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AN APPROXIMATE SOLUTION FOR DIFFERENT TYPES OF WAVE PROBLEMS

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Abstract: In this article, He's variational iteration method (VIM) is implemented to solve the non-homogeneous dissipative wave, Helmholtz and some nonlinear fifth-order Korteweg-de Vries (FKdV) partial differential equations with specified initial conditions. The initial approximations can be freely chosen with possible unknown constants which can be determined by imposing the boundary or initial conditions after few iterations. Comparison of the results with those obtained by exact solution and Adomian's decomposition method reveals that VIM is very effective, convenient and quite accurate to both linear and nonlinear problems. It is predicted that VIM can be widely applied in engineering.

Key words: Variational iteration method, Helmholtz equation, FKdV equation, nonlinear partial differential equations

DALGA PROBLEMLERİNİN FARKLI TİPLERİ İÇİN BİR YAKLAŞIK ÇÖZÜM

Özet: Bu makalede, He'nin varyasyonel iterasyon yöntemi (VIM), belli başlangıç koşulları ile homojen olmayan dissipative dalga, Helmholtz ve bazı lineer olmayan beşinci mertebeden Korteweg-de Vries (FKdV) kısmi diferansiyel denklemlerini çözmek için uygulanmıştır. Başlangıç yaklaşımları, birkaç iterasyon sonra başlangıç ve sınır koşullarının uygulanmasıyla belirlenebilen mümkün bilinmeyen sabitler ile keyfi olarak seçilebilir. Analitik çözüm ve Adomian'ın ayrıştırma yöntemi ile elde edilen sonuçların karşılaştırılması, VIM'in çok etkili, uygun ve hem lineer hem de lineer olmayan problemler için oldukça hatasız olduğunu ortaya koymaktadır. VIM'in mühendislikte yaygın olarak uygulanabildiği tahmin edilmektedir.

Anahtar kelimeler: Varyasyonel iterasyon yöntemi, Helmholtz denklemi, FKdV denklemi, lineer olmayan kısmi diferansiyel denklemler

1. INTRODUCTION

It is well-known that there are many linear nonlinear partial differential equations in the study of physics, mechanics and biology, etc. The solution of these equations can guide authors to know the described process deeply. But because of the complexity and

limitations of classical mathematical methods, it is sometimes difficult for us to achieve exact solutions for the problems. In the recent decades, there has been great development in the numerical analysis (BURDEN & FAIRES 1993) and exact solutions for partial differential equations.

Many different methods have recently introduced to solve linear and nonlinear problems, such as variational iteration method (VIM) (HE 1998a; 1998b; 1999a; 2006; 2007a; 2007b; 2008, ABDOU & SOLIMAN 2005, MIANSARI et al. 2008), the Adomian's decomposition method (ADM) (WAZWAZ 2002), and homotopy perturbation method (HE 1999b; 2000; 2004; 2005, GANJI & RAJABI 2006). The VIM is strongly and simply capable of solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also is it reduces the size of calculations besides, its interations are direct and straightforward. In this paper, we implement the VIM for finding the exact solutions of some derived linear non-homogeneous dissipative wave, Helmholtz and nonlinear fifth-order Korteweg-de Vries (FKdV) partial differential equations. The homogeneous and inhomogeneous solutions of the equations will be handled more easily and quickly by implementing the VIM rather than the traditional methods for the exact solutions as well as approximate solutions, without suffering from traditional difficulty. Unlike classical techniques, the nonlinear equations are solved easily without transforming the equation by using the VIM. The technique has many advantages over the classical ones. Mainly, it avoids linearization and perturbation in order to find solutions of a given nonlinear equations, On the other hand, the VIM provides explicit and numerical solutions with high accuracy, minimal calculations and strong operability, avoiding physically unrealistic assumptions.

The Helmholtz equation is

$$\nabla^2 u + f(x,y)u = g(x,y) \tag{1}$$

with the boundary and initial conditions of

$$u(0,y) = \psi_1(y), \qquad u_x(0,y) = \psi_2(y), \tag{2}$$

$$u(x,0) = \psi_3(x), \qquad u_y(x,0) = \psi_4(x), \tag{3}$$

where $\psi_1(x), \psi_2(x), \psi_3(x)$ and $\psi_4(x)$ are given functions. The Helmholtz equation appears in such diverse phenomena as elastic waves in solids including vibrating string, bars, membranes, sound or acoustics, electromagnetic waves, and nuclear reactors (DRAZIN & JOHNSON 1989, HE 2004). The FKdV equation (DRAZIN & JOHNSON 1989) is

$$u_t - u_{xxxxx} = F(x, tu, u^2, u_x, u_{xx}, u_{xxx}). \tag{4}$$

This occurs, for example in the theory of magneto- acoustic waves in plasmas (AKYLAS & YANG 1995) and in the theory of shallow water waves with surface tension (HUNTER & SCHEURLE 1988). The FKdV equation has been investigated extensively over the last decade. It has been shown that the traveling- wave solutions of this equation do not vanish at infinity (BEALE 1991, BODY 1991).

2. BASIC IDEA OF VIM

To illustrate basic concepts of the VIM, we consider the following differential equation

$$Lu + Nu = g(x), \tag{5}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{u_n(s) + N\tilde{u}_n(s) - g(s)\}ds \tag{6}$$

where λ is a general Lagrange multiplier , which can be identified optimally via the variational theory. The subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$ (HE 2007b). For linear problems, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified and for nonlinear equation, the Lagrange multiplier is difficult to be identified. To overcome the difficulty, we apply restricted variations to nonlinear terms.

3. APPLICATIONS

In this section, we will apply the VIM to four linear and nonlinear examples.

Example 1. We consider the following linear non-homogeneous dissipative wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial t}u(x,t) - \frac{\partial}{\partial x}u(x,t) + 2(t-x), \quad 0 \le x \le 1, \ t > 0. \tag{7}$$

The initial and boundary conditions posed are

$$u(x,0) = x^2, \ 0 < x < 1, \ u_t(x,0) = 0, \ u(0,t) = t^2, \ t \ge 0. \tag{8}$$

Exact solution of this equation (INC et al. 2004) is

$$u(x,t) = x^2 + t^2. \tag{9}$$

The variational iteration formula is obtained in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(\frac{\partial^2 u_n(x,\tau)}{\partial\tau^2} - \frac{\partial u_n(x,\tau)}{\partial\tau} + \frac{\partial u_n(x,\tau)}{\partial x} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2\tau + 2x\right)d\tau. \tag{10}$$

The Lagrangian multiplier can there be identified as

$$\lambda = e^{(\tau - t)} - 1. \tag{11}$$

As a result, we obtain the following iteration formula

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &+ \int_0^t (e^{(\tau-t)} - 1)\left(\frac{\partial^2 u_n(x,\tau)}{\partial\tau^2} - \frac{\partial u_n(x,\tau)}{\partial\tau} + \frac{\partial u_n(x,\tau)}{\partial x} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} - 2\tau + 2x\right)d\tau. \end{aligned} \tag{12}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition

$$u_0(x,t) = x^2. \tag{13}$$

Using the above variational formula (12), we have

$$u_1(x,t) = u_0(x,t) + \int_0^t (e^{(\tau-t)} - 1)\left(\frac{\partial^2 u_0(x,\tau)}{\partial \tau^2} - \frac{\partial u_0(x,\tau)}{\partial \tau} + \frac{\partial u_0(x,\tau)}{\partial x} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2} - 2\tau + 2xx \right) d\tau. \tag{14}$$

Substituting Eq. (13) in to Eq. (14) and after simplification, we have

$$u_1(x,t) = x^2 - 2e^{(-t)}x + t^2 - 2xt + 2x. \tag{15}$$

In the same way, we obtain $u_2(x,t)$ as follows

$$u_2(x,t) = x^2 - 10e^{-t}x + 3t^2 - 6xt + 10x - 4e^{-t} - 4xte^{-t} - 4t + 4 \tag{16}$$

and so on. In the same way the rest of the components of the iteration formula can be obtained.

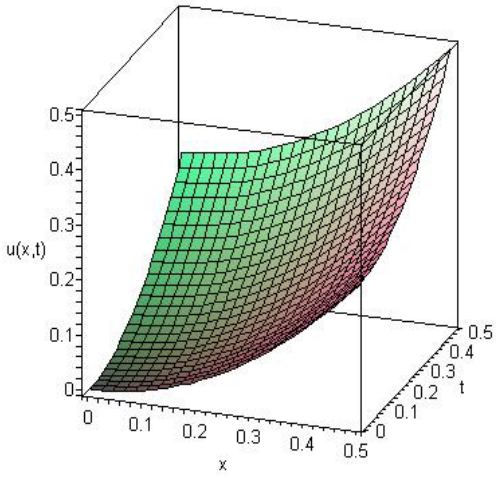

Figure 1. Numerical results obtained by exact solution

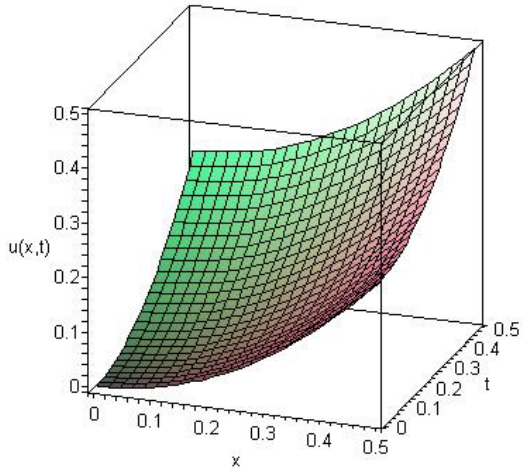

Figure 2. Numerical results obtained by VIM

Example 2. We consider the Helmholtz equation, as follows

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} + 5u(x,y) = 0, \tag{17}$$

with the initial conditions

$$u(0,y) = 0, \quad u_x(0,y) = 3\sinh(2y). \tag{18}$$

Now we being with an arbitrary initial approximation: $u_0(x,y) = A + Bx$, where A and B are constants in x to be determined using the initial conditions (18), thus we have

$$u_0(x,y) = 3x\sinh(2y). \tag{19}$$

According to the VIM, we can construct the correction functional of (17) as follows

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda(\ddot{u}_n(t,y) + u_n''(t,y) + 5u_n(t,y))dt, \tag{20}$$

where ''dot ''denotes differentiation with respect to x and ''prime ''denotes differentiation with respect to y and λ is general Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary condition

$$\begin{aligned} &\lambda''(t) + 5\lambda(t) = 0, \\ &1 - \lambda'(t)\big|_{t=x} = 0, \\ &\lambda(t)\big|_{t=x} = 0. \end{aligned} \tag{21}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda = \frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(t\sqrt{5}) - \sin(x\sqrt{5})\cos(t\sqrt{5})). \tag{22}$$

Substituting Eq. (22) into Eq. (20)

$$\begin{aligned} u_{n+1}(x,y) = u_n(x,y) + \int_0^x \left[\frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(t\sqrt{5}) - \sin(x\sqrt{5})\cos(t\sqrt{5}))\right][\ddot{u}_n(t,y) \\ + u_n''(t,y) - u_n(t,y)]dt. \end{aligned} \tag{23}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition

$$u_0(x,y) = 3x\sinh(2y). \tag{24}$$

Using the above variational formula (23), we have

$$\begin{aligned} u_1(x,y) = u_0(x,y) + \int_0^x \left[\frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(t\sqrt{5}) - \sin(x\sqrt{5})\cos(t\sqrt{5}))\right][\ddot{u}_0(t,y) \\ + u_0''(t,y) - u_0(t,y)]dt. \end{aligned} \tag{25}$$

Substituting Eq. (24) into Eq. (25) and after simplifications, we have

$$\begin{aligned} u_1(x,y) = 3x\sinh(2y) + \frac{54\sqrt{5}}{25}\sin(x\sqrt{5})\sinh(y)\cosh(y) - \frac{54}{5}\cos^2(x\sqrt{5})x\sinh(y)\cosh(y) \\ - \frac{54}{5}\sin^2(x\sqrt{5})x\sinh(y)\cosh(y). \end{aligned} \tag{26}$$

In the same way, other components can be obtained.

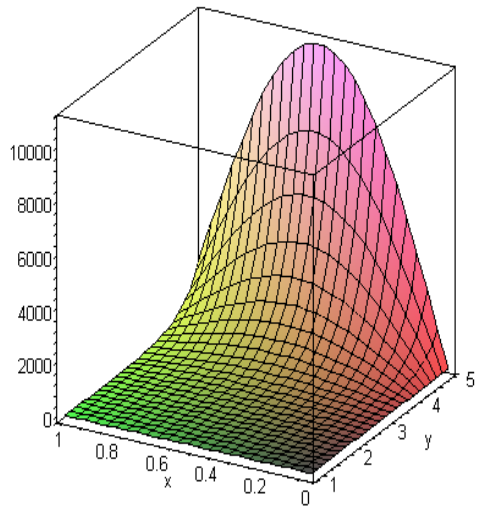

Figure 3. Numerical results obtain by VIM.

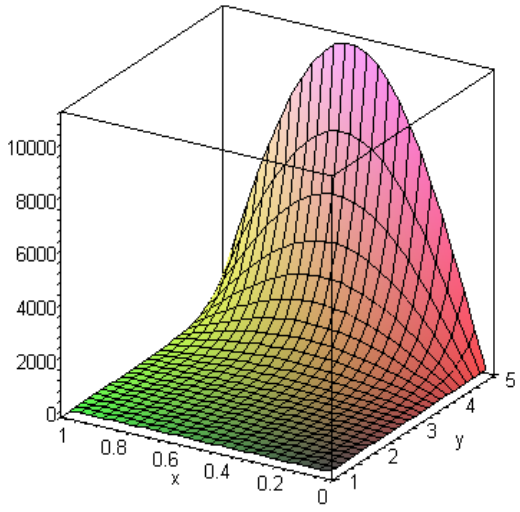

Figure 4. Numerical results obtain by ADM (EL-SAYED & KAYA 2004) and HPM (RAFEI & GANJI 2006).

Example 3. Let's consider a special case of the FKDV equation as follows (DRAZIN & JOHNSON 1989)

$$u_t + u_x + u^2 u_{xx} + u_x u_{xx} - 20u^2 u_{xxx} + u_{xxxxx} = 0, \quad u(x,0) = \frac{1}{x}. \tag{27}$$

According to the VIM, we can construct the correction functional of (27) as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \lambda(\dot{u}_n + u_n' + u_n^2 u_n'' + u_n' u_n'' - 20u_n^2 u_n''' + u_n^{(5)})d\tau, \tag{28}$$

where ''dot ''denotes differentiation with respect to t and ''prime ''denotes differentiation with respect to x and λ is general Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary condition

$$\begin{aligned} &\lambda'(\tau) = 0, \\ &1 + \lambda(\tau)\big|_{\tau=t} = 0. \end{aligned} \tag{29}$$

The Lagrange multiplier can be identified as

$$\lambda = -1. \tag{30}$$

Substituting Eq. (30) into Eq. (28)

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^\tau (\dot{u}_n + u_n' + u_n^2 u_n'' + u_n' u_n'' - 20u_n^2 u_n''' + u_n^{(5)})d\tau. \tag{31}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t) = \frac{1}{x}. \tag{32}$$

Using the above variational formula (31), we have

$$u_1(x,t) = u_0(x,t) - \int_0^\tau (\dot{u}_0 + u_0' + u_0^2 u_0'' + u_0' u_0'' - 20u_0^2 u_0''' + u_n^{(5)})d\tau. \tag{33}$$

Substituting (32) into (33) and after simplifications, we have

$$u_1(x,t) = \frac{1}{x} + \frac{t}{x^2} \tag{34}$$

which is exactly the same as those obtained by Adomian's decomposition method (KAYA 2003) and HPM (RAFEI & GANJI 2006). In the same way, other components can be obtained.

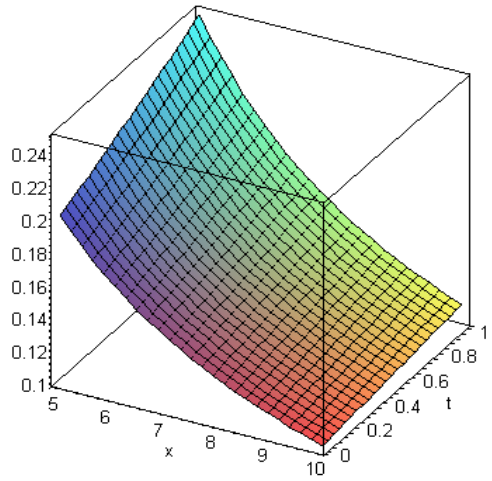

Figure 5. Numerical results obtain by VIM.

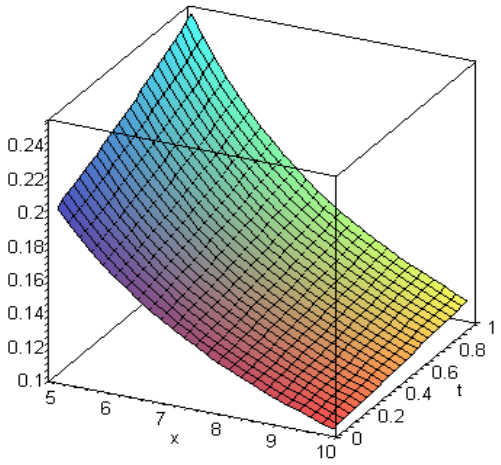

Figure 6. Numerical results obtain by ADM (EL-SAYED & KAYA 2004) and HPM (RAFEI & GANJI 2006).

Example 4. We consider an equation with initial condition is given by

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0, \qquad u(x,0) = e^x. \tag{35}$$

According to the VIM, we can construct the correction functional of (35) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \lambda(\dot{u}_n + u_n u_n' - u_n u_n''' + u_n^{(5)})d\tau, \tag{36}$$

where ''dot ''denotes differentiation with respect to t and ''prime ''denotes differentiation with respect to x and λ is general Lagrange multiplier. Making the above correction functional stationary, we can obtain following stationary condition:

$$\lambda'(\tau) = 0, \qquad 1 + \lambda(\tau)\big|_{\tau=t} = 0. \tag{37}$$

The Lagrange multiplier, therefore, can be identified as

$$\lambda = -1. \tag{38}$$

Substituting Eq. (38) into Eq. (36)

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^\tau (\dot{u}_n + u_n u_n' - u_n u_n''' + u_n^{(5)}) d\tau. \tag{39}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition

$$u_0(x,t) = e^{(x)}. \tag{40}$$

Using the above variational formula (39), we have

$$u_1(x,t) = u_0(x,t) - \int_0^\tau (\dot{u}_0 + u_0 u_0' - u_0 u_0''' + u_0^{(5)})\, d\tau. \tag{41}$$

Substituting Eq. (40) into Eq. (41) and after simplifications, we have

$$u_1(x,t) = e^{(x)} - t\, e^{(x)}, \tag{42}$$

which is exactly the same as obtained by Adomian's decomposition method (KAYA 2003) and HPM (RAFEI & GANJI 2006).

In the same way, other components can be obtained.

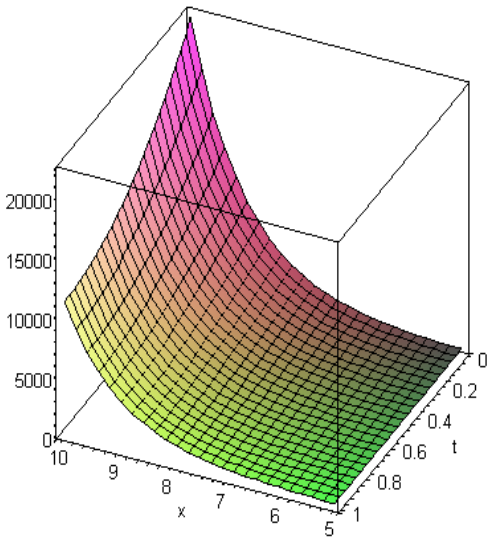

Figure 7. Numerical results obtain by VIM.

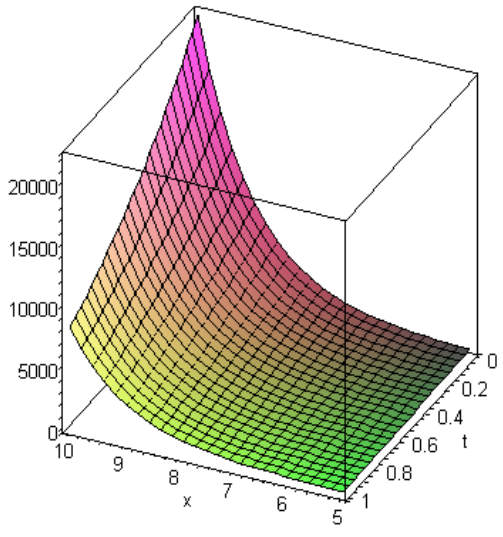

Figure 8. Numerical results obtain by ADM (EL-SAYED & KAYA 2004) and HPM (RAFEI & GANJI 2006).

4. CONCLUSION

In this paper, He's variational iteration method has been successfully applied to finding the exact solutions of the linear dissipative wave, Helmholtz and fifth-order Korteweg-de Vries (FKdV) partial differential equations with specified initial conditions. The obtained solutions are compared with the exact solution, Adomian's decomposition method and Homotopy perturbation Method. All the examples show that the results of the present method are in excellent accordance with those obtained by the other ones. An interesting point about VIM is that with the fewest number of iterations or even in some cases, once, it can converge to correct results. The results show that the VIM is a powerful mathematical tool for solving linear and nonlinear partial differential equations, and therefore can be widely applied in engineering. The variational iteration method can be easily comprehended with only a basic knowledge of Advanced Calculus, even the reader has no knowledge of calculus of variations in pure mathematics.

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