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On Orlicz Difference Sequence Spaces

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Abstract: The main aim of this article is to generalize the famous Orlicz sequence space by using difference operators and a sequence of non-zero scalars and investigate some topological structure relevant to this generalized space.

Key words: Difference sequence space, multiplier sequence space, Orlicz function, *AK*-*BK* space, topological isomorphism and Köthe-Toeplitz dual.

Orlicz Fark Dizi Uzayları Üzerine

Özet: Bu makalenin amacı, sıfırdan farklı skalerlerden oluşan bir diziyi ve fark operatörlerini kullanarak Orlicz dizi uzaylarını genelleştirmek ve bu yeni tanımladığımız uzayın topolojik yapısını incelemektir.

Anahtar kelimeler: Fark dizi uzayı, çok indisli dizi uzayı, Orlicz fonksiyonu, *AK*-*BK* uzayı, toplojik izomorfizm, Köthe-Toeplitz duali.

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1. Introduction

Throughout this paper $w, \ell_\infty, \ell_1,$ c and $c_\circ$ denote the spaces of *all*, *bounded*, *absolutely summable*, *convergent* and *null* sequences $x=(x_k)$ with complex terms respectively. The notion of difference sequence space was introduced by Kizmaz [1], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, where

$$Z(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in Z\},$$

where $\Delta x=(\Delta x_k)=(x_k-x_{k+1})$ and $\Delta^0 x_k=x_k$ for all *k*, for *Z*= ℓ_∞, *c* and c_0.

An Orlicz function $M:[0,\infty)\to[0,\infty)$ is a function, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$, as $x\to\infty$.

An Orlicz function *M* can always be represented in the following integral form:

$$M(x)=\int_0^x p(t)\,dt,$$

where *p*, known as kernel of *M*, is right differentiable for $t\geq 0$, $p(0)=0$, $p(t)>0$ for $t>0$, p is non-decreasing, and $p(t)\to\infty$ as $t\to\infty$.

Consider the kernel $p(t)$ associated with the Orlicz function $M(t)$, and let

$$q(s) = \sup\{t: p(t) \le s\}$$

Then q possesses the same properties as the function p. Suppose now

$$\Phi(x) = \int_0^x q(s)\,ds$$

Then Φ is an Orlicz function. The functions M and Φ are called mutually complementary Orlicz functions.

Now we state the following well known results which can be found in [2].
Let M and F are mutually complementary Orlicz functions. Then we have (Young's inequality)

$$(i)\ \text{For } x, y \ge 0,\ xy \le M(x) + \Phi(y) \tag{1}$$

We also have

$$(ii)\ \text{For } x \ge 0,\ xp(x) = M(x) + \Phi(p(x)) \tag{2}$$

$$(iii)\ M(\lambda x) < \lambda M(x) \tag{3}$$

for all $x \ge 0$ and λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for small x or at 0 if for each $k>0$ there exist $R_k>0$ and $x_k>0$ such that

$$M(kx) \le R_k M(x)$$

for all $x \in (0, x_k]$.

Moreover an Orlicz function M is said to satisfy the Δ_2-condition if and only if

$$\limsup_{x\to 0} \frac{M(2x)}{M(x)} < \infty .$$

Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and x_0 such that

$$M_1(\alpha x) \le M_2(x) \le M_1(\beta x) \tag{4}$$

for all x with $0 \le x \le x_0$.

Lindenstrauss and Tzafriri [3] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}.$$

For more details about Orlicz functions and sequence spaces associated with Orlicz functions one may refer to [2-5].

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for a sequence space E, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence Λ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [6] defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus it also covers a larger class of sequences for study. In the present article we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

The notion of duals of sequence spaces was introduced by Köthe and Toeplitz [7]. Later on it was studied by Kizmaz [1], Kamthan [8] and many others.

Let E and F be two sequence spaces. Then the F dual of E is defined as

$$E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all} (y_k) \in E\}.$$

For $F = \ell_1$, the dual is termed as Köthe-Toeplitz or α-dual of E and denoted by E^α. More precisely, we have the following definition of Köthe Toeplitz dual of E:

$$E^\alpha = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty, \text{ for all } x \in E \right\}.$$

It is known that if $X \subset Y$, then $Y^\alpha \subset X^\alpha$. If $E^{FF} = E$, where $E^{FF} = (E^F)^F$, then E is said to be F-reflexive or F-perfect. In particular, if $E^{\alpha\alpha} = E$, then E is also said to be a Köthe space.

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then we define the following spaces.

Definition 1.1. Let M be any Orlicz function. Then we define

$$\tilde{\ell}_M(\Delta, \Lambda) = \left\{ x \in w : \delta_\Delta^\Lambda(M, x) = \sum_{k=1}^{\infty} M\left(\left|\Delta \lambda_k x_k\right|\right) < \infty \right\},$$

where $\Delta \lambda_k x_k = \lambda_k x_k - \lambda_{k+1} x_{k+1}$ for all $k \geq 1$.

We can write $\tilde{\ell}_M(\Delta^0, \Lambda) = \tilde{\ell}_M(\Lambda)$ and if $\lambda_k = 1$ for all $k \geq 1$, then we write

$$\tilde{\ell}_M(\Delta^0, \Lambda) = \tilde{\ell}_M.$$

Similarly we can define $\tilde{\ell}_M(\nabla, \Lambda)$, where $\nabla \lambda_k x_k = \lambda_k x_k - \lambda_{k-1} x_{k-1}$ for all $k \geq 1$.

Definition 1.2. Let M and Φ be mutually complementary functions. Then we define

$$\ell_M(\Delta, \Lambda) = \left\{ x \in w : \sum_{k=1}^{\infty} (\Delta \lambda_k x_k) y_k \text{ converges for all } y \in \tilde{\ell}_\Phi \right\}.$$

We call this sequence space as Orlicz difference sequence space associated with the multiplier sequence $\Lambda = (\lambda_k)$.

We can write $\ell_M(\Delta^0, \Lambda) = \ell_M(\Lambda)$ and if $\lambda_k = 1$ for all $k \geq 1$, then we write

$$\ell_M\left(\Delta^0,\Lambda\right)=\ell_M .$$

Similarly we can define $\ell_M\left(\nabla,\Lambda\right)$ where $\nabla\lambda_k x_k=\lambda_k x_k-\lambda_{k-1}x_{k-1}$ for all $k\geq 1$.

One can easily observe in the special case $M(x) = x^p$ with $0<p<\infty$ and $\Lambda=\left(\lambda_k\right)=\left(1,1,1,...\right)=e$, the sequence space $\ell_M\left(\nabla,\Lambda\right)$ is reduced in the case $1\leq p<\infty$ to the Banach space bv_p introduced by Başar and Altay [9] and is reduced in the case $0<p<1$ to the p-normed complete space bv_p introduced by Altay and Başar [10], where bv_p denotes the space of all sequences $x = (x_k)$ such that

$$\nabla x = \left(x_k-x_{k-1}\right)\in\ell_p .$$

2. Main Results

In this section we investigate the main results of this article.

Proposition 2.1. For any Orlicz function M,

$$(i)\ \tilde{\ell}_M\left(\Delta,\Lambda\right)\subset\ell_M\left(\Delta,\Lambda\right),$$

$$(ii)\ \tilde{\ell}_M\left(\nabla,\Lambda\right)\subset\ell_M\left(\nabla,\Lambda\right).$$

Proof. (i) Let $x\in\tilde{\ell}_M\left(\Delta,\Lambda\right)$. Then $\sum_{k=1}^{\infty}M\left(\left|\Delta\lambda_k x_k\right|\right)<\infty$. Now using (1), we have

$$\left|\sum_{k=1}^{\infty}(\Delta\lambda_k x_k)y_k\right|\leq\sum_{k=1}^{\infty}\left|(\Delta\lambda_k x_k)y_k\right|\leq\sum_{k=1}^{\infty}M\left(\left|\Delta\lambda_k x_k\right|\right)+\sum_{k=1}^{\infty}\Phi\left(\left|y_k\right|\right)<\infty,$$

for every $y=(y_k)$ with $y\in\tilde{\ell}_\Phi$. Thus $x\in\ell_M\left(\Delta,\Lambda\right)$.

(ii) Since the proof is similar to the proof of part (i), we omit it.

Proposition 2.2. (i) For each $x\in\ell_M\left(\Delta,\Lambda\right)$, $\sup\left\{\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i\right|:\delta\left(\Phi,y\right)\leq 1\right\}<\infty$,

(ii) For each $x\in\ell_M\left(\nabla,\Lambda\right)$, $\sup\left\{\left|\sum_{i=1}^{\infty}(\nabla\lambda_i x_i)y_i\right|:\delta\left(\Phi,y\right)\leq 1\right\}<\infty$.

Proof. (i) Suppose that the result is not true. Then for each $n\geq 1$, there exists y^n with $\delta\left(\Phi,y^n\right)\leq 1$ such that

$$\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i^n\right|>2^n.$$

Without loss of generality we may assume that $(\Delta\lambda_i x_i)$, $y^n\geq 0$. Now, we can define a sequence $z=\{z_i\}$ by

$$z_i=\sum_{n=1}^{\infty}\frac{1}{2^n}y_i^n .$$

By the convexity of Φ,

$$\Phi\left(\sum_{n=1}^{l}\frac{1}{2^n}y_i^n\right)\le\frac{1}{2}\left[\Phi(y_i^1)+\Phi(\frac{y_i^2}{2}+...+\frac{y_i^l}{2^{l-1}})\right]\le...\le\sum_{n=1}^{l}\frac{1}{2^n}\Phi(y_i^n)$$

and hence, using the continuity of Φ, we have

$$\delta(\Phi,z)=\sum_{i=1}^{\infty}\Phi(z_i)\le\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2^n}\Phi(y_i^n)\le\sum_{n=1}^{\infty}\frac{1}{2^n}=1.$$

But for every $l\ge1$,

$$\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)z_i\ge\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)\sum_{n=1}^{l}\frac{1}{2^n}y_i^n=\sum_{n=1}^{l}\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)\frac{y_i^n}{2^n}\ge l.$$

Hence $\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)z_i$ diverges and this implies that $x\notin \ell_M(\Delta,\Lambda)$. This contradiction leads us to the required result.

(*ii*) Proof is similar to that of part (*i*).

The preceding result encourage us to introduce the following norms $\|.\|_M^{\Delta}$ and $\|.\|_M^{\nabla}$ on $\ell_M(\Delta,\Lambda)$ and $\ell_M(\nabla,\Lambda)$, respectively.

Proposition 2.3.

(*i*) $\ell_M(\Delta,\Lambda)$ is a normed linear space under the norm $\|.\|_M^{\Delta}$ defined by

$$\|x\|_M^{\Delta}=|\lambda_1 x_1|+\sup\left\{\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i\right|:\delta(\Phi,y)\le1\right\} \tag{5}$$

(*ii*) $\ell_M(\nabla,\Lambda)$ is a normed linear space under the norm $\|.\|_M^{\nabla}$ defined by

$$\|x\|_M^{\nabla}=\sup\left\{\left|\sum_{i=1}^{\infty}(\nabla\lambda_i x_i)y_i\right|:\delta(\Phi,y)\le1\right\}. \tag{6}$$

Proof. (*i*) It is easy to verify that $\ell_M(\Delta,\Lambda)$ is a linear space. Now we show that $\|.\|_M^{\Delta}$ is a norm on $\ell_M(\Delta,\Lambda)$.

If $x=\theta$, then obviously $\|x\|_M^{\Delta}=0$. Conversely assume $\|x\|_M^{\Delta}=0$. Then using the definition of norm, we have

$$|\lambda_1 x_1|+\sup\left\{\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i\right|:\delta(\Phi,y)\le1\right\}=0.$$

This implies

$$|\lambda_1 x_1|=0 \tag{7}$$

and

$$\sup\left\{\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i\right|:\delta(\Phi,y)\le1\right\}=0.$$

This implies that $\left|\sum_{i=1}^{\infty}(\Delta\lambda_i x_i)y_i\right| = 0$ for all y such that $\delta(\Phi, y) \le 1$.

Now considering $y = \{e_i\}$ if $\Phi(1) \le 1$ otherwise considering $y = \{e_i/_{\Phi(1)}\}$ so that

$$\Delta\lambda_i x_i = 0 \text{ for all } i \ge 1. \tag{8}$$

Combining (7) and (8), we have $x_i = 0$ for all $i \ge 1$, since (λ_k) is a sequence of non-zero scalars and thus $x = \theta$.

It is easy to show

$$\|\alpha x\|_M^\Delta = |\alpha|\ \|x\|_M^\Delta \text{ and } \|x+y\|_M^\Delta \le \|x\|_M^\Delta + \|x\|_M^\Delta .$$

(*ii*) Let $x = \theta$, then obviously $\|x\|_M^\nabla = 0$. Conversely assume $\|x\|_M^\nabla = 0$. Then using the definition of norm, we have

$$\sup\left\{\left|\sum_{i=1}^{\infty}(\nabla\lambda_i x_i)y_i\right| : \delta(\Phi, y) \le 1\right\} = 0.$$

This implies $\left|\sum_{i=1}^{\infty}(\nabla\lambda_i x_i)y_i\right| = 0$ for all y such that $\delta(\Phi, y) \le 1$.

Now considering $y = \{e_i\}$ if $\Phi(1) \le 1$ otherwise considering $y = \{e_i/_{\Phi(1)}\}$ so that

$$\nabla\lambda_i x_i = 0 \text{ for all } i \ge 1.$$

Taking $i=1$, we have

$$\nabla\lambda_1 x_1 = \lambda_1 x_1 - \lambda_0 x_0 = 0.$$

This implies $\lambda_1 x_1 = 0$, by taking $x_0 = 0$. Proceeding in this way we have $\lambda_i x_i = 0$ for all $i \ge 1$ and so $x_i = 0$ for all $i \ge 1$, since (λ_k) is a sequence of non-zero scalars. Thus $x = \theta$.
It is easy to show

$$\|\alpha x\|_M^\nabla = |\alpha|\ \|x\|_M^\nabla \text{ and } \|x+y\|_M^\nabla \le \|x\|_M^\nabla + \|x\|_M^\nabla .$$

This completes the proof.

Remark. $\sum_{k=1}^{\infty}(\Delta\lambda_k x_k)y_k < \infty$ for all $y \in \tilde{\ell}_\Phi$ if and only if $\sum_{k=1}^{\infty}(\nabla\lambda_k x_k)y_k < \infty$ for all $y \in \tilde{\ell}_\Phi$.

Also it is obvious that the norms $\|.\|_M^\Delta$ and $\|.\|_M^\nabla$ are equivalent.

Proposition 2.4. (*i*) $\ell_M(\Delta, \Lambda)$ is a Banach space under the norm $\|.\|_M^\Delta$,

(*ii*) $\ell_M(\nabla, \Lambda)$ is a Banach space under the norm $\|.\|_M^\nabla$.

Proof. We shall give proof of part (*i*). Proof of part (*ii*) is easy than part (*i*).

Let (x^i) be any Cauchy sequence in $\ell_M(\Delta, \Lambda)$. Then for any $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left\| x^i - x^j \right\|_M^{\Delta} < \varepsilon ,$$

for all $i, j \geq n_0$. Using the definition of norm, we get

$$\left|\lambda_1(x_1^i - x_1^j)\right| + \sup\left\{\left|\sum_{k=1}^{\infty}(\Delta\lambda_k(x_k^i - x_k^j))y_k\right| : \delta(\Phi, y) \leq 1\right\} < \varepsilon,$$

for all $i, j \geq n_0$. This implies that $\left|\lambda_1(x_1^i - x_1^j)\right| < \varepsilon$, for all $i, j \geq n_0$. Thus $\left(\lambda_1 x_1^i\right)$ is a Cauchy sequence in C and hence it is a convergent sequence in C.

Let

$$\lim_{i\to\infty} \lambda_1 x_1^i = z_1. \tag{9}$$

Again we have

$$\sup\left\{\left|\sum_{k=1}^{\infty}(\Delta\lambda_k(x_k^i - x_k^j))y_k\right| : \delta(\Phi, y) \leq 1\right\} < \varepsilon$$

for all $i, j \geq n_0$ and so

$$\left|\sum_{k=1}^{\infty}(\Delta\lambda_k(x_k^i - x_k^j))y_k\right| < \varepsilon$$

for all y with $\delta(\Phi, y) \leq 1$ and $i, j \geq n_0$.

Now considering $y = \{e_i\}$ if $\Phi(1) \leq 1$ otherwise considering $y = \{e_i/_{\Phi(1)}\}$ we have $\left(\Delta\lambda_k x_k^i\right)$ is a Cauchy sequence in C for all $k \geq 1$ and hence it is a convergent sequence in C for all $k \geq 1$.

Let

$$\lim_{i\to\infty} \Delta\lambda_k x_k^i = y_k \tag{10}$$

for all $k \geq 1$. Using (9) and (10) we have $\lim_{i\to\infty} \lambda_k x_k^i$ exists for each $k \geq 1$ and so $\lim_{i\to\infty} x_k^i = x_k$, say exists for each $k \geq 1$.

Now

$$\lim_{j\to\infty}\left|\lambda_1(x_1^i - x_1^j)\right| = \left|\lambda_1(x_1^i - x_1)\right| < \varepsilon$$

for all $i \geq n_0$. Also we can have

$$\sup\left\{\left|\sum_{k=1}^{\infty}(\Delta\lambda_k(x_k^i - x_k))y_k\right| : \delta(\Phi, y) \leq 1\right\} < \varepsilon$$

for all $i \geq n_0$ as $j \to \infty$. Thus

$$\left|\lambda_1(x_1^i - x_1)\right| + \sup\left\{\left|\sum_{k=1}^{\infty}(\Delta\lambda_k(x_k^i - x_k))y_k\right| : \delta(\Phi, y) \leq 1\right\} < 2\varepsilon$$

for all $i \geq n_0$ and as $j \to \infty$. It follows that $(x^i - x) \in \ell_M(\Delta, \Lambda)$ and $\ell_M(\Delta, \Lambda)$ is a linear space and hence $x = (x_k) \in \ell_M(\Delta, \Lambda)$.

From above proof we can easily conclude that $\|x^i\|_M^\Delta \to 0$ implies that $x_k^i \to 0$ for each $i \geq 1$. Hence we have the following Proposition.

Proposition 2.5. $\ell_M(\Delta,\Lambda)$ and $\ell_M(\nabla,\Lambda)$ are *BK* spaces under the norms defined by (5) and (6), respectively.

Our next aim is to show that $\ell_M(\Delta,\Lambda)$ and $\ell_M(\nabla,\Lambda)$ can be made *BK* spaces under different but equivalent norms.

Proposition 2.6.

(*i*) $\ell_M(\Delta,\Lambda)$ is a normed linear space under the norm $\|.\|_{(M)}^\Delta$ defined by

$$\|x\|_{(M)}^\Delta = |\lambda_1 x_1| + \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta\lambda_k x_k|}{\rho}\right) \leq 1\right\}, \tag{11}$$

(*ii*) $\ell_M(\nabla,\Lambda)$ is a normed linear space under the norm $\|.\|_{(M)}^\nabla$ defined by

$$\|x\|_{(M)}^\nabla = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\nabla\lambda_k x_k|}{\rho}\right) \leq 1\right\}. \tag{12}$$

Proof. (*i*) Clearly $\|x\|_{(M)}^\Delta = 0$ if $x=\theta$. Next suppose $\|x\|_{(M)}^\Delta = 0$. Then from (11) we have

$$|\lambda_1 x_1| = 0 \text{ and so } \lambda_1 x_1 = 0. \tag{13}$$

Again $\inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta\lambda_k x_k|}{\rho}\right) \leq 1\right\} = 0$. This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon)$ such that

$$\sup_k M\left(\frac{|\Delta\lambda_k x_k|}{\rho_\varepsilon}\right) \leq 1.$$

This implies that $M\left(\frac{|\Delta\lambda_k x_k|}{\rho_\varepsilon}\right) \leq 1$ for all $k \geq 1$. Thus

$$M\left(\frac{|\Delta\lambda_k x_k|}{\varepsilon}\right) \leq M\left(\frac{|\Delta\lambda_k x_k|}{\rho_\varepsilon}\right) \leq 1$$

for all $k \geq 1$.

Suppose $\Delta\lambda_{n_i} x_{n_i} \neq 0$, for some i. Let $\varepsilon \to 0$, then $\frac{|\Delta\lambda_{n_i} x_{n_i}|}{\varepsilon} \to \infty$. It follows that $M\left(\frac{|\Delta\lambda_{n_i} x_{n_i}|}{\varepsilon}\right) \to \infty$ as $\varepsilon \to 0$ for some $n_i \in N$. This is a contradiction. Therefore

$$\Delta\lambda_k x_k = 0 \tag{14}$$

for all $k \geq 1$. Thus, by (13) and (14), it follows that $\lambda_k x_k=0$ for all $k \geq 1$. Hence $x=\theta$, since (λ_k) is a sequence of non-zero scalars.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $\ell_M(\Delta,\Lambda)$. Then there exist ρ_1, $\rho_2>0$ such that

$$\sup_k M\left(\frac{|\Delta\lambda_k x_k|}{\rho_1}\right) \leq 1 \quad \text{and} \quad \sup_k M\left(\frac{|\Delta\lambda_k y_k|}{\rho_2}\right) \leq 1 .$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of M, we have

$$\sup_k M\left(\frac{|\Delta\lambda_k (x_k + y_k)|}{\rho}\right) \leq \frac{\rho_1}{\rho_1+\rho_2}\sup_k M\left(\frac{|\Delta\lambda_k x_k|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2}\sup_k M\left(\frac{|\Delta\lambda_k y_k|}{\rho_2}\right) \leq 1 .$$

Hence we have

$$\|x+y\|_{(M)}^{\Delta} = |\lambda_1(x_1+y_1)| + \inf\left\{\rho>0 : \sup_k M\left(\frac{|\Delta\lambda_k(x_k+y_k)|}{\rho}\right) \leq 1\right\}$$

$$\leq |\lambda_1 x_1| + \inf\left\{\rho_1>0 : \sup_k M\left(\frac{|\Delta\lambda_k x_k|}{\rho_1}\right) \leq 1\right\} + |\lambda_1 y_1|$$

$$+\inf\left\{\rho_2>0 : \sup_k M\left(\frac{|\Delta\lambda_k y_k|}{\rho_2}\right) \leq 1\right\}.$$

This implies $\|x+y\|_{(M)}^{\Delta} \leq \|x\|_{(M)}^{\Delta} + \|x\|_{(M)}^{\Delta}$.

Finally, let ν be any scalar. Then

$$\|\nu x\|_{(M)}^{\Delta} = |\nu\lambda_1 x_1| + \inf\left\{\rho>0 : \sup_k M\left(\frac{|\Delta \nu\lambda_k x_k|}{\rho}\right) \leq 1\right\}$$

$$= |\nu||\lambda_1 x_1| + \inf\left\{r|\nu|>0 : \sup_k M\left(\frac{|\Delta\lambda_k x_k|}{r}\right) \leq 1\right\}$$

$$= |\nu|\, \|x\|_{(M)}^{\Delta}$$

where $r = \dfrac{\rho}{|\nu|}$. This completes the proof.

(*ii*) Proof is easy than part (*i*).

Remark. It is obvious that the norms $\|.\|_{(M)}^{\Delta}$ and $\|.\|_{(M)}^{\nabla}$ are equivalent.

Proposition 2.7. For $x \in \ell_M(\nabla,\Lambda)$, we have

$$\sum_{k=1}^{\infty} M\left(\frac{|\nabla \lambda_k x_k|}{\| x \|_{(M)}^{\Delta^{-1}}}\right) \leq 1 .$$

Proof. Proof is immediate from (12).

Now we show that the norms $\|.\|_{(M)}^{\nabla}$ and $\|.\|_{M}^{\nabla}$ are equivalent. To prove this some other results are required. First we prove those results.

Proposition 2.8. Let $x \in \ell_M(\nabla, \Lambda)$ with $\|x\|_M^{\nabla} \leq 1$. Then $\{p(|\nabla \lambda_n x_n|)\} \in \tilde{\ell}_{\Phi}$ and $\delta(\Phi, \{p(|\nabla \lambda_n x_n|)\}) \leq 1$.

Proof. For any $z \in \tilde{\ell}_{\Phi}$, we may write

$$\left|\sum_{i=1}^{\infty} (\nabla \lambda_i x_i) z_i\right| \leq \begin{cases} \| x \|_M^{\nabla} & \text{if } \delta(\Phi, z) \leq 1 \\ \delta(\Phi, z) \| x \|_M^{\nabla} & \text{if } \delta(\Phi, z) > 1 \end{cases} . \qquad (15)$$

Let now $x \in \ell_M(\nabla, \Lambda)$ with $\|x\|_M^{\nabla} \leq 1$. Also $x^{(n)} = (x_1, \ldots x_n, 0, 0, \ldots..) \in \ell_M(\nabla, \Lambda)$ for $n \geq 1$. We observe that

$$\|x\|_M^{\nabla} \geq \left|\sum_{i=1}^{\infty} (\nabla \lambda_i x_i) y_i^{(n)}\right| = \left|\sum_{i=1}^{\infty} (\nabla \lambda_i x_i^{(n)}) y_i\right|, \qquad n \geq 1$$

for every $y \in \tilde{\ell}_{\Phi}$ with $\delta(\Phi, y) \leq 1$ and thus

$$\|x^{(n)}\|_M^{\nabla} \leq \|x\|_M^{\nabla} \leq 1.$$

Since

$$\sum_{i=1}^{n} \Phi(p(|\nabla \lambda_i x_i|)) = \sum_{i=1}^{\infty} \Phi(p(|\nabla \lambda_i x_i^{(n)}|)).$$

We find that $\{p(|\nabla \lambda_i x_i^{(n)}|)\} \in \tilde{\ell}_{\Phi}$ for each $n \geq 1$. Let $l \geq 1$ be an integer such that

$$\sum_{i=1}^{l} \Phi(p(|\nabla \lambda_i x_i|)) > 1.$$

Then $\sum_{i=1}^{\infty} \Phi(p(|\nabla \lambda_i x_i^{(l)}|)) > 1$. Using (2), we have

$$\begin{aligned} \Phi(p(|\nabla \lambda_i x_i^{(l)}|)) &< M(|\nabla \lambda_i x_i^{(l)}|) + \Phi(p(|\nabla \lambda_i x_i^{(l)}|)) \\ &= |\nabla \lambda_i x_i^l| \, p(|\nabla \lambda_i x_i^l|) \end{aligned}$$

for all $i, l \geq 1$. So by (15), we get

$$\sum_{i=1}^{\infty} \Phi(p(|\nabla \lambda_i x_i^{(l)}|)) < \| x^{(l)} \|_M^{\nabla} \, \delta(\Phi, \{p(|\nabla \lambda_i x_i^l|)\}).$$

This implies that $\| x^{(l)} \|_M^{\nabla} > 1$, a contradiction. This contradiction implies that

$$\sum_{i=1}^{l} \Phi\left(p\left(\left|\nabla\lambda_i x_i\right|\right)\right) \leq 1$$

for all $l \geq 1$. Hence $\left\{p\left(\left|\nabla\lambda_i x_i\right|\right)\right\} \in \tilde{\ell}_\Phi$ and $\delta\left(\Phi, \left\{p\left(\left|\nabla\lambda_i x_i\right|\right)\right\}\right) \leq 1$.

Proposition 2.9. Let $x \in \ell_M(\nabla, \Lambda)$ with $\|x\|_M^{\nabla} \leq 1$. Then $x \in \tilde{\ell}_M(\nabla, \Lambda)$ and $\delta_{\nabla}^{\Lambda}(M, x) \leq \|x\|_M^{\nabla}$.

Proof. Let $y = \left\{p\left(\left|\nabla\lambda_i x_i\right|\right) / \operatorname{sgn}(\nabla\lambda_i x_i)\right\}$. Then from Proposition 2.8, $y \in \tilde{\ell}_\Phi$ and $\delta(\Phi, y) \leq 1$. By (2), we get

$$\begin{aligned} \sum_{i=1}^{\infty} M\left(\left|\nabla\lambda_i x_i\right|\right) &\leq \sum_{i=1}^{\infty} M\left(\left|\nabla\lambda_i x_i\right|\right) + \sum_{i=1}^{\infty} \Phi\left(p\left(\left|\nabla\lambda_i x_i\right|\right)\right) \\ &= \sum_{i=1}^{\infty} \left|\nabla\lambda_i x_i\right| p\left(\left|\nabla\lambda_i x_i\right|\right) \\ &= \left|\sum_{i=1}^{\infty} (\nabla\lambda_i x_i) y_i\right| \leq \|x\|_M^{\nabla}. \end{aligned}$$

This implies that $\delta_{\nabla}^{\Lambda}(M, x) \leq \|x\|_M^{\nabla}$.

Proposition 2.10. For $x \in \ell_M(\nabla, \Lambda)$, we have $\sum_{k=1}^{\infty} M\left(\frac{\left|\nabla\lambda_k x_k\right|}{\| x \|_M^{\nabla}}\right) \leq 1$.

Proof. Proof is immediate from Proposition 2.9.

Theorem 2.11. For $x \in \ell_M(\nabla, \Lambda)$, $\| x \|_{(M)}^{\nabla} \leq \| x \|_M^{\nabla} \leq 2 \| x \|_{(M)}^{\nabla}$.

Proof. We have

$$\|x\|_{(M)}^{\nabla} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{\left|\nabla\lambda_k x_k\right|}{\rho}\right) \leq 1\right\}.$$

Then using Proposition 2.10, we get

$$\| x \|_{(M)}^{\nabla} \leq \| x \|_M^{\nabla}.$$

Let us suppose that $x \in \ell_M(\nabla, \Lambda)$ with $\|x\|_{(M)}^{\nabla} \leq 1$. Then $x \in \tilde{\ell}_M(\nabla, \Lambda)$ and $\delta_{\nabla}^{\Lambda}(M, x) \leq 1$. Indeed,

$$\frac{1}{\| x \|_{(M)}^{\nabla}} \sum_{i=1}^{\infty} M\left(\left|\nabla\lambda_i x_i\right|\right) \leq \sum_{i=1}^{\infty} M\left(\frac{\left|\nabla\lambda_i x_i\right|}{\| x \|_{(M)}^{\nabla}}\right) \leq 1,$$

by Proposition 2.7.

Thus $\frac{x}{\| x \|_{(M)}^{\nabla}} \in \tilde{\ell}_M(\nabla,\Lambda)$ with $\delta\left(M, \frac{x}{\| x \|_{(M)}^{\nabla}}\right) \le 1$. We further observe that for an arbitrary $z \in \tilde{\ell}_M(\nabla,\Lambda)$,

$$\| z \|_M^{\nabla} = \sup\left\{\left|\sum_{i=1}^{\infty}(\nabla\lambda_i z_i) y_i\right| : \delta(\Phi, y) \le 1\right\} \le 1 + \delta_{\nabla}^{\Lambda}(M, z)$$

using (1). Hence taking $z = \frac{x}{\| x \|_{(M)}^{\nabla}}$, we have

$$\left\| \frac{x}{\| x \|_{(M)}^{\nabla}} \right\|_M^{\nabla} \le 1 + \sum_{i=1}^{\infty} M\left(\frac{|x|}{\| x \|_{(M)}^{\nabla}}\right) \le 2$$

by Proposition 2.7. Thus $\| x \|_M^{\nabla} \le 2 \| x \|_{(M)}^{\nabla}$. This completes the proof.

Proposition 2.12. For any Orlicz function *M*, $\ell_M(\nabla,\Lambda) = \ell'_M(\nabla,\Lambda)$, where

$$\ell'_M(\nabla,\Lambda) = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|\nabla\lambda_k x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

Proof. Proof follows from Proposition 2.10.

In view of above Proposition we give the following definition.

Definition 2.13. For any Orlicz function *M*,

$$h_M(\nabla,\Lambda) = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|\nabla\lambda_k x_k|}{\rho}\right) < \infty, \quad \text{for each } \rho > 0 \right\}.$$

Clearly $h_M(\nabla,\Lambda)$ is a subspace of $\ell_M(\nabla,\Lambda)$. Henceforth we shall write $\|.\|$ instead of $\|.\|_{(M)}^{\nabla}$ provided it does not lead to any confusion. The topology of $h_M(\nabla,\Lambda)$ is the one it inherits from $\|.\|$.

Proposition 2.14. Let *M* be an Orlicz function. Then ($h_M(\nabla,\Lambda)$, $\|.\|$) is an *AK-BK* space.

Proof. First we show that $h_M(\nabla,\Lambda)$ is an *AK* space. Let $x \in h_M(\nabla,\Lambda)$. Then for each ε, $0 < \varepsilon < 1$, we can find an n_0 such that

$$\sum_{i \ge n_0} M\left(\frac{|\nabla\lambda_i x_i|}{\varepsilon}\right) \le 1.$$

Hence for $n \ge n_0$,

$$\|x - x^{(n)}\| = \inf\left\{\rho > 0 : \sum_{i \ge n+1} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) \le 1\right\} \le \inf\left\{\rho > 0 : \sum_{i \ge n} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) \le 1\right\} < \varepsilon.$$

Thus we can conclude that $h_M(\nabla,\Lambda)$ is an *AK* space.

Next to show $h_M(\nabla,\Lambda)$ is an *BK* space it is enough to show $h_M(\nabla,\Lambda)$ is a closed subspace of $h_M(\nabla,\Lambda)$. For this let $\{x^n\}$ be a sequence in $h_M(\nabla,\Lambda)$ such that

$$\|x^n - x\| \to 0,$$

where $x \in h_M(\nabla,\Lambda)$. To complete the proof we need to show that $x \in h_M(\nabla,\Lambda)$, i.e.,

$$\sum_{i\geq 1} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) < \infty$$

for every $\rho>0$. To $\rho>0$ there corresponds an l such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then using convexity of M,

$$\sum_{i\geq 1} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) = \sum_{i\geq 1} M\left(\frac{2|\nabla\lambda_i x_i^l| - 2\left(|\nabla\lambda_i x_i^l| - |\nabla\lambda_i x_i|\right)}{2\rho}\right)$$

$$\leq \frac{1}{2}\sum_{i\geq 1} M\left(\frac{2|\nabla\lambda_i x_i^l|}{\rho}\right) + \frac{1}{2}\sum_{i\geq 1} M\left(\frac{2|\nabla\lambda_i (x_i^l - x_i)|}{\rho}\right)$$

$$\leq \frac{1}{2}\sum_{i\geq 1} M\left(\frac{2|\nabla\lambda_i x_i^l|}{\rho}\right) + \frac{1}{2}\sum_{i\geq 1} M\left(\frac{2|\nabla\lambda_i (x_i^l - x_i)|}{\| x^l - x\|}\right) < \infty$$

by proposition 2.7. Thus $x \in h_M(\nabla,\Lambda)$ and consequently $h_M(\nabla,\Lambda)$ is a *BK* space.

Proposition 2.15. Let M be an Orlicz function. If M satisfies the Δ_2-condition at 0, then $\ell_M(\nabla,\Lambda)$ is an *AK* space.

Proof. In fact we shall show that if M satisfies the Δ_2-condition at 0, then $\ell_M(\nabla,\Lambda) = h_M(\nabla,\Lambda)$ and the result follows. Therefore it is enough to show that $\ell_M(\nabla,\Lambda) \subset h_M(\nabla,\Lambda)$. Let $x \in \ell_M(\nabla,\Lambda)$, then $\rho > 0$,

$$\sum_{i\geq 1} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) < \infty .$$

This implies that

$$M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) \to 0 \text{ as } i \to \infty. \tag{16}$$

Choose an arbitrary $l > 0$. If $\rho \leq l$, then $\sum_{i\geq 1} M\left(\frac{|\nabla\lambda_i x_i|}{l}\right) < \infty$. Let now $l < \rho$ and put $k=\frac{\rho}{l}$.

Since M satisfies Δ_2-condition at 0, there exist $R \equiv R_k>0$ and $r \equiv r_k > 0$ with $M(kx) \leq RM(x)$ for all $x \in (0, r]$. By (16) there exists a positive integer n_1 such that

$$M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) < \frac{1}{2} rp\left(\frac{r}{2}\right)$$

for all $i \geq n_1$. We claim that $\frac{|\nabla\lambda_i x_i|}{\rho} \leq r$ for all $i \geq n_1$. Otherwise, we can find $j > n_1$ with $\frac{|\nabla\lambda_j x_j|}{\rho} > r$, and thus

$$M\left(\frac{|\nabla\lambda_j x_j|}{\rho}\right) \geq \int_{r/2}^{\frac{|\nabla\lambda_j x_j|}{\rho}} p(t)\,dt > \frac{1}{2} r p\left(\frac{r}{2}\right)$$

Is a contradiction. Hence our claim is true. Then we can find that

$$\sum_{i\geq n_1} M\left(\frac{|\nabla\lambda_i x_i|}{l}\right) \leq \sum_{i\geq n_1} M\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right),$$

and hence

$$\sum_{i\geq 1} M\left(\frac{|\nabla\lambda_i x_i|}{l}\right) < \infty$$

for every $l > 0$. This completes our proof.

Proposition 2.16. Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent then $\ell_{M_1}(\nabla,\Lambda) = \ell_{M_2}(\nabla,\Lambda)$ and the identity map

$$I: \left(\ell_{M_1}(\nabla,\Lambda), \|.\|_{M_1}^{\nabla}\right) \to \left(\ell_{M_2}(\nabla,\Lambda), \|.\|_{M_2}^{\nabla}\right)$$

is a topological isomorphism.

Proof. Let M_1 and M_2 are equivalent and so satisfy (4). Suppose $x \in \ell_{M_2}(\nabla,\Lambda)$, then

$$\sum_{i=1}^{\infty} M_2\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) < \infty$$

for some $\rho > 0$. Hence for some $l \geq 1$, $\frac{|\nabla\lambda_i x_i|}{l\rho} \leq x_0$ for all $i \geq 1$. Therefore,

$$\sum_{i=1}^{\infty} M_1\left(\frac{\alpha|\nabla\lambda_i x_i|}{l\rho}\right) \leq \sum_{i=1}^{\infty} M_2\left(\frac{|\nabla\lambda_i x_i|}{\rho}\right) < \infty.$$

Thus $\ell_{M_2}(\nabla,\Lambda) \subset \ell_{M_1}(\nabla,\Lambda)$. Similarly $\ell_{M_1}(\nabla,\Lambda) \subset \ell_{M_2}(\nabla,\Lambda)$. Let us abbreviate here $\|.\|_{M_1}^{\nabla}$ and $\|.\|_{M_2}^{\nabla}$ by $\|.\|_1$ and $\|.\|_2$, respectively. For $x \in \ell_{M_2}(\nabla,\Lambda)$,

$$\sum_{i=1}^{\infty} M_2\left(\frac{|\nabla\lambda_i x_i|}{\|x\|_2}\right) \leq 1.$$

One can find $\mu > 1$ with $\left(\frac{x_0}{2}\right)\mu p_2\left(\frac{x_0}{2}\right) \geq 1$, where p_2 is the kernel associated with M_2. Hence

$$M_2\left(\frac{|\nabla\lambda_i x_i|}{\|x\|_2}\right) \leq \left(\frac{x_0}{2}\right)\mu p_2\left(\frac{x_0}{2}\right)$$

for all $i \geq 1$. This implies that $\dfrac{|\nabla\lambda_i x_i|}{\mu\|x\|_2} \leq x_0$ for all $i \geq 1$. Therefore

$$\sum_{i=1}^{\infty} M_1\left(\frac{\alpha|\nabla\lambda_i x_i|}{\mu\|x\|_2}\right) < 1$$

and so $\|x\|_1 \leq \left(\dfrac{\mu}{\alpha}\right)\|x\|_2$. Similarly we can show $\|x\|_2 \leq \beta\gamma\|x\|_1$ by choosing γ with $\gamma\beta > 1$ such that $\gamma\beta\left(\dfrac{x_0}{2}\right)p_1\left(\dfrac{x_0}{2}\right) \geq 1$. Thus $\alpha\mu^{-1}\|x\|_1 \leq \|x\|_2 \leq \beta\gamma\|x\|_1$ which establishes that I is a topological isomorphism.

Proposition 2.17. (*i*) $\ell_M(\Lambda) \subset \ell_M(\nabla,\Lambda)$,

(*ii*) $\ell_M(\Lambda) \subset \ell_M(\Delta,\Lambda)$.

Proof. (*i*) Proof follows from the following inequality:

$$\sum_{i=1}^{\infty} M\left(\frac{|\nabla\lambda_i x_i|}{2\rho}\right) \leq \frac{1}{2}\sum_{i=1}^{\infty} M\left(\frac{|\lambda_i x_i|}{\rho}\right) + \frac{1}{2}\sum_{i=1}^{\infty} M\left(\frac{|\lambda_{i-1} x_{i-1}|}{\rho}\right),$$

(*ii*) Proof is similar to that of part (*i*).

Proposition 2.18. Let M be an Orlicz function and p the corresponding kernel. If $p(x) = 0$ for all x in $[0, x_0]$ where x_0 is some positive number, then $\ell_M(\nabla,\Lambda)$ is topologically isomorphic to $\ell_\infty(\nabla,\Lambda)$ and $h_M(\nabla,\Lambda)$ is topologically isomorphic to $c_0(\nabla,\Lambda)$.

Proof. Let $p(x) = 0$ for all x in $[0, x_0]$. If $y \in \ell_\infty(\nabla,\Lambda)$, then we can find a $\rho > 0$ such that $\dfrac{|\nabla\lambda_i y_i|}{\rho} \leq x_0$ for $i \geq 1$, and so $\sum_{i=1}^{\infty} M\left(\dfrac{|\nabla\lambda_i y_i|}{\rho}\right) < \infty$, giving thus $y \in \ell_M(\nabla,\Lambda)$. On the other hand let $y \in \ell_M(\nabla,\Lambda)$, then $\sum_{i=1}^{\infty} M\left(\dfrac{|\nabla\lambda_i y_i|}{\rho}\right) < \infty$, for some $\rho > 0$ and so $|\nabla\lambda_i y_i| < \infty$ for all $i \geq 1$, giving thus $y \in \ell_\infty(\nabla,\Lambda)$. Hence $y \in \ell_\infty(\nabla,\Lambda)$ if and only if $y \in \ell_M(\nabla,\Lambda)$. We can easily find an x_1 with $M(x_1) \geq 1$. Let $y \in \ell_\infty(\nabla,\Lambda)$ and $\alpha = \|y\|_\infty = \sup_i\left(|\nabla\lambda_i y_i|\right) > 0$. (It is easy to show that $\|y\|_\infty = \sup_i\left(|\nabla\lambda_i y_i|\right)$ is a norm on $\ell_\infty(\nabla,\Lambda)$). For every ε, $0 < \varepsilon < \alpha$, we can determine y_j with $|\nabla\lambda_j y_j| > \alpha - \varepsilon$ and so

$$\sum_{i=1}^{\infty} M\left(\frac{|\nabla\lambda_i y_i| x_1}{\alpha}\right) \geq M\left(\frac{(\alpha-\varepsilon)x_1}{\alpha}\right).$$

Since M is continuous, we find $\sum_{i=1}^{\infty} M\left(\frac{|\nabla \lambda_i y_i| x_1}{\alpha}\right) \geq 1$, and so $\|y\|_\infty \leq x_1 \|y\|$, for otherwise $\sum_{i=1}^{\infty} M\left(\frac{|\nabla \lambda_i y_i|}{\|y\|}\right) > 1$ is a contradiction by Proposition 2.7. Again, $\sum_{i=1}^{\infty} M\left(\frac{|\nabla \lambda_i y_i| x_0}{\alpha}\right) = 0$ and it follows that $\|y\| \leq \frac{1}{x_0}\|y\|_\infty$. Thus the identity map

$$I: \left(\ell_M(\nabla, \Lambda), \|.\|\right) \rightarrow \left(\ell_\infty(\nabla, \Lambda), \|.\|\right)$$

is a topological isomorphism.

For the last part, let $y \in h_M(\nabla, \Lambda)$, then for any $\varepsilon > 0$, $|\nabla \lambda_i y_i| \leq \varepsilon x_1$, for all sufficiently large i, where x_1 is some positive number with $p(x_1) > 0$. Hence $y \in c_0(\nabla, \Lambda)$. Next let $y \in c_0(\nabla, \Lambda)$. Then for any $\rho > 0$, $\frac{|\nabla \lambda_i y_i|}{\rho} < \frac{1}{2} x_0$ for all sufficiently large i. Thus $M\left(\frac{|\nabla \lambda_i y_i|}{\rho}\right) < \infty$ for all $\rho > 0$ and so $y \in h_M(\nabla, \Lambda)$. Hence $h_M(\nabla, \Lambda) = c_0(\nabla, \Lambda)$ and we are done.

Corollary 2.19. Let M be an Orlicz function and p the corresponding kernel. If $p(x) = 0$ for all x in $[0, x_0]$ where x_0 is some positive number, then $\ell_M(\nabla, \Lambda)$ is topologically isomorphic to ℓ_∞ and $h_M(\nabla, \Lambda)$ is topologically isomorphic to c_0.

Proof. Let us define the mapping for $Z = \ell_\infty,\ c_0$

$$T: Z(\nabla, \Lambda) \rightarrow Z$$

by $Tx = (\nabla \lambda_k x_k)$, for every $x \in Z(\nabla, \Lambda)$. Then clearly T is a linear homeomorphism.

Hence the proof follows from Proposition 2.18.

Lemma 2.20. Let M be an Orlicz function. Then $x \in \ell_M(\Delta, \Lambda)$ implies $(k^{-1}\lambda_k x_k) \in \ell_\infty$.

Proof. Let $x \in \ell_M(\Delta, \Lambda)$. Then, one can easily prove that $(\Delta \lambda_k x_k) \in \ell_\infty$ which gives the result $(k^{-1}\lambda_k x_k) \in \ell_\infty$.

Proposition 2.21. Let M be an Orlicz function and p be the corresponding kernel of M. If $p(x) = 0$ for all x in $[0, x_0]$, where x_0 is some positive number, then

(*i*) Köthe-Toeplitz dual of $\ell_M(\Delta, \Lambda)$ is D_1, where

$$D_1 = \left\{(a_k) : \sum_{k=1}^{\infty} k \left|\lambda_k^{-1} a_k\right| < \infty\right\},$$

(ii) Köthe-Toeplitz dual of D_1 is D_2, where

$$D_2=\left\{(b_k): \sup_k k^{-1}\left|\lambda_k b_k\right| < \infty\right\}.$$

Proof. (i) Let $a\in D_1$ and $x\in \ell_M(\Delta,\Lambda)$. Then

$$\sum_{k=1}^{\infty}\left|a_k x_k\right| = \sum_{k=1}^{\infty} k\left|\lambda_k^{-1}a_k\right|k^{-1}\left|\lambda_k x_k\right| \le \sup_k k^{-1}\left|\lambda_k x_k\right|\sum_{k=1}^{\infty} k\left|\lambda_k^{-1}a_k\right| < \infty.$$

Hence $a\in\left[\ell_M(\Delta,\Lambda)\right]^{\alpha}$. Thus, the inclusion $D_1\subset\left[\ell_M(\Delta,\Lambda)\right]^{\alpha}$ holds.

Conversely suppose that $a\in\left[\ell_M(\Delta,\Lambda)\right]^{\alpha}$. Then $\sum_{k=1}^{\infty}\left|a_k x_k\right|<\infty$ for every $x\in \ell_M(\Delta,\Lambda)$. So we can take $x_k=\lambda_k^{-1}k$ for all $k\geq 1$, because then $(x_k)\in \ell_\infty(\Delta,\Lambda)$ and hence $(x_k)\in \ell_M(\Delta,\Lambda)$ as shown in Proposition 2.18.

Now $\sum_{k=1}^{\infty} k\left|\lambda_k^{-1}a_k\right| = \sum_{k=1}^{\infty}\left|a_k x_k\right| < \infty$ and thus $a\in D_1$. Hence, the inclusion $\left[\ell_M(\Delta,\Lambda)\right]^{\alpha}\subset D_1$ holds.

(ii) Proof follows by similar arguments used in the prove of case (i).

Proposition 2.22. Let M be an Orlicz function and p be the corresponding kernel of M. If $p(x) = 0$ for all x in $[0, x_0]$, where x_0 is some positive number, then Köthe-Toeplitz dual of $h_M(\Delta,\Lambda)$ is D_1, where D_1 is defined as in Proposition 2.21.

Proof. Let $a\in D_1$ and $x\in h_M(\Delta,\Lambda)$. Then

$$\sum_{k=1}^{\infty}\left|a_k x_k\right| = \sum_{k=1}^{\infty} k\left|\lambda_k^{-1}a_k\right|k^{-1}\left|\lambda_k x_k\right| \le \sup_k k^{-1}\left|\lambda_k x_k\right|\sum_{k=1}^{\infty} k\left|\lambda_k^{-1}a_k\right| < \infty.$$

Hence $a\in\left[h_M(\Delta,\Lambda)\right]^{\alpha}$, that is the inclusion $D_1\subset\left[h_M(\Delta,\Lambda)\right]^{\alpha}$ holds.

Conversely suppose that $a\in\left[h_M(\Delta,\Lambda)\right]^{\alpha}$ and $a\notin D_1$. Then there exists a strictly increasing sequence (n_i) of positive integers such that $n_1 < n_2 <\ldots$, such that

$$\sum_{k=n_i+1}^{n_{i+1}}\left|\lambda_k\right|^{-1} k\left|a_k\right| > i.$$

Define (x_k) by

$$x_k=\begin{cases}0 & , \quad 1\le k\le n_1\\ k\lambda_k^{-1}\operatorname{sgn} a_k / i & , \quad n_i < k\le n_{i+1}\end{cases}$$

Then $(x_k)\in c_0(\Delta,\Lambda)$ and so by Proposition 2.18, $(x_k)\in h_M(\Delta,\Lambda)$. Then we have

$$\sum_{k=1}^{\infty}|a_k x_k| = \sum_{k=n_1+1}^{n_2}|a_k x_k| + \ldots + \sum_{k=n_i+1}^{n_{i+1}}|a_k x_k| + \ldots$$

$$= \sum_{k=n_1+1}^{n_2} k\left|\lambda_k^{-1}a_k\right| + \ldots + \frac{1}{i}\sum_{k=n_i+1}^{n_{i+1}} k\left|\lambda_k^{-1}a_k\right| + \ldots > 1+1+\ldots = \infty.$$

This contradicts to $a \notin \left[h_M(\Delta,\Lambda)\right]^\alpha$. Hence $a \in D_1$, i.e. the inclusion $\left[h_M(\Delta,\Lambda)\right]^\alpha \subset D_1$ also holds. This completes the proof.

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