On an Extension of the Exponential and Weibull Distributions[*]

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Abstract: In this study, the Poisson distribution is used to add a new parameter to the exponential and Weibull distributions, and some results on survival distributions of the new extended distributions are given.

Key words: Survival function, exponential, Weibull and Poisson distributions.

Üstel ve Weibull Dağılımlarının Bir Genişletmesi Üzerine

Özet: Bu çalışmada Poisson dağılımı, üstel ve Weibull dağılımlarına yeni bir parametre eklenmesi amacıyla kullanılmış ve yeni genişletilen dağılımların sağkalım fonksiyonuna ilişkin bazı sonuçlara yer verilmiştir.

Anahtar kelimeler: Sağkalım fonksiyonu, üstel, Weibull ve Poisson dağılımları.

1. Introduction

In survival analysis, the exponential and Weibull distributions are the most frequently used parametric models. The importance of the exponential distribution emerges from its lack of memory property and having a constant hazard rate function.

The Weibull distribution contains the exponential distribution and constitutes a more general model for the survival analysis since it does not assume a constant hazard rate.

In [1], a new method is given, which uses the geometric distribution for adding a new parameter to the families of exponential and Weibull distributions. A bivariate version is also considered in the same study. In this study, the Poisson distribution with drifted supporting set {1,2,3,...} is used for extending the families of exponential and Weibull distributions.

2. The New Survival Function

Let $\overline{F}$ be a one-parameter survival function and N be a Poisson random variable with parameter λ, having the drifted supporting set {1,2,3,...}, or equivalently let M be a Poisson variable with parameter λ and N = M + 1.

For all $x \in \mathbf{R}$, define a new survival function by

$$\overline{G}(x) = \sum_{n=1}^{\infty} \overline{F}^n(x) P\{N=n\} = \sum_{n=1}^{\infty} \overline{F}^n(x) \frac{e^{-\lambda} \lambda^{n-1}}{(n-1)!} = \overline{F}(x) e^{-\lambda[1-\overline{F}(x)]} ,$$ (1)

or equivalently by

$$\left. \begin{array}{ll} \overline{G}(x) = \overline{F}(x) e^{-\lambda F(x)} & , \qquad x \in R, \ \lambda \geq 0 \\[2mm] G(x) = 1 - [1 - F(x)] e^{-\lambda F(x)}, & \qquad x \in R, \ \lambda \geq 0. \end{array} \right\}$$ (2)

or

For the new survival function, it is true that $\overline{G}(x)$ is decreasing in λ, as λ becomes nearer 0, $\overline{G}(x)$ takes after $\overline{F}(x)$, otherwise we have

$$\overline{G}(x) \leq \overline{F}(x), \ x \in R ,$$

or equivalently $G(x) \geq F(x), \ x \in R$ which means that in the new distribution much more portion of the corresponding random variable is cumulated on the "left side" of the distribution whatever the distribution is. In fact, it is easy to see that the equation (1) is the survival function of a random variable X_i having the shortest lifetime; i.e.,

$$\overline{G}(x) = P\left(\underset{1 \leq i \leq N}{\text{Min}} X_i > x \right).$$ (3)

For this reason, as an example, over the integer values for λ, G may be thought to use as an approximation to the life of the patient who has the shortest life among its "Poisson" sample peers and has been taking some therapy on a given disease. It is also known that the Poisson sampling preserves many basic properties of stochastic processes [2].

In this study, the survival function (1) will be used to obtain the new extended classes of exponential and Weibull distributions.

3. Density and Hazard Rate Functions

Let the distribution function F in (2) have a density function f and denote its hazard rate function by r_F, and similarly of G by r_G. The density function corresponding to G is given by

$$g(x) = \{1 + \lambda[1 - F(x)]\} f(x) e^{-\lambda F(x)}, \ x \in R; \ \lambda \geq 0,$$ (4)

and the hazard rate function is

$$r_G(x) = \left[\lambda + \frac{1}{1 - F(x)} \right] f(x), \ x \in R; \ \lambda \geq 0;$$

or in terms of r_F,

$$r_G(x) = \lambda f(x) + r_F(x), \ x \in R; \ \lambda \geq 0.$$ (5)

Equation (5) states that the shape of the new hazard function n depends on the shapes of old density and old hazard rate function. In addition, we have $r_G(x) \geq r_F(x)$, $x \in R$; i.e., the new family has a failure rate at least as the old family.

4. The New Extended Exponential Family

When $\overline{F}(x) = e^{-\gamma x}$, $\gamma > 0$; $x \geq 0$, (1.1) gives the survival function of a new two-parametric exponential family as

$$\overline{G}(x; \lambda, \gamma) = \exp\{\lambda e^{-\gamma x} - \gamma x - \lambda\}, x \geq 0, \ \lambda \geq 0, \ \gamma > 0. \tag{6}$$

The corresponding density is given by

$$g(x; \lambda, \gamma) = \gamma(1 + \lambda e^{-\gamma x}) \exp\{\lambda e^{-\gamma x} - \gamma x - \lambda\}, \ x \geq 0, \ \lambda \geq 0, \ \gamma > 0,$$

and its hazard rate function and Laplace-Stieltjes transform are

$$r_G(x; \lambda, \gamma) = \gamma + \lambda\gamma e^{-\gamma x}, \ x \geq 0, \ \lambda \geq 0, \ \gamma > 0,$$

$$\gamma_G(s) = E[e^{-sX}] = \frac{1}{\lambda}\left\{ e^{-\lambda}(-\lambda)^{-\frac{s}{\gamma}} \left[\Gamma\left(\tfrac{\gamma+s}{\gamma}, -\lambda\right) - \Gamma\left(2 + \tfrac{s}{\gamma}, -\lambda\right) - \Gamma\left(\tfrac{\gamma+s}{\gamma}\right) + \Gamma\left(2 + \tfrac{s}{\gamma}\right) \right] \right\},$$

respectively, where $\Gamma(.,.)$ is the upper incomplete gamma function. $r_G(x;\lambda,\gamma)$ is decreasing for all x≥0, $r_G(x;0,\gamma) = \gamma$, and is a Gompertz-Makeham hazard function with $\rho_0 = \gamma$, $\rho_1 = \lambda\gamma$ and $\rho_2 = -\gamma$ and is as a compound exponential distribution in [3]. The compound exponential distributions are geometric infinitely divisible and hence infinitely divisible, and satisfy the stability equation of the form

$$X =^d B(X + S)$$

where S is a R^+-valued random variable, B is a mixed Bernoulli variable with mixing variable W taking values in (0,1), all of the variables X, B and S are independent [4] and $=^d$ shows the equality of distributions. As stated in [5], the distribution functions having decreasing hazard rate are new worse than used, we have

$$P(X > u + v \mid X > u) \geq P(X > v).$$

It is also easy to show that

$$P(X > u + v \mid X > u) = \overline{G}(v; \lambda e^{-\gamma u}, \gamma),$$

i.e., in the conditional survival probability λ is replaced by $\lambda e^{-\gamma u}$ which tends to λ as u→0 and tends to 0 as u→∞, which means that as u→∞ the new distribution behaves as its original distribution, i.e. as the exponential distribution.

The new exponential distribution has a mode at 0, and for any fixed value of λ its median decreases as γ increases, and conversely the same property holds for increasing values of λ versus any fixed value γ as seen in Table 1. In fact, the median value is completely indirectly proportional to γ for any fixed value of λ, i.e., the median value

can be predicted from any of its given value versus any fixed λ. As an example, from Table 1, for $\lambda = 1$ fixed, and $\gamma = 10$, the median value is 0,038 while it is $2 \times 0,038 = 0,076$ for $\gamma = 5$.

Table 1. Median values of X for various values of λ and γ.

λ	γ					
	0,2	0,5	1	5	10	20
0,2	3,013	1,205	0,603	0,121	0,060	0,030
0,5	2,486	0,994	0,497	0,099	0,050	0,025
1	1,891	0,756	0,378	0,076	0,038	0,019
5	0,607	0,243	0,121	0,024	0,012	0,006
10	0,324	0,130	0,065	0,013	0,006	0,003
20	0,168	0,067	0,034	0,007	0,003	0,002

The moments of the new extended exponential distribution are given by

$$E[X^r] = r\int_0^\infty \overline{G}(x)x^{r-1}dx = re^{-\lambda}\int_0^\infty x^{r-1}e^{-\gamma x}\sum_{n=0}^\infty \frac{(\lambda e^{-\gamma x})^n}{n!}dx = \frac{\Gamma(r+1)e^{-\lambda}}{\gamma^r}\sum_{n=0}^\infty \frac{\lambda^n}{n!(n+1)^r} \qquad (7)$$

or shortly

$$E[X^r] = \frac{\Gamma(r+1)}{\gamma^r}E[N^{-r}],$$

which is $E[N^{-r}]$ times the rth moment of exponential distribution. From (7) it is easy to show that

$$E[X] = \frac{1-e^{-\lambda}}{\gamma\lambda}, \quad Var(X) \le \frac{1}{3}\frac{\pi^2}{\gamma^2} - \frac{(1-e^{-\lambda})^2}{\gamma^2\lambda^2};$$

since $\lambda^n/n! < e^\lambda$ gives $E[X^r] < (\Gamma(r+1)/\gamma^r)\zeta(r)$, where $\zeta(r)$ is the real Riemann zeta function. The real Riemann zeta function $\zeta(r)$ is a regular function for all values of r except for a simple pole at r = 1 with residue 1. Some additional properties of the real Riemann zeta function $\zeta(r)$ can be found in [6] and [7].

From another point of view, the variance of this distribution can be written as

$$Var(X) = \frac{2}{\gamma^2}E[N^{-2}] - \frac{1}{\gamma^2}\left[E[N^{-1}]\right]^2 \qquad (8)$$

by (7). For fixed values of γ, numerical computations showed that E[X] attains its maximum value at $\lambda=1$ over the integers for λ. It is also true for this distribution that

$$\lim_{\lambda,\gamma\to\infty} Med(X)/E[X] \to 1.$$

Note that $\lim_{x \to 0} r_G(x; \lambda, \gamma) = \gamma(1 + \lambda)$ while $\lim_{x \to \infty} r_G(x; \lambda, \gamma) = \gamma;$ i.e., the new distribution behaves like its successor, exponential distribution.
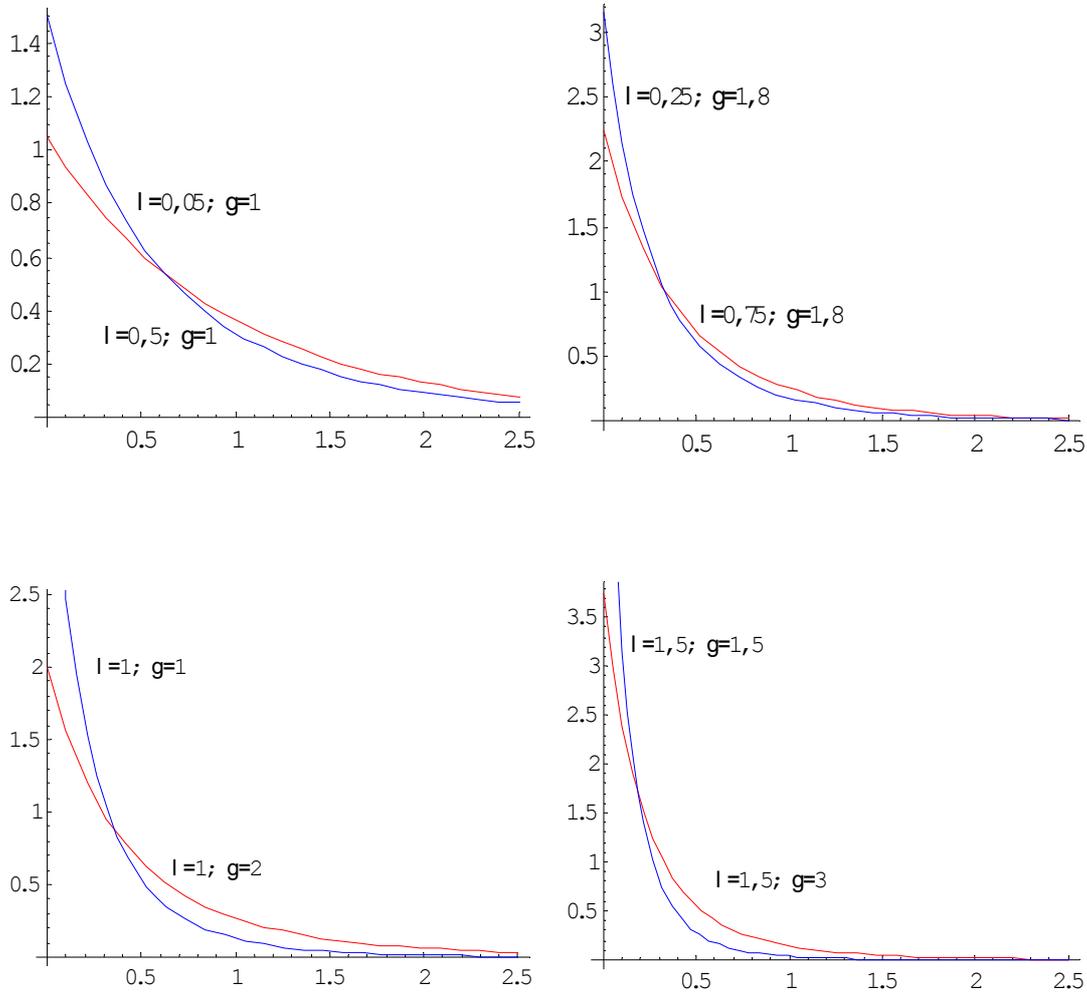
Figure 1. The density functions for the new exponential family.

For some values of λ, γ and β, the density functions are drawn for the new exponential distribution in Figure 1 and their hazard rate functions are drawn in Figure 2. Their graphics show that the shapes of the new densities take after the exponential distribution. As $\lambda \to 0$, it is seen that this similarity increases. The shapes of hazard functions of the new family seem like their density functions.
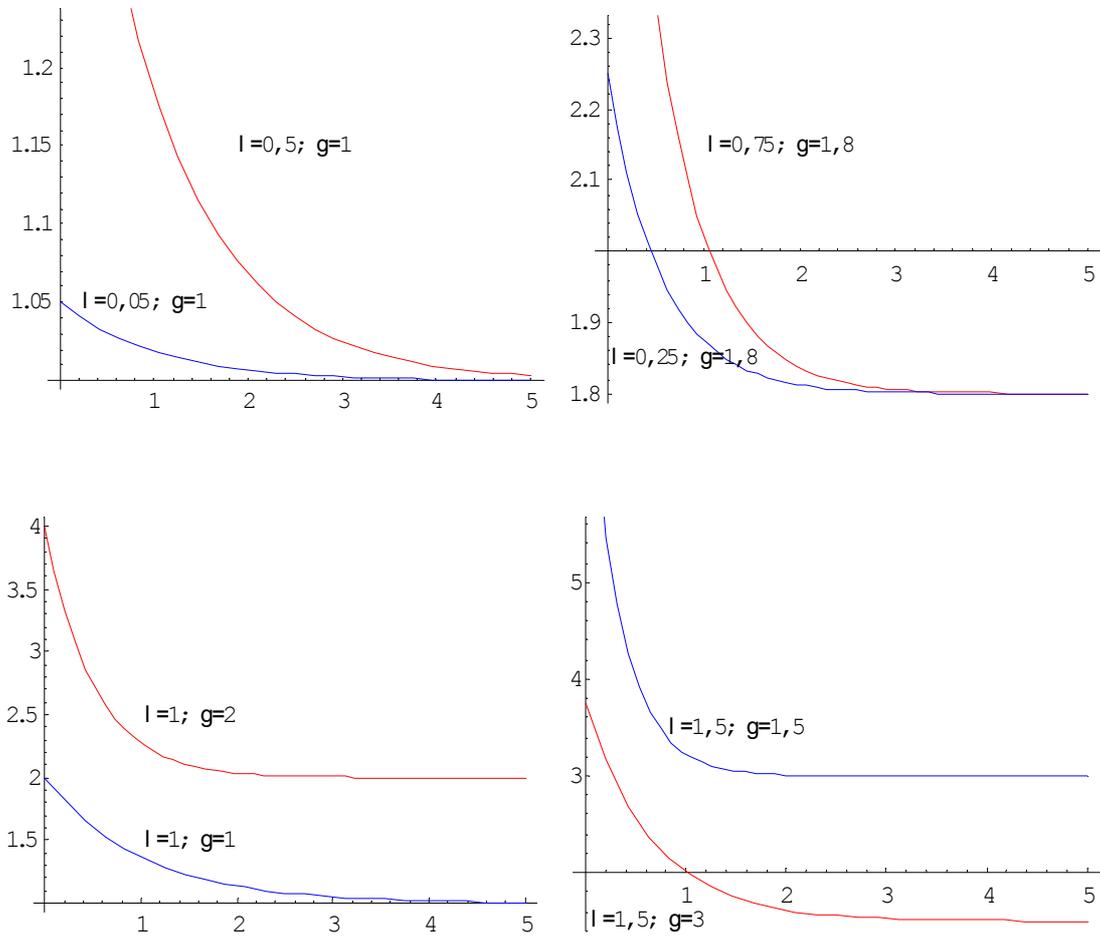
Figure 2. The hazard rate functions for the new exponential distribution.

5. The New Extended Weibull Distribution

If the Weibull survival function

$$\overline{F}(x) = \exp\{-(\gamma x)^{\beta}\}, \ \ x \geq 0, \ \ \gamma > 0, \ \ \beta > 0 \tag{9}$$

is substituted in (1.2) we have the new three-parameter survival function

$$\overline{G}(x; \lambda, \gamma, \beta) = \exp\{\lambda e^{-(\gamma x)^{\beta}} - (\gamma x)^{\beta} - \lambda\}, \ \ x \geq 0, \ \ \gamma, \beta > 0, \ \ \lambda \geq 0. \tag{10}$$

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If X has an exponential distribution with parameter 1, then $X^{1/\beta}/\gamma$ has the survival function (9), i.e., $X^{1/\beta}/\gamma$ has the Weibull distribution with parameter γ and β, so (10) may also be obtained from (6) using the transformation $X^{1/\beta}/\gamma$.

The new extended Weibull distribution has the density given by

$$g(x;\lambda,\gamma,\beta) = (1 + \lambda \exp\{-(\gamma x)^{\beta}\})\beta\gamma(\gamma x)^{\beta-1} \exp\{\lambda \exp\{-(\gamma x)^{\beta}\} - (\gamma x)^{\beta} - \lambda\}$$

for $x \geq 0, \gamma, \beta > 0, \lambda \geq 0,$ and the hazard rate function is

$$r_G(x;\lambda,\gamma,\beta) = (1 + \lambda \exp\{-(\gamma x)^{\beta}\})\beta\gamma(\gamma x)^{\beta-1} ; \quad x \geq 0, \quad \gamma, \beta > 0, \quad \lambda \geq 0.$$

Notice that $\lim_{x \to \infty} r_G(x;\lambda,\gamma,\beta) = 0$ for $\beta<1$, and $\lim_{x \to \infty} r_G(x;\lambda,\gamma,\beta) = \infty$ for $\beta>1$. The new Weibull distribution has median values increasing as β increases, decreasing as λ and γ increases. For fixed $\lambda = \gamma = 1$, the median value tends to 1 as $\beta \to \infty$. Similarly, for fixed $\beta=1, \lambda \to \infty$ (fixed γ) or $\gamma \to \infty$ (fixed λ) the median value tends to 0. The moments of the new Weibull distribution are given by

$$E[X^r] = r\int_0^{\infty}\overline{G}(x)x^{r-1}dx = \frac{re^{-\lambda}}{\beta\gamma^r}\Gamma\left(\frac{r}{\beta}\right)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!(n+1)^{r/\beta}} = \frac{\Gamma\left(\frac{r}{\beta}+1\right)}{\gamma^r}E[N^{-\frac{r}{\beta}}], \quad (11)$$

and these moments cannot be put in a simpler form. These moments have also a similar form given in (7). As an example, for the same parametric values in Table 2, the expected values of X are given in Table 3. Similar to (7) we have an upper bound for the moments related to the real Riemann zeta function as follows

$$E[X^r] < \left[\Gamma\left(\tfrac{r}{\beta}+1\right)/\gamma^r\right]\zeta\left(\tfrac{r}{\beta}\right).$$

As it is expected, the expected values of X show the same behavior as its median values in respect of the proportionality to γ for fixed values of λ and β. The expected and median values show the same behavior as the expected and median values of the new exponential distribution.

Table 2. Median values of X for various values of λ, γ and β.

$\lambda \to$	0,05			0,2			0,5			1		
β $\downarrow$	γ											
	0,5	1	1,5	0,5	1	1,5	0,5	1	1,5	0,5	1	1,5
0,5	0,894	0,447	0,298	0,726	0,363	0,242	0,496	0,248	0,165	0,286	0,143	0,095
1	1,338	0,669	0,446	1,205	0,603	0,402	0,994	0,497	0,331	0,756	0,378	0,252
1,5	1,529	0,765	0,510	1,427	0,713	0,476	1,255	0,628	0,418	1,046	0,523	0,349
2	1,636	0,818	0,545	1,553	0,776	0,516	1,410	0,705	0,470	1,230	0,615	0,410
5	1,845	0,923	0,615	1,807	0,904	0,602	1,739	0,870	0,579	1,647	0,823	0,549

Table 3. Expected values of X for various values of λ, γ and β .

λ→	0,05			0,2			0,5			1		
β	γ											
↓	0,5	1	1,5	0,5	1	1,5	0,5	1	1,5	0,5	1	1,5
0,5	3,853	1,927	1,284	3,446	1,723	1,149	2,767	1,383	0,922	1,939	0,970	0,646
1	1,951	0,975	0,650	1,813	0,906	0,604	1,574	0,787	0,525	1,264	0,632	0,421
1,5	1,773	0,886	0,591	1,679	0,840	0,560	1,516	0,758	0,505	1,298	0,649	0,433
2	2,640	1,320	0,880	2,530	1,265	0,843	2,336	1,168	0,779	2,071	1,036	0,690
5	1,825	0,912	0,608	1,791	0,895	0,597	1,730	0,865	0,577	1,645	0,823	0,548

When the graphics of the densities and hazard rate functions of the new Weibull family are drawn, it is seen that the shapes of the density functions resemble the corresponding hazard rate functions. When the β value exceeds 1, the density and hazard rate functions leave the simple exponentialwise form.
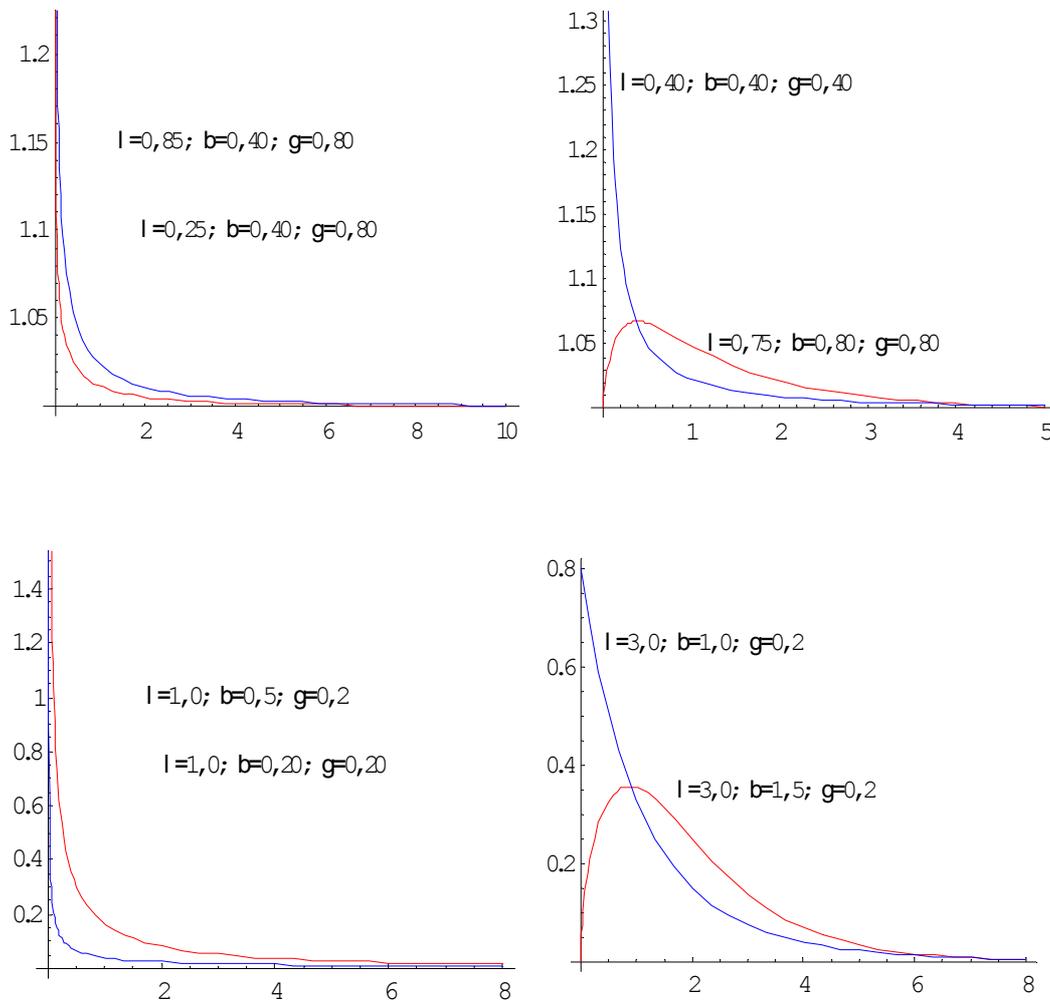
Figure 3. The density functions for the new Weibull distribution.
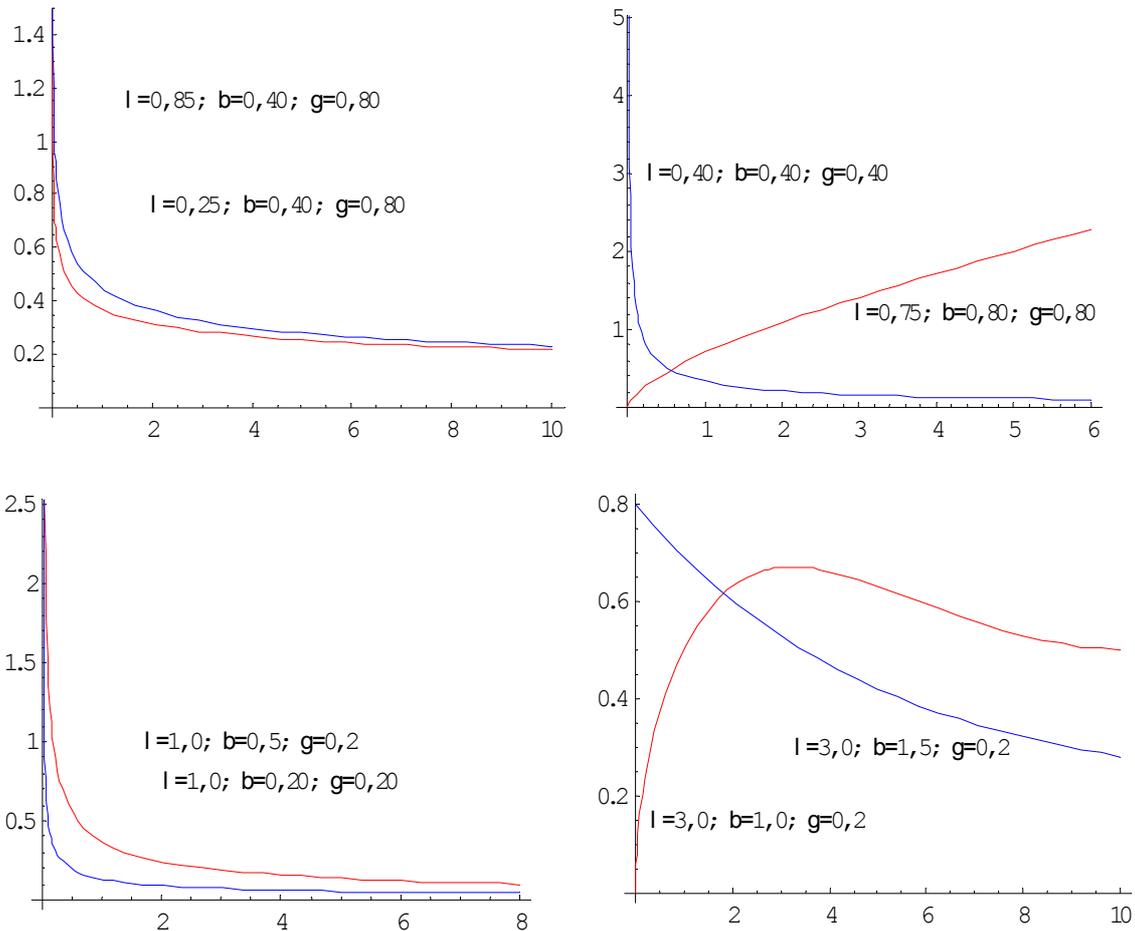
Figure 4. The hazard rate functions for the new Weibull distribution.

6. Conclusion

The new exponential and Weibull distributions given in this study are extended versions of these distributions, and have some nice properties. These distributions can be seen as the distribution of first order statistic of a random sample which is drawn from the same exponential or Weibull distribution, having a random sample size, and consequently can be used as an approximation to the lifetimes of patients who are subject to the same disease.

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