ON GENERALIZED $\phi$ – RECURRENT KENMOTSU MANIFOLDS

Ash BAŞARI, Cengizhan MURATHAN

Department of Mathematics, Faculty of Science, University of Uludağ
Görükle Campus, 16059 Bursa, TURKEY.
e-mail: aslibasari@gmail.com; cengiz@uludag.edu.tr
Received: 12 February 2008, Accepted: 07 March 2008

Abstract: The purpose of this paper is to study generalized $\phi$ – recurrent Kenmotsu manifolds.

Key words: Kenmotsu manifold, generalized recurrent, $\phi$ – recurrent manifold, Einstein manifold.

1. INTRODUCTION

A Riemannian manifold $(M^n, g)$ is called generalized recurrent (DE & GUHA 1991) if its curvature tensor $R$ satisfies the condition

$$(\nabla_X R)(Y, Z)W = \alpha(X)R(Y, Z)W + \beta(X)[g(Z, W)Y - g(Y, W)Z],$$

where, $\alpha$ and $\beta$ are two 1-forms, $\beta$ is non-zero and these are defined by:

$\alpha(X) g(X, A), \beta(X) = g(X, B),$

A and B are vector fields associated with 1-forms $\alpha$ and $\beta$, respectively.

ÖZGÜR (2007) studied generalized recurrent Kenmotsu manifolds. He showed that for a generalized recurrent Kenmotsu manifold $\alpha = \beta$.

In their study VENKATESHA & BAGEWADI (2006) studied pseudo-projective $\phi$ – recurrent Kenmotsu manifolds. It was shown that for a pseudo-projective $\phi$ – recurrent Kenmotsu manifold is an Einstein manifold and also a space of constant curvature.

Motivated by the above studies, in this paper, we define generalized $\phi$ – recurrent and generalized concircular $\phi$ – recurrent Kenmotsu manifolds and obtain some interesting results.
The paper is organized as follows. In Preliminaries, we give a brief account of Kenmotsu manifolds. In Section 3, we show that a generalized $\phi -$recurrent or a generalized concircular $\phi -$recurrent Kenmotsu manifold $(M^{2n+1}, g)$ is an Einstein manifold. We also find some relations between the associated 1-froms $\alpha$ and $\beta$ for a generalized $\phi -$recurrent and a generalized concircular $\phi -$recurrent Kenmotsu manifold.

2. PRELIMINARIES

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $2n+1$-dimensional almost contact Riemannian manifold, where $\phi$ is a $(1, 1)$-tensor field, $\xi$ is the structure vector field, $\eta$ is a 1-form and $g$ is the Riemannian metric. It is well known $(\phi, \xi, \eta, g)$-structure satisfy the conditions (BLAIR 1976)

\begin{align}
(2.1) & \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\
(2.2) & \quad \phi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X), \\
(2.3) & \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\end{align}

for any vector fields $X$ and $Y$ on $M^n$. If moreover

\begin{align}
(2.4) & \quad (\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \\
(2.5) & \quad \nabla_X \xi = X - \eta(X)\xi,
\end{align}

where $\nabla$ denotes the Riemannian connection of $g$ hold, then $(M^{2n+1}, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold.

In this case, it is well known that KENMOTSU (1972)

\begin{align}
(2.6) & \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \\
(2.7) & \quad S(X, \xi) = -2n\eta(X),
\end{align}

where $S$ denotes the Ricci tensor. From (2.6), it easily follows that

\begin{align}
(2.8) & \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \\
(2.9) & \quad R(X, \xi)\xi = \eta(X)\xi - X, \\
(2.10) & \quad \eta(R(X, Y)V) = \eta(Y)g(X, V) - \eta(X)g(Y, V).
\end{align}

Since $S(X, Y) = g(QX, Y)$, we have $S(\phi X, \phi Y) = g(Q\phi X, \phi Y)$, where $Q$ is the Ricci operator. Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.2) and (2.7), we get

\begin{align}
(2.11) & \quad S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y).
\end{align}
Also we have KENMOTSU (1972)

\[(2.12) \quad (\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y).\]

Kenmotsu manifold $M^{2n+1}$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

\[(2.13) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),\]

for any vector fields $X$ and $Y$, where $a$ and $b$ are functions on $M^n$.

3. GENERALIZED $\phi$ – RECURRENT KENMOTSU MANIFOLDS

**Definition 3.1.** Kenmotsu manifold \((M^{2n+1}, g)\) is called generalized $\phi$–recurrent if its curvature tensor $R$ satisfies the condition

\[(3.1) \quad \phi^2((\nabla_W R)(X, Y)Z) = \alpha(W)R(X, Y)Z + \beta(W)[g(Y, Z)X - g(X, Z)Y]\]

where, $\alpha$ and $\beta$ are two 1-forms, $\beta$ is non-zero and these are defined by:

\[(3.2) \quad \alpha(W) = g(W, A), \quad \beta(W) = g(W, B)\]

and $A, B$ are vector fields associated with 1-forms $\alpha$ and $\beta$, respectively (TAKAHASHI 1977, DE & GUHA 1991).

From (3.1), using (2.2) we have

\[(3.3) \quad -g((\nabla_W R)(X, Y)Z) + \eta((\nabla_W R)(X, Y)Z)\xi = \alpha(W)R(X, Y)Z + \beta(W)[g(Y, Z)X - g(X, Z)Y]\]

from which it follows that

\[(3.4) \quad -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) = \alpha(W)[g(R(X, Y)Z, U) + \beta(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]\]

Let \(\{e_i\}, \quad i = 1, 2, \ldots, 2n+1\), be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over $i$, $1 \leq i \leq 2n + 1$, we get

\[(3.5) \quad -(\nabla_W S)(Y, Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = \alpha(W)S(Y, Z) + 2n\beta(W)g(Y, Z).\]
The second term of (3.5) is reduced to
\[ \sum_{i=1}^{2n+1} \eta((\nabla_w R)(e_i, Y)Z)\eta(e_i) = g((\nabla_w R)(\xi, Y)Z, \xi) \]
Using (2.5) and (2.6), we get
\[ g((\nabla_w R)(\xi, Y)Z, \xi) = 0. \]
So, the equation (3.5) has following form:
\[ (\nabla_w S)(Y, Z) = -\alpha(W)S(Y, Z) - 2n\beta(W)g(Y, Z). \]
Replacing \( Z \) by \( \xi \) in (3.5) and using (2.7) we have
\[ -(\nabla_w S)(Y, \xi) = 2n\alpha(W)\eta(Y) - 2n\beta(W)\eta(Y). \]
Now we have \( (\nabla_w S)(Y, \xi) = \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi) \). Using (2.5) and (2.7) in the above relation, it follows that
\[ (\nabla_w S)(Y, \xi) = -2ng(Y, W) - S(Y, W). \]
In view of (3.6) and (3.7) we obtain
\[ -2ng(Y, W) - S(Y, W) = 2n\eta(Y)(\alpha(W) - \beta(W)). \]
Replacing \( Y \) by \( \xi \) in (3.8) and then using (2.7), we get
\[ \beta(W) = \alpha(W). \]
So using (3.9) in (3.8) we get
\[ S(Y, W) = -2ng(Y, W) \]
This leads to the following results:

**Theorem 3.1.** A generalized \( \phi \)-recurrent Kenmotsu manifold \((M^{2n+1}, g)\) is an Einstein manifold.

**Theorem 3.2.** Let \((M^{2n+1}, g)\) be a generalized \( \phi \)-recurrent Kenmotsu manifold. Then \( \beta = \alpha \).

Now from (3.1) we have
\[ (3.11) \quad (\nabla_w R)(X,Y)Z = \eta((\nabla_w R)(X,Y)Z)\xi - \alpha(W)R(X,Y)Z - \beta(W)[g(Y,Z)X - g(X,Z)Y] \]

Then by the use of second Bianchi identity, (3.11) and (3.9) we have

\[ (3.12) \quad \alpha(W)R(X,Y)Z + \alpha(W)[g(Y,Z)X - g(X,Z)Y] \\
+ \alpha(X)R(Y,W)Z + \alpha(X)[g(W,Z)Y - g(Y,Z)W] \\
+ \alpha(Y)R(W,X)Z + \alpha(Y)[g(X,Z)W - g(W,Z)X] = 0. \]

So by a suitable contraction from (3.12) we get

\[ (3.13) \quad \alpha(W)S(X,U) + 2n\alpha(W)g(X,U) - \alpha(X)S(W,U) - 2n\alpha(X)g(W,U) \\
- \alpha(R(W,X)U) + \alpha(X)g(W,U) - \alpha(W)g(X,U) = 0. \]

Hence by the use of (3.9), (3.10) in (3.13) it can be easily seen that:

\[ (3.14) \quad -\alpha(R(W,X)U) + \alpha(X)g(W,U) - \alpha(W)g(X,U) = 0. \]

Replacing \( X, U \) by \( \xi \) in (3.14), we have

\[ \alpha(W) = \alpha(\xi)\eta(W) \]

or,

\[ (3.15) \quad \alpha(W) = \eta(A)\eta(W). \]

This leads to the following result:

**Theorem 3.2.** In a a generalized \( \phi \)–recurrent Kenmotsu manifold \((M^{2n+1}, g)\), the characteristic vector field \( \xi \) and the vector field \( A \) associated to the 1-form \( \alpha \) are co-directional and the 1-form \( \alpha \) is given by (3.15).

**Definition 3.2.** A Kenmotsu manifold \((M^{2n+1}, g)\) is called generalized concircular \( \phi \)–recurrent if its concircular curvature tensor \( \overline{C} \) (YANO & KON 1984)

\[ (3.17) \quad \overline{C}(X,Y)W = R(X,Y)W - \frac{r}{(2n+1)2n}[g(Y,W)X - g(X,W)Y], \]

satisfies the condition

\[ (3.18) \quad \phi^2(\nabla_w \overline{C})(X,Y)Z = \alpha(W)\overline{C}(X,Y)Z + \beta(W)[g(Y,Z)X - g(X,Z)Y], \]

[5] where \( \alpha \) and \( \beta \) are defined as in (3.2) and \( r \) is the scalar curvature of \( \left(M^n, g\right)\).

Let us consider a generalized concircular \( \phi \)–recurrent Kenmotsu manifold. Then by virtue of (2.2) and (2.16) we have

\[ (3.19) \]
\[-(\nabla_wC)(X,Y)Z + \eta(\nabla_w\bar{C})(X,Y)Z\xi = \alpha(W)\bar{C}(X,Y)Z + \beta(W)\left[g(Y,Z)X - g(X,Z)Y\right],\]

from which it follows that

\[
(3.20) \quad -(\nabla_wC)(X,Y)Z + \eta(\nabla_w\bar{C})(X,Y)Z\eta(U) = \alpha(W)\bar{C}(X,Y)Z + \beta(W)\left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right].
\]

Let \(\{e_i\}, \ i = 1, 2, \ldots, 2n + 1\), be an orthonormal basis of the tangent space at any point of the manifold. Then putting \(Y = Z = e_i\) in (3.20) and taking summation over \(i\), \(1 \leq i \leq 2n + 1\), we get

\[
(3.21) \quad -(\nabla_wS)(X,U) + \frac{W(r)}{(2n+1)2n}2ng(X,U) + (\nabla_w\bar{S})(X,\xi)\eta(U) - \frac{W(r)}{(2n+1)2n}2n\eta(X)\eta(U) = \alpha(W)\left[S(X,U) - \frac{r}{2n+1}g(X,U)\right] + 2n\beta(W)g(X,U).
\]

Replacing \(U\) by \(\xi\) in (3.3) and using (2.1) and (2.7), we have

\[
(3.22) \quad 0 = \alpha(W)\left[2n + \frac{r}{2n+1}\right]\eta(X) - 2n\beta(W)\eta(X).
\]

Putting \(X = \xi\) in (3.22), we obtain

\[
(3.22) \quad \alpha(W)\left[2n + \frac{r}{2n+1}\right] - 2n\beta(W) = 0.
\]

This leads to the following results:

**Theorem 3.3.** Let \((M^{2n+1}, g)\) be a generalized concircular \(\phi\)-recurrent Kenmotsu manifold. Then \(\alpha(W)\left[2n + \frac{r}{2n+1}\right] - 2n\beta(W) = 0\).

**REFERENCES**


