A DECOMPOSITION OF CONTINUITY ON F*- SPACES
AND MAPPINGS ON SA*- SPACES

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Abstract: An ideal topological space (X, τ, I) is said to be an F* – space if A=Cl*(A) for every open set A ⊂ X. In this paper, a decomposition of continuity on F* – spaces is introduced. An ideal topological space (X, τ, I) is said to be an SA* – space if (A)* ⊂ A for every set A⊂X. It is shown that δI– r – continuity (resp. pre – I – continuity, semi – δ – I – continuity, * – perfect continuity) is equivalent to R – I – continuity (resp. R – I – continuity, t – I – continuity, * – dense – in – itself continuity) if the domain is an SA* – space.


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1. INTRODUCTION

Recently, ACIKGOZ et al. (2004) introduced the notion of a “δ – I – open set” in an ideal topological space, investigated some of its properties and obtained a decomposition of a α – I – continuous function using this set. HATIR & NOIRI
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(2002) introduced the notions of \( t-I \) – sets, \( \alpha^* - I \) – sets, \( B_I \) – sets and \( C_I \) – sets. YÜKSEL et al. (2005) introduced the notion of an \( R-I \) – open set and obtained some of its properties.

The purpose of this paper is to introduce a decomposition of continuity on \( F^* \) – spaces and also to show that \( \delta_t - r \) – continuity ( resp. \( \text{pre} - I \) – continuity, \( \text{semi} - \delta - I \) – continuity, \( * - \text{perfect continuity} \) ) is equivalent to \( R-I \) – continuity ( resp. \( R-I \) – continuity, \( t-I \) – continuity, \( * - \text{dense} - \text{in} - \text{itself continuity} \) ) if the domain is an \( SA^* \) – space.

2. PRELIMINARIES

Let \((X, \tau)\) be a topological space, and \( A \subset X \). Throughout this paper \( \text{Cl}(A) \) and \( \text{Int}(A) \) denote the closure and the interior of \( A \) with respect to \( \tau \), respectively.

An ideal, \( I \) is defined as a nonempty collection of subsets of \( X \) satisfying the following two conditions: (1) If \( A \in I \) and \( B \subset A \), then \( B \in I \); (2) If \( A \in I \) and \( B \in I \), then \( A \cup B \in I \). An ideal topological space is a topological space \((X, \tau, I)\) with an ideal \( I \) on \( X \) and is denoted by \((X, \tau, I)\). For a subset \( A \) of \( X \), \( A^* (I) = \{ x \in X \mid \exists U \cap A \notin I \text{ for each neighborhood } U \text{ of } x \} \) is called the local function of \( A \) with respect to \( I \) (KESKİN et al. 2004). We simply write \( A^* \) instead of \( A^*(I) \) when there is no chance for confusion. Note that \( X^* \) is often a proper subset of \( X \). The hypothesis that \( X = X^* \) (HATIR & NOIRI 2005) is equivalent to the hypothesis that \( \tau \cap I = \emptyset \) (Levine, 1963). The ideal topological spaces which satisfy this hypothesis are called Hayashi – Samuels space (ANKOVIĆ & HAMLETT 1990). For every ideal topological space \((X, \tau, I)\), there exists a topology \( \tau^*(I) \), finer than \( \tau \), generated by \( \beta(I, \tau) = \{ U \setminus I \mid U \in \tau \text{ and } I \in I \} \), but in general \( \beta(I, \tau) \) is not always a topology (JANKOVIĆ & HAMLETT 1990).

Additionally, \( \text{Cl}^*(A) = A \cup A^* \) defines a Kuratowski closure operator for \( \tau^*(I) \).

First we shall recall some definitions that will be used in the sequel.

DEFINITION 1. A subset \( A \) of an ideal topological space \((X, \tau)\) is said to be regular open (DUGUNDJI 1966) ( semi – open (KURATOWSKI 1966)) if \( A = \text{Int} \text{(Cl}(A)) \) (\( A \subset \text{Cl}(\text{Int}(A)) \)).

DEFINITION 2. A subset \( A \) of an ideal topological space \((X, \tau, I)\) is said to be

a) \( \alpha - I \) – open (HATIR & NOIRI 2002) if \( A \subset \text{Int}(\text{Cl}^*(\text{Int}(A))) \),
b) \( \alpha^* - I \) – set (HATIR & NOIRI 2002) if \( A = \text{Int}(\text{Cl}^*(\text{Int}(A))) \),
c) \( \text{pre} - I \) – open (DONTCHEV 1996) if \( A \subset \text{Int}(\text{Cl}^*(A)) \),
d) \( R - I \) – open (YUKSEL et. al. 2005) if \( A = \text{Int}(\text{Cl}^*(A)) \),
e) \( t - I \) – set (HATIR & NOIRI 2002) if \( \text{Int}(A) = \text{Int}(\text{Cl}^*(A)) \),
f) \( \delta - I \) – open (ACİKGOZ et. al. 2004) if \( \text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A)) \),
g) \( \text{regular} - I \) – closed (SAMUELS 1975) if \( A = (\text{Int}(A))^* \),
h) \( I - \text{open} \) (ABD EL – MONSEF et. al. 1992) if \( A \subset \text{Cl}(\text{Int}(A)) \),
i) \( fi - \text{set} \) (KESKİN et. al. 2004) if \( A \subset \text{Cl}(\text{Int}(A))^* \),
j) \( \text{semi} - I - \text{open} \) (HATIR & NOIRI 2002) if \( A \subset \text{Cl}(\text{Int}(A)) \),
k) $\beta-I$–open (HATIR & NOIRI 2002) if $A \subset \text{Cl} (\text{Int}(\text{Cl}^*(A)))$,
l) $\delta - I$–perfect (HAYASHI 1964) if $A = A^*$,
m) $\delta - I$–dense in itself (HAYASHI 1964) if $A \subset A^*$,
n) $I$–locally closed (DONTCHEV 1999) if $A = U \cap V$, where $U$ is open and $V$ is $\delta - I$–perfect,
o) $B_I$–set (HATIR & NOIRI 2002) if $A = U \cap V$, where $U$ is open and $V$ is $t - I$–set,
p) $C_I$–set (HATIR & NOIRI 2002) if $A = U \cap V$, where $U$ is open and $V$ is $\alpha^*-I$–set.

The family of all $R-I$–open (resp. $\alpha-I$–open, pre $- I$–open, $t-I$–set, $\delta-I$–open, $\alpha^*$–perfect set, $\delta^*$–dense in itself) sets in an ideal topological space $(X, \tau, I)$ is denoted by $RIO (X, \tau)$ (resp. $\alpha IO (X, \tau)$, $PIO (X, \tau)$, $tIO (X, \tau)$, $\delta IO (X, \tau)$, $\alpha^* PI (X, \tau)$, $\delta^* DI (X, \tau)$).

**DEFINITION 3.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\delta-I$–regular (ACIKGOZ & YUKSEL 2006) if $A$ is both a pre $- I$–open set and a $\delta-I$–open set.

The family of all $\delta- I$–regular sets of $(X, \tau, I)$ is denoted by $\delta IR (X, \tau)$, when there is no chance for confusion with the ideal.

The following diagram is given by Acikgoz et al. (ACIKGOZ & YUKSEL 2006).

3. ON $F^*$–SPACES AND $SA^*$–SPACES

**PROPOSITION 1.** Let $(X, \tau, I)$ be an ideal topological space and $A$ a subset of $X$. Then the following properties hold:

a) If $A$ is an $R-I$–open set and $(X, \tau, I)$ is a Hayashi-Samuels space, then $A$ is an $I$–locally closed set,

b) If $A$ is an $R-I$–open set, then $A$ is a $B_I$–set.

c) If $A$ is a $B_I$–set, then $A$ is a $C_I$–set.

**PROOF.** a) Let $A$ be an $R-I$–open set. Since $(X, \tau, I)$ is a Hayashi-Samuels space, then $X^* = X$. Since every $R-I$–open set is an open set by (ACIKGOZ & YUKSEL 2006) and $X$ is a $*-$perfect set, $A = A \cap X$ is an $I$–locally closed set.
b) Let $A$ be an $R - I$ open set. Hence $A$ is a $t - I$ set by (ACIKGOZ & YUKSEL 2006). Since $X$ is an open set, $A = A \cap X$ is a $B_1$ set.

c) The proof is obvious from (HATIR & NOIRI 2002).

REMARK 1. The converse of Proposition 1(b) need not be true as shown in the following example. Also, HATIR & NOIRI (2002) showed that $C_1$ set is not a $B_1$ set, in general.

EXAMPLE 1. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, $I = \{\emptyset, \{c\}\}$. Set $A = \{d\}$. Then $A$ is a $B_1$ set which is not an $R - I$ open set. For $A = \{d\}$, since $Cl^*(A) = \{c, d\}$, $Int(Cl^*(A)) = \emptyset$ and $Int(A) = Int(Cl^*(A))$, $A$ is a $t - I$ set. By [7, Proposition 3.1(c)] since every $t - I$ set is a $B_1$ set, $A$ is a $B_1$ set. On the other hand, since $Cl^*(Int(A)) = \emptyset \neq A$, $A$ is not an $R - I$ open set.

DIAGRAM II has been expanded by using DIAGRAM I. ACIKGOZ et al. (2004) also have showed that every semi $- I$ open set is a $\delta - I$ open set and definitions of pre $- I$ open sets and $\delta - I$ open sets are independent concepts. HATIR & NOIRI (2002) have already showed that every open set is a $B_1$ set and every $B_1$ set is a $C_1$ set.

PROPOSITION 2. For a subset, $A$ of an ideal topological space, $(X, \tau, I)$, the following properties hold:

a) Every regular $I$ closed set is a $\delta - I$ open set,

b) Every $\delta - I$ regular set is a $\beta - I$ open set,

c) Every $* - \mu$ perfect set is a $\delta - I$ open set.

PROOF. a) Let $A$ be a regular $I$ closed set. Then $A = (Int(A))^*$ and so $A \subset Cl^*(Int(A))$. By Definition 2, $A$ is a semi $- I$ open set. Hence, $A$ is a $\delta - I$ open set using Diagram II.
b) Let \( A \) be a \( \delta - I \) – regular set. By Definition 3, \( A \) is a pre – \( I \) – open set. Then \( A \subset \text{Int} (\text{Cl}^*(A)) \subset \text{Cl}(\text{Int}(\text{Cl}^*(A))) \). Hence \( A \) is a \( \beta - I \) – open set.

c) Let \( A \) be a * – perfect set. Then \( A = (A)^* \) and so \( \text{Cl}^*(A) = A \cup A^* = A \). Hence \( \text{Int}(\text{Cl}^*(A)) = \text{Int}(A) \), and therefore, \( A \) is a \( t - I \) – set. Since every \( t - I \) – set is a \( \delta - I \) – open set by Diagram I, \( A \) is \( \delta - I \) – open set.

REMARK 2. The converses of Proposition 2 need not be true as shown in the following examples.

EXAMPLE 2. Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{a,c\}, \{a,c,d\}\}, I = \{\emptyset, \{a\}\} \). Set \( A = \{a,c\} \). Then \( A \) is a \( \delta - I \) – open set which is not a regular \( I \) – closed set. For \( A = \{a,c\} \), since \( \text{Cl}^*(A) = \{a,b,c\} \), \( \text{Int}(\text{Cl}^*(A)) = \{a,c\} \), \( \text{Cl}^*(\text{Int}(A)) = \text{Cl}^*(\{a,c\}) = \{a,b,c\} \) and \( \text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A)) \), \( A \) is a \( \delta - I \) – open set. On the other hand, since \( A \not\subset \text{Int}(\text{Cl}^*(A)) \), \( A \) is not a regular \( I \) – closed set.

EXAMPLE 3. Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{a,c\}, \{a,c,d\}\}, I = \{\emptyset, \{a\}, \{c\}, \{a,c\}\} \). Set \( A = \{c,d\} \). Then \( A \) is a \( \beta - I \) – open set which is not a \( \delta - I \) – regular set. For \( A = \{c,d\} \), since \( \text{Cl}^*(A) = \{b,c,d\} \), \( \text{Int}(\text{Cl}^*(A)) = \{c\} \), \( \text{Cl}(\text{Int}(\text{Cl}^*(A))) = \{b,c,d\} \) and so \( A \subset \text{Cl}(\text{Int}(\text{Cl}^*(A))) \). This shows that \( A \) is not a \( \beta - I \) – open set.

EXAMPLE 4. Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{a,b\}, \{a,b,c\}\}, I = \{\emptyset, \{a\}, \{d\}\} \). Set \( A = \{c,d\} \). Then \( A \) is a \( \delta - I \) – open set which is not a * – perfect set. For \( A = \{c,d\} \), since \( \text{Cl}^*(A) = \{c\} \), \( \text{Cl}^*(\text{Int}(A)) = \{c\} \) and so \( \text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A)) \). This shows that \( A \) is a \( \delta - I \) – open set. On the other hand, since \( (A)^* = \{c\} \neq A \), \( A \) is not a * – perfect set.

REMARK 3. Using the two examples presented below, it is shown that \( R - I \) – open sets and regular \( I \) – closed sets are independent of each other.

EXAMPLE 5. Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{b\}, \{a,c\}, \{a,b,c\}\}, I = \{\emptyset, \{a\}, \{d\}\} \). Set \( A = \{b\} \). Then \( A \) is a \( R - I \) – open set which is not a regular \( I \) – closed set. For \( A = \{b\} \), since \( \text{Cl}^*(A) = \{b,d\} \) and \( \text{Int}(\text{Cl}^*(A)) = \{b\} \). This shows that \( A \) is not an \( R - I \) – open set. On the other hand, since \( \text{Int}(A)^* = \{b,d\} \neq A \), \( A \) is not a regular \( I \) – closed set.

EXAMPLE 6. Let \( X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}, I = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \). Set \( A = \{b,c,d\} \). Then \( A \) is a regular \( I \) – closed set which is not an \( R - I \) – open set. For \( A = \{b,c,d\} \), since \( \text{Int}(A)^* = \{b,c,d\} \). This shows that \( A \) is a regular \( I \) – closed set. On the other hand, since \( \text{Cl}^*(\text{Int}(A)) = \{b,c,d\} \neq A \), \( A \) is not an \( R - I \) – open set.

REMARK 4. The following diagram showing the relationship among several sets defined above, is obtained using DIAGRAM II, Proposition 2 and the Diagram in (KESKİN et. al. 2004).
DEFINITION 4. An ideal topological space \((X, \tau, I)\) is said to be a \(F^* - space\) if \(A=\text{Cl}^*(A)\) for every open set \(A \subset X\).

EXAMPLE 7. Let \(X = \{a,b,c\}, \tau = \{\emptyset, X, \{c\}\}, I = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}.\) Then \(X\) is a \(F^* - space\). For every \(A \in \tau\), we have since \((A)^* \subset A\).

EXAMPLE 8. Let \(X = \{a,b,c,d\}, \tau = \{\emptyset, X, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,c,d\}\}, I = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.\) For every \(A \in \tau\), since \((A)^* \not\subset A\), we have \(X\) is not \(F^* - space\).

DEFINITION 5. A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be

\(a)\) \(A\) semi – \(I\) – closed if (HATIR & NOIRI 2005) \(\text{Int(Cl}^*(A)) \subset A\),

\(b)\) \(A\) SC – \(I\) – open set if \(A = U \cap V\), where \(U \in \tau\) and \(A\) is semi – \(I\) – closed set.

THEOREM 1. For a subset \(A\) of an ideal topological space \((X, \tau, I)\), the following property holds: \(A\) is an open set if and only if \(A\) is an \(\alpha - I\) – open set and a SC – \(I\) – open set.

PROOF. The necessity is obvious.
Sufficiency: Let \(A\) be an \(\alpha - I\) – open and a SC – \(I\) – open set. Then \(A \subset \text{Int(Cl}^*(\text{Int}(A)))\) and \(A = U \cap V\), where \(U \in \tau\) and \(V\) is semi – \(I\) – closed.
We have $A \subset \text{Int}(\text{Cl}^*(\text{Int}(A))) = \text{Int}(\text{Cl}^*(\text{Int}(U \cap V))) \subset \text{Int}(\text{Cl}^*(U \cap V)) \subset \text{Int}(\text{Cl}^*(U)) \cap \text{Int}(\text{Cl}^*(V)) = \text{Int}(\text{Cl}^*(U)) \cap \text{Int}(V)$. Thus we obtain $A = U \cap A \subset U \cap \text{Int}(\text{Cl}^*(U)) \subset \text{Int}(U \cap V) = \text{Int}(A)$ and hence $A$ is open – set.

PROPOSITION 3. For a subset, $A$ of a F* – space $(X,\tau,I)$, the following properties hold: $A$ is an open set if and only if $A$ is a pre – I – open set and a $\delta$ – I – open set.

PROOF. Necessity: The proof is obvious from DIAGRAM I.
Sufficiency: Let $A$ be a pre – I – open set. Then $A \subset \text{Int}(\text{Cl}^*(A))$. Since $A$ is a $\delta$ – I – open set, $\text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A))$. Furthermore by hypothesis, since $X$ is also an F* – space, $\text{Cl}^*(\text{Int}(A)) \subset \text{Int}(A)$. Consequently, $A \subset \text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A)) \subset \text{Int}(A)$, that is, $A = \text{Int}(A)$ and hence $A$ is an open set.

DEFINITION 6. An ideal topological $(X,\tau,I)$ is said to be an $SA^*$ – space if $(A)^* \subset A$ for every set $A \subset X$.

EXAMPLE 9. Let $X = \{a,b,c\}$, $\tau = \{\emptyset,X,\{a,c\}\}$, $I = \{\emptyset,\{a\},\{c\},\{a,c\}\}$. Then $X$ is an $SA^*$ – space. For every $A \subset X$ since $(A)^* \subset A$, $X$ is an $SA^*$ – space.

THEOREM 2. Every $SA^*$ – space is F* – space.

REMARK 5. The converse of Theorem 2 need to be true as Example 7 shows.

In any ideal topological space $(X,\tau,I)$ $SA^*$ – space, we have the following fundamental relationships between the classes of subsets of $X$ considered:

PROPOSITION 4. For a subset $A$ of an ideal topological space $(X,\tau,I)$ $SA^*$ – space, the following properties hold:

a) $\delta R (X,\tau) = \text{RIO} (X,\tau)$,
b) $tIO (X,\tau) = \delta IO (X,\tau)$,
c) $*PI (X,\tau) = *DI (X,\tau)$.

PROOF. a) Necessity: By [3, Proposition 3(a)], we have $\text{RIO} (X,\tau) \subset \delta R (X,\tau)$.
Sufficiency: Let $A$ be a $\delta$ – I – regular set. According to Definition 3, $A$ is both a $\delta$ – I – open set and a pre – I – open set. Hence $A \subset \text{Int}(\text{Cl}^*(A)) \subset \text{Cl}^*(\text{Int}(A))$. Furthermore by hypothesis, since $X$ is also an $SA^*$ – space, $(\text{Int}(A))^* \subset \text{Int}(A)$, $(\text{Cl}^*(\text{Int}(A))) = (\text{Int}(A) \cup (\text{Int}(A)^*)) \subset \text{Int}(A)$ and $\text{Cl}^*(\text{Int}(A)) \subset \text{Int}(A)$. Consequently, $A = \text{Int}(\text{Cl}^*(A))$ and hence $A$ is an $R – I$ – open set.

b) and c) are analogous to the Proof of (a) and are thus omitted.

PROPOSITION 5. For a subset $A$ of an $SA^*$ ideal topological space $(X,\tau,I)$, the following properties hold:

a) Every $I$ – open set is an $R$ – I – open set,
b) Every $f_I$ – set is an open set,
c) Every $\beta$ – I – open set is a semi – open set.
PROOF. a) Let A be an I – open set. Then $A \subset \text{Int}((A)^*)$. Furthermore by hypothesis, since X is also an SA* – space, $(A)^* \subset A$ and $\text{Cl}^*(A) \subset A$. Consequently, $A \subset \text{Int}((A)^*) \subset \text{Int}(\text{Cl}^*(A)) \subset \text{Int}(A)$ and so $A = \text{Int}(\text{Cl}^*(A))$. Hence A is an R – I – open set.

b) Let A be an $f_I$ – set. By Definition 2, we have $A \subset (\text{Int}(A))^*$. Since X is also an SA* – space, $(\text{Int}(A))^* \subset \text{Int}(A)$ and so $A \subset (\text{Int}(A))^* \subset \text{Int}(A)$. Hence A is an open set.

c) Let A be an $\beta$ – I – open set. Then $A \subset \text{Cl}(\text{Int}(\text{Cl}^*(A)))$. Since X is an SA* – space, $A \subset \text{Cl}(\text{Int}(\text{Cl}^*(A))) \subset \text{Cl}(\text{Int}(A))$ and so $A \subset \text{Cl}(\text{Int}(A))$. Hence A is a semi – open set.

DEFINITION 7. A function $f : (X, \tau) \rightarrow (Y, \varphi)$ between two topological spaces is said to be semi – continuous (KURATOWSKI 1966) if for every $V \in \varphi$, $f^{-1}(A)$ is semi – open of $(X, \tau)$.


We are now able to provide a decomposition of continuity in this setting.

THEOREM 3. Let $(X, \tau, I)$ be an F* – space. For a function $f : (X, \tau, I) \rightarrow (Y, \varphi)$, the following properties are equivalent:

a) $f$ is continuous,

b) $f$ is pre – I – continuous and semi – $\delta$ – I – continuous.

PROOF. This follows from Proposition 3.

THEOREM 4. Let $(X, \tau, I)$ be an SA* – space. For a function $f : (X, \tau, I) \rightarrow (Y, \varphi)$, the following equivalences hold:

a) $f$ is $\delta_I – r – continuous$ if and only if it is R – I – continuous,

b) $f$ is semi – $\delta$ – I – continuous if and only if it is $t – I – continuous$,

c) $f$ is $* – perfect continuous$ and if and only if it is $* – dense – in – itself continuous$.

PROOF. This follows from Proposition 4.

THEOREM 5. Let $(X, \tau, I)$ be an SA* – space. For a function $f : (X, \tau, I) \rightarrow (Y, \varphi)$, the following implications hold:

a) If $f$ is I – continuous, then it is R – I – continuous,

b) If $f$ is $f_I$ – continuous, then it is continuous,

c) If $f$ is $\beta$ – I – continuous, then it is semi – continuous.
PROOF. This follows from Proposition 5.

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REFERENCES


HAYASHI E, 1964. Topologies defined by local properties, Mathematische Annalen, 156, 205–215.


