g_i(s, x(\alpha_i(s)) \| x(s) | ds \\
&\quad + \int_0^T | e(s) \| x(s) | \\
&\le \left((r_0 + \varepsilon) + \sum_{k=1}^{m} (r_k + \varepsilon) \| \beta_k{}' \|_\infty^{q/(q+1)} \right) \int_0^T | x(s) |^{q+1} ds \\
&\quad + \left(\int_0^T | e(s) |^{(q+1)/q} ds \right)^{q/(q+1)} \left(\int_0^T | x(s) |^{q+1} ds \right)^{1/(q+1)} \\
&\quad + \sum_{k=0}^{m} g_{\delta,k} T^{q/(q+1)} \left(\int_0^T | x(s) |^{q+1} ds \right)^{1/(q+1)} \\
&\le \left((r_0 + \varepsilon) + \sum_{k=1}^{m} (r_k + \varepsilon) \| \beta_k{}' \|_\infty^{q/(q+1)} \right) M + \left(\int_0^T | e(s) |^{(q+1)/q} ds \right)^{q/(q+1)} M^{1/(q+1)} \\
&\quad + \sum_{k=0}^{m} g_{\delta,k} T^{q/(q+1)} M^{1/(q+1)} \\
&=: M_2.
\end{aligned}$$

Now, we see for $t \in [0, T]$ from (A_7) that

$$\begin{aligned}
| x(t) | &= \left| x(\xi) + \lambda \sum_{t \le t_k < \xi \text{ or } \xi \le t_k < t} I_{0,k}(x(t_k), \cdots, x^{(2n-1)}(t_k)) + \int_\xi^t x'(s)ds \right| \\
&\le (M/T)^{1/(q+1)} + \sum_{k=1}^{p} \alpha_{0,k} \| x \|_\infty + \int_0^T | x'(s) | ds.
\end{aligned}$$

Hence

$$\| x \|_\infty \le \frac{1}{1 - \sum_{k=1}^{p} \alpha_{0,k}} \left((M/T)^{1/(q+1)} + \int_0^T | x'(s) | ds \right).$$

For $i = 1, \cdots, n-1$, it is easy from (A_6) to get that there is $\xi_i \in [0, T]$ such that $x^{(i)}(\xi_i) = 0$, then from (A_7), we have

$$\begin{aligned}|x^{(i)}(t)| &= \left| x^{(i)}(\xi_i)+\lambda\sum_{\xi_i\le t_k<t \text{ or } t\le t_k<\xi_i} I_{i,k}(x_0,\cdots,x_{2n-1})+\int_{\xi_i}^{t}x^{(i+1)}(s)ds\right| \\ &\le \sum_{k=1}^{p}\alpha_{i,k}\|x^{(i)}\|_\infty+\int_0^T|x^{(i+1)}(s)|ds.\end{aligned}$$

This implies that

$$\|x^{(i)}\|_\infty\le\frac{1}{1-\sum_{k=1}^{p}\alpha_{i,k}}\int_0^T|x^{(i+1)}(s)|ds.$$

Hence

$$\begin{aligned}\|x\|_\infty &\le \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}}\left((M/T)^{1/(q+1)}+T\prod_{j=1}^{n-1}\frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}}\int_0^T|x^{(n)}(s)|ds\right) \\ &\le \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}}\left[(M/T)^{1/(q+1)}+T^{n-1}\prod_{j=1}^{n-1}\frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}}\left(\int_0^T|x^{(n)}(s)|ds\right)^{1/2}\right] \\ &\le \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}}\left[(M/T)^{1/(q+1)}+T^{n-1}\prod_{j=1}^{n-1}\frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}}M_2^{1/2}\right] \\ &=: M_1.\end{aligned}$$

It follows that Ω_1 is bounded.

Step 2. Let

$$\Omega_2=\{x\in \mathrm{Ker}L,\ Nx\in \mathrm{Im}L\}.$$

We prove Ω_2 is bounded. Suppose $x\in\Omega_2$, then $x(t)=c\in R$ and

$$\int_0^T f(t,c,c,\cdots,c)dt+\sum_{k=1}^{p}I_{n-1,k}(c,0,\cdots,0)=0.$$

It follows from (A_3) that $|c|\le M_0$.

Step 3. If the first case in (A_3) holds, let

$$\Omega_3=\{x\in \mathrm{Ker}L,\ \lambda\wedge x+(1-\lambda)QNx=0,\ \lambda\in[0,1]\},$$

where $\wedge:\mathrm{Ker}L\to \mathrm{Im}Q$ is the linear isomorphism given by $\wedge(c)=(c,0,\cdots,0)$ for all $c\in R$. Now we show that Ω_3 is bounded. Suppose $x_n(t)=c_n\in\Omega_3$ and $|c_n|\to+\infty$ as n tends to infinity. Then

$$\lambda\wedge(c_n)+(1-\lambda)\left(\frac{1}{T}\int_0^T f(t,c_n,\cdots,c_n)dt+\sum_{k=1}^{p}I_{n-1,k}(c_n,0,\cdots,0)\right)=0.$$

So

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$$\lambda c_n^2 = -(1-\lambda)c_n\left(\frac{1}{T}\int_0^T f(t,c_n,c_n,\cdots,c_n)dt + \sum_{k=1}^{p} I_{n-1,k}(c_n,0,\cdots,0)\right).$$

If $\lambda = 1$, then $c_n = 0$. If $\lambda \in [0,1)$ and $|c_n| > M_0$, then $\lambda c_n^2 < 0$, a contradiction. Hence $|c_n| \le M_0$. Ω_3 is bounded.

If the second case in (A_3) holds, let

$$\Omega_3 = \{x \in \mathrm{KerL}, \lambda \wedge x - (1-\lambda)QNx = 0, \lambda \in [0,1]\}.$$

Similar to above discussion, we get Ω_3 is bounded.

In the following, we shall show that all conditions of Lemma 2.1 are satisfied. Let Ω be a non-empty open bounded subset of X centered at zero point such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$ centered at zero. By Lemma 2.1, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(a). $Lx \ne \lambda Nx$ for $x \in (\mathrm{domL} \setminus \mathrm{KerL}) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b). $Nx \notin \mathrm{ImL}$ for $x \in \mathrm{KerL} \cap \partial\Omega$.

Step 4. We prove **(c).** $\deg(QN|_{\mathrm{KerL}}, \Omega \cap \mathrm{KerL}, 0) \ne 0$.

In fact, let $H(x,\lambda) = \lambda \wedge x \pm (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \ne 0$ for $x \in \partial\Omega \cap \mathrm{KerL}$, thus by the homotopy property of degree,

$$\begin{aligned} &deg(QN \mid KerL, \Omega \cap KerL, 0) = deg(H(\cdot,0), \Omega \cap KerL, 0) \\ = \quad &deg(H(\cdot,1), \Omega \cap KerL, 0) = deg(I, \Omega \cap KerL, 0) \ne 0 \ since\ 0 \in \Omega. \end{aligned}$$

Thus by Lemma 2.1, $Lx = Nx$ has at least one solution in $\mathrm{domL} \cap \overline{\Omega}$, which is a ω periodic solution of equation (17). The proof is complete.

4. Existence Results for the Odd Order Case

The even order case of PBVP(1)-(2) is as follows:

$$\begin{cases} x^{(2n+1)}(t)) = f(t,x(t),x(\alpha_1(t))\cdots,x(\alpha_m(t))), \text{ a.e. } t \in [0,T], \\ \Delta x^{(i)}(t_k) = I_{i,k}(x(t_k),\cdots,x^{(2n)}(t_k)), \ k = 1,\cdots,p, \\ x^{(i)}(0) = x^{(i)}(T), \ i = 0,\cdots,2n, \end{cases} \tag{22}$$

where $n \ge 1$ is an integer. Suppose

(A_8). For all $(x_0,\cdots,x_{2n}) \in R^{2n+1}$ and $i = 0,\cdots,n$ we have

$$\begin{aligned} &I_{n,k}(x_0,\cdots,x_{2n})(2x_n + I_{n,k}(x_0,\cdots,x_{2n})) \ge 0, \\ &(-1)^{i+n}\Big(x_{2n-i}I_{i,k}(x_0,\cdots,x_{2n}) + x_i I_{2n-i,k}(x_0,\cdots,x_{2n}) \\ &\qquad + I_{i,k}(x_0,\cdots,x_{2n})I_{2n-i,k}(x_0,\cdots,x_{2n})\Big) \ge 0. \end{aligned}$$

(A_9). For all $(x_0,\cdots,x_{2n}) \in R^{2n+1}$ and $i = 0,\cdots,n$ we have

$$\begin{aligned} &I_{n,k}(x_0,\cdots,x_{2n})(2x_n + I_{n,k}(x_0,\cdots,x_{2n})) \le 0, \\ &(-1)^{i+n}\Big(x_{2n-i}I_{i,k}(x_0,\cdots,x_{2n}) + x_i I_{2n-i,k}(x_0,\cdots,x_{2n}) \\ &\qquad + I_{i,k}(x_0,\cdots,x_{2n})I_{2n-i,k}(x_0,\cdots,x_{2n})\Big) \le 0. \end{aligned}$$

(A_{10}). For all $(x_0,\cdots,x_{2n}) \in R^{2n+1}$ and $i = 1,\cdots,2n-1$ and $k = 1,\cdots,p$ we have

$x_i(x_i + I_{i,k}(x_0,\cdots,x_{2n})) \ge 0$;

(A_{11}). There exist constants $\alpha_{i,k} \ge 0$ such that $|I_{i,k}(x_0,\cdots,x_{2n})| \le \alpha_{i,k}|x_i|$ with $\sum_{k=1}^{p}\alpha_{i,k} < 1$, $i = 0,\cdots,2n$ and $k = 1,\cdots,p$;

Theorem 4.1. Suppose (A_2), (A_3), (A_4), (A_8), (A_9), (A_{10}) and (A_{11}) hold. Then problem (22) has at least one solution if

$$r_0 + \sum_{k=1}^{m} r_k \|\beta_{k'}\|_{\infty}^{q/(q+1)} < \beta, \tag{23}$$

where $s = \beta_k(u)$ is the inverse function of $u = \alpha_k(s)$, $k = 1,\cdots,m$.

Proof: To apply Lemma 2.1, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Lemma 2.1 hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in domL \setminus KerL,\ Lx = \lambda Nx \ \ for\ some\ \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. Suppose $x \in \Omega_1$. Then

$$\begin{cases} x^{(2n+1)}(t) = \lambda f(t, x(t), x(\alpha_1(t)),\cdots, x(\alpha_m(t))),\ t \in [0,T],\ t \ne t_k,\ k = 1,\cdots,p, \\ \Delta x^{(i)}(t_k) = \lambda I_{i,k}(x(t_k),\cdots, x^{(2n-1)}(t_k)),\ k = 1,\cdots,p, i = 0,\cdots,2n, \\ x^{(i)}(0) = x^{(i)}(T),\ \ i = 0,\cdots,2n. \end{cases} \tag{24}$$

Substep 1.1. Prove that there is a constant $M > 0$ so that $\int_0^T |x(s)|^{q+1}ds \le M$.

Multiplying two sides of the first equation of (24) by $x(t)$, integrating it from 0 to T, we get from (A_2) that

$$\begin{aligned} &\sum_{i=1}^{n-1}(-1)^{i+1}\left(x^{(2n-i)}(T)x^{(i)}(T) - x^{(2n-i)}(0)x^{(i)}(0)\right) \\ &+\sum_{i=1}^{n-1}(-1)^{i+1}\sum_{k=1}^{p}\left(x^{(2n-i)}(t_k^+)x^{(i)}(t_k^+) - x^{(2n-i)}(t_k)x^{(i)}(t_k)\right) \\ &+\frac{(-1)^{n+1}}{2}\sum_{k=1}^{p}\left([x^{(n)}(t_k^+)]^2 - [x^{(n)}(t_k)]^2\right) + \frac{(-1)^n}{2}[x^2(T) - x^2(0)] \\ = \ \ & \lambda\int_0^T f(s, x(s), x(\alpha_1(s)),\cdots, x(\alpha_n(s)))x(s)ds \\ = \ \ & \lambda\left(\int_0^T h(s, x(s), x(\alpha_1(s)),\cdots, x(\alpha_n(s)))x(s)ds + \int_0^T g_0(s, x(s))x(s)ds\right. \\ &\left. + \sum_{i=1}^{n}\int_0^T g_i(s, x(\alpha_i(s))x(s)ds + \int_0^T e(s)x(s)ds\right). \end{aligned}$$

It follows from (A_8), for $i = 1,\cdots,n$, that

$$\begin{aligned} &(-1)^{i+n}\left[x^{(2n-i)}(t_k^+)x^{(i)}(t_k^+) - x^{(2n-i)}(t_k)x^{(i)}(t_k)\right] \\ = \ \ & (-1)^{i+n}\left(x_{2n-i}I_{i,k}(x_0,\cdots,x_{2n}) + x_i I_{2n-i,k}(x_0,\cdots,x_{2n})\right. \\ &\left. + I_{i,k}(x_0,\cdots,x_{2n})I_{2n-i,k}(x_0,\cdots,x_{2n})\right) \\ \ge \ \ & 0, \end{aligned}$$

and

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$$[x^{(n)}(t_k^+)]^2-[x^{(n)}(t_k)]^2=I_{n,k}(x_0,\cdots,x_{2n})(2x_n+I_{n,k}(x_0,\cdots,x_{2n}))\geq 0.$$

Hence we get

$$(-1)^{n+1}\left(\int_0^T h(s,x(s),x(\alpha_1(s)),\cdots,x(\alpha_n(s)))x(s)ds+\int_0^T g_0(s,x(s))x(s)ds\right.$$
$$\left.+\sum_{i=1}^{n}\int_0^T g_i(s,x(\alpha_i(s))x(s)ds+\int_0^T e(s)x(s)ds\right)\geq 0.$$

It follows from (A_2) that

$$\begin{aligned}&\beta\int_0^T |x(s)|^{q+1}ds\\ \leq\ & -(-1)^{n+1}\int_0^T g_0(s,x(s))x(s)ds-(-1)^{n+1}\sum_{i=1}^{n}\int_0^1 g_i(s,x(\alpha_i(s))x(s)ds-(-1)^{n+1}\int_0^T e(s)x(s)ds\\ \leq\ & \int_0^T |g_0(s,x(s))||x(s)|ds+\sum_{i=1}^{n}\int_0^T |g_i(s,x(\alpha_i(s))||x(s)|ds+\int_0^T |e(s)||x(s)|ds.\end{aligned}$$

Let $\varepsilon>0$ satisfy (20). For such $\varepsilon>0$, there is $\delta>0$ so that for every $i=0,1,\cdots,n$,

$$|g_i(t,x)|<(r_i+\varepsilon)|x|^q \textit{ uniformly for } t\in[0,T] \textit{ and } |x|>\delta. \tag{25}$$

Let, for $i=1,\cdots,n$,

$\Delta_{1,i}=\{t:t\in[0,T],|x(\alpha_i(t))|\leq\delta\}$, $\Delta_{2,i}=\{t:t\in[0,T],|x(\alpha_i(t))|>\delta\}$, $g_{\delta,i}=\max_{t\in[0,T],|x|\leq\delta}|g_i(t,x)|$, and $\Delta_1=\{t\in[0,T],|x(t)|\leq\delta\}$, $\Delta_2=\{t\in[0,T],|x(t)|>\delta\}$. Similar to that of Substep 1.1 in the proof of Theorem 3.1, we get that there is a constant $M>0$ so that $\int_0^T|x(s)|^{q+1}ds\leq M$.

Substep 1.2. Prove that there is a constant $M_1>0$ such that $\|x\|_\infty\leq M_1$.

It follows from Subcase 1.1 that there is $\xi\in[0,T]$ such that $|x(\xi)|\leq(M/T)^{1/(q+1)}$. (A_7) implies

$$\begin{aligned}|x(t)| &\leq \left|x(\xi)+\lambda\sum_{t\leq t_k<\xi \text{ or } \xi\leq t_k<t} I_{0,k}(x(t_k),\cdots,x^{(2n)}(t_k))+\int_\xi^t x'(s)ds\right|\\ &\leq (M/T)^{1/(q+1)}+\sum_{k=1}^{p}\alpha_{0,k}\|x\|_\infty+\int_0^T|x'(s)|ds.\end{aligned}$$

So

$$\|x\|_\infty\leq\frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}}\left((M/T)^{1/(q+1)}+\int_0^T|x'(s)|ds\right).$$

Similar to that of the Substep 1.2, from (A_{10}), there is $\xi_i\in[0,T]$ such that $x^{(i)}(\xi_i)=0$ for $i=1,\cdots,2n$. Thus we get

$$|x^{(i)}(t)|\leq\sum_{k=1}^{p}\alpha_{i,k}\|x^{(i)}\|_\infty+\int_0^1|x^{(i+1)}(s)|ds,\ i=1,\cdots,2n.$$

So

$$\| x^{(i)} \|_\infty \le \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \int_0^T | x^{(2n+1)}(s) | ds, \ \ i = 1,\cdots,2n.$$

Hence

$$\begin{aligned}
\| x \|_\infty \ \ \le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \left((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \int_0^T | x^{(2n+1)}(s) | ds \right) \\
\le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \Bigg((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \times \\
& \int_0^T \left(p_0(s) | x^q(s) | + \sum_{i=1}^{m} p_i(s) | x^q(\alpha_i(s)) | + r(s) \right) ds \Bigg) \\
\le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \Bigg((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \times \\
& \| p_0 \|_\infty \int_0^T | x(s) |^q \, ds + \sum_{i=1}^{m} \| p_i \|_\infty \int_0^T | x(\alpha_i(s)) |^q \, ds + \int_0^T r(s) ds \Bigg) \\
\le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \Bigg((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \times \\
& \| p_0 \|_\infty \int_0^T | x(s) |^q \, ds + \sum_{i=1}^{m} \| p_i \|_\infty \| \beta_{i'} \|_\infty \int_0^T | x(s) |^q \, ds + \int_0^T r(s) ds \Bigg) \\
\le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \Bigg((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \times \\
& \left(\| p_0 \|_\infty \ T + \sum_{i=1}^{m} \| p_i \|_\infty \| \beta_{i'} \|_\infty \ T \right) \left(\int_0^T | x(s) |^{1+q} \, ds \right)^{q/(q+1)} + \int_0^T r(s) ds \Bigg) \\
\le\ & \frac{1}{1-\sum_{k=1}^{p}\alpha_{0,k}} \Bigg((M/T)^{1/(q+1)} + T^{2n} \prod_{j=i}^{2n} \frac{1}{1-\sum_{k=1}^{p}\alpha_{j,k}} \times \\
& \left(\| p_0 \|_\infty \ T + \sum_{i=1}^{m} \| p_i \|_\infty \| \beta_{i'} \|_\infty \ T \right) M^{q/(q+1)} + \int_0^T r(s) ds \Bigg).
\end{aligned}$$

It follows that Ω_1 is bounded.

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The remainder of the proof are similar to those of the proof of the Theorem 3.1 and are omitted. The proof is complete.

Theorem 4.2. Suppose (A_1), (A_3), (A_4), (A_8), (A_9), (A_{10}) and (A_{11}) hold. Then problem (22) has at least one solution if (23) holds.
Proof. It is similar to that of Theorem 4.1 and is omitted.

5. Examples

In this section, two examples are discussed to illustrate the obtained results.

Example 5.1. Consider the problem

$$\begin{cases} x''(t) = e(t) + [\beta + x^2(t)][x(t)]^{2q+1} + \sum_{i=1}^{m} \beta_i [x(\frac{t}{i})]^{2q+1}, \\ \Delta x(t_k) = a_k x(t_k), \ k = 1, \cdots, p, \\ \Delta x'(t_k) = b_k x'(t_k), \ k = 1, \cdots, p, \\ x(0) = x(T), \ x'(0) = x'(T). \end{cases} \tag{26}$$

Corresponding to problem (17), we have $n = 1$, $\alpha_i(t) = \frac{t}{i}$, $e \in C([0,T],[0,T])$, $\beta, \beta_i, a_k, b_k \in R$ and

$$f(t, x_0, \cdots, x_m) = e(t) + [\beta + x_0^2] x_0^{2q+1} + \sum_{i=1}^{m} \beta_i x_i^{2q+1}, \ I_{0,k}(x) = a_k x, \ I_{1,k}(x) = b_k x.$$

Let

$$h(t, x_0, \cdots, x_m) = [\beta + x_0^2] x_0^{2q+1}, \ g_i(t, x) = \beta_i x^{2q+1}, i = 1, \cdots, m.$$

$\beta < 0$ implies that (A_1) holds.
$b_k(2 + b_k) \geq 0$ implies that (A_5) holds.
Since $n = 1$, (A_6) holds.
$\sum_{i=1}^{p} |a_k| < 1$ implies that (A_7) holds.

$$c\left[\int_0^T f(t, c, \cdots, c)dt + \sum_{k=1}^{p} I_{1,k}(c, 0, \cdots, 0)\right] = c\left[\int_0^T e(s)ds + \left(\beta + \sum_{k=1}^{p} \beta_k\right) c^{2q+1}\right],$$

it is easy to see that there is a constant $M_0 > 0$ such that (A_3) holds if $\beta + \sum_{k=1}^{p} \beta_k \neq 0$.
It follows from Theorem 3.1 that problem (26) has at least one solution if

$$\beta < 0, \sum_{k=1}^{p} |a_k| < 1, \ \beta + \sum_{k=1}^{p} \beta_k \neq 0, a_k b_k + a_k + b_k = 0, \ \sum_{k=1}^{m} |\beta_k| k^{(2q+1)/(2q+2)} < -\beta.$$

Example 5.2. Consider the problem

$$\begin{cases} x'''(t) = e(t) + \beta[x(t)]^{2q+1} + \sum_{i=1}^{m} \beta_i [x(\frac{t}{i})]^{2q+1}, \\ \Delta x(t_k) = a_k x(t_k), \ k = 1, \cdots, p, \\ \Delta x'(t_k) = b_k x'(t_k), \ k = 1, \cdots, p, \\ \Delta x''(t_k) = c_k x''(t_k), \ k = 1, \cdots, p, \\ x(0) = x(T), \ x'(0) = x'(T), \ x''(0) = x''(T). \end{cases} \tag{27}$$

Corresponding to problem (22), we have $n = 1$, $\alpha_i(t) = \frac{t}{i}$, $e \in C([0,T],[0,T])$, $\beta, \beta_i, a_k, b_k \in R$ and

$$f(t, x_0, \cdots, x_m) = e(t) + \beta x_0^{2q+1} + \sum_{i=1}^{m} \beta_i x_i^{2q+1}, \ I_{0,k}(x) = a_k x, \ I_{1,k}(x) = b_k x.$$

Let

$$h(t, x_0, \cdots, x_m) = \beta x_0^{2q+1}, \ g_i(t, x) = \beta_i x^{2q+1}, i = 1, \cdots, m.$$

$\beta < 0$ implies that (A_2) holds.

$b_k \geq -1$ and $b_k(2 + b_k) \geq 0$ implies that (A_{10}) holds.

$a_k + b_k + a_k b_k = 0$ implies that (A_8) holds.

Since $n = 1$, (A_9) holds.

$\sum_{i=1}^{p} |a_k| < 1$, $\sum_{i=1}^{p} |b_k| < 1$ and $\sum_{i=1}^{p} |c_k| < 1$ imply that (A_{11}) holds.

$$c\left[\int_0^T f(t, c, \cdots, c)dt + \sum_{k=1}^{p} I_{1,k}(c, 0, \cdots, 0)\right] = c\left[\int_0^T e(s)ds + \left(\beta + \sum_{k=1}^{p} \beta_k\right) c^{2q+1}\right],$$

it is easy to see that there is a constant $M_0 > 0$ such that (A_3) holds if $\beta + \sum_{k=1}^{p} \beta_k \neq 0$.

It is easy to see that (A_4) holds.

It follows from Theorem 4.1 that problem (27) has at least one solution if

$$\beta < 0, \sum_{k=1}^{p} |a_k| < 1, \ \sum_{i=1}^{p} |b_k| < 1 \ \sum_{i=1}^{p} |c_k| < 1, \ b_k \geq -1, \ a_k b_k + a_k + b_k = 0,$$

and

$$\sum_{k=1}^{m} |\beta_k| k^{(2q+1)/(2q+2)} < -\beta.$$

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