Uluslararası Mühendislik Araştırma ve Geliştirme Dergisi
International Journal of Engineering Research and Development

UMAGD, (2021) 13(1), 127-139.

10.29137/umagd.756455

Cilt/Volume:13 Sayı/Issue:1 Ocak/January 2021

Araştırma Makalesi / Research Article

The New Screw Interpolations and Their Geometric Properties in the Dual Spherical Mechanisms

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***Başvuru/Received**: 22/06/2020* ***Kabul / Accepted**: 09/12/2020* ***Çevrimiçi Basım / Published Online:** 18/01/2021*

***Son Versiyon/Final Version**: 18/01/2021*

Abstract
In this paper, some new Sclerp interpolation motions are defined in the dual spherical mechanisms by developing the Sclerp interpolation given by Ravani (1994). These new methods are the sequential Sclerp interpolation, fast dual spherical interpolation, and the fast screw linear interpolation and they will lead to some advantages to Sclerp interpolation. Also, the lengths between the joints in the spherical dual mechanisms are designed by Sclerp interpolation and the Blaschke frame of this dual spherical interpolation curve is analyzed. At the end of the study, a numeric example is given.

Key Words
"Dual Quaternion, Sclerp Interpolation, Fast Sclerp, Mechanism, Dual Space, Screw motion"

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1. Introduction

An Irish mathematician Sir William Rowan Hamilton (1843) defines the mathematical statement of the *quaternion* which is an extension of the complex numbers as $i^2 = j^2 = k^2 = ijk = -1$ by the square structure comprising of one real and three imaginary terms $s + ix + jy + kz$ (Hamilton (1848), Hacısalihoğlu(1983), Vince (2011)). Because it can easily perform the rotational motion of a point in the space, the quaternions which have a non-commutative real algebra structure, have quite a wide range of application areas. Particularly, in such areas involving geometric motions as mechanics, kinematics, robotics, and physics, the quaternions have a very advantageous usage in terms of representation and process. Sheomake (1985) is the first person forming the SLERP interpolation between two quaternions. While Jafari et al. (2004) gave the linear and geometric SLERP interpolation definitions, Kremer (2008) gave the normalized linear interpolation definition. Hast et al. (2003) defined the fast incremental SLERP by using the orthogonalization step of Gram-Schmidt (Hast et al. (2003), Jafari et al.(2014), Kremer (2008), Hast et al. (2004), Sheomake(1985), Dam et al.(1998)). James (2006) considered the transition matrices of the transformations between the unit quaternions, rotation matrices, and Euler angles (Diebel (2006)). On the other hand, the dual numbers were firstly defined by W. Clifford (1873) such as $A = a + \varepsilon a*$ where $a,\ a^* \in \mathbb{R}$ and $\varepsilon \neq 0,\ \varepsilon^2 = 0$. Since the early part of the 21st century, E. Study has analyzed the correlation between the dual points and directed line; thus, he has proven that a unit dual vector on a unit dual sphere corresponds to a directed line in the Euclidean-3 space. Afterward, Baky (2002) worked on some characterization of the spherical closed dual curves (Baky (2002)). Ravani et al. (1994) defined the geometric screw linear interpolation for the first time (Ge et al. (1994)). Smitth (2013) carried on studies by using the dual quaternion interpolations. On the other hand, Kavan et al. (2006) conducted their studies depending on DLB and DIB interpolations (Smitth (2013), Kavan et al. (2006)). Yaylı et al. (2012) defined the fast De-Moivre rule for the spline split quaternion interpolations, fast spherical linear interpolations, and the spherical linear interpolation in the Lorentzian sphere with the help of the Bezier algorithm (Ghademi et al. (2012)).

In our paper, the links of the dual spherical mechanism between the joints were formed with the help of dual Sclerp interpolation defined by Ravani (1994). Sequential Sclerp interpolation and some new alternative dual interpolation motions were first defined as mechanism motion in this generated dual spherical mechanism. We also identified fast Sclerp interpolation for the first time. Using the links of the dual spherical mechanism with the Blaschke frame, the invariants at each point were geometrically calculated. At the end of our study, several examples related to the subject were included. Consequently, we believe that the theoretical results we obtained from our study will contribute to researches in scientific studies such as computer graphics, mechanics, and robotics.

2. Materials & Method

In 1873, W.K. Clifford defined the number $A = a + \varepsilon a*$ as a dual number where $a,\ a^* \in \mathbb{R}$ and $\varepsilon \neq 0,\ \varepsilon^2 = 0$. ε is assumed as a dual unit. Since the ring of the dual numbers isn't with zero divisors, ie. $\varepsilon a * . \varepsilon b* = 0$, the elements $\varepsilon a*$ haven't an inverse. Therefore, the dual numbers do not indicate a field, they just show an algebra. The set of the dual numbers is denoted by

$$D = \left\{ A = a + \varepsilon a^* : a, a^* \in \mathbb{R},\ \varepsilon^2 = 0 \right\}.$$

Definition 2.1. Let $A = a + \varepsilon a*$ and $B = b + \varepsilon b*$ be the dual numbers for $a, a*, b, b* \in \mathbb{R}$ and $\varepsilon^2 = 0.$ The operators of these dual numbers are defined by

Addition and Subtraction: $\mp : D \times D \to D$, $A \mp B = \left(a \mp b\right) + \varepsilon\left(a * \mp b*\right)$

Multiplication $. : D \times D \to D \quad A.B = ab + \varepsilon(ab * + a * b)$

Division $\div : D \times D \to D \quad \dfrac{A}{B} = \dfrac{a}{b} + \varepsilon \dfrac{a * b - ab*}{b^2}.$

Also, the equality of two dual numbers is described by $A = B \Leftrightarrow a = b$ and $\varepsilon a* = \varepsilon b*$. Thus, the set of the dual numbers $D = \left\{ A : A = a + \varepsilon a* \ \ a, \varepsilon a* \in R \right\}$ formed a commutative ring (Hacısalihoğlu(1983)).

Definition 2.2. Let D be a dual number set. $(D^3 = D \times D \times D, +)$ is an abelian group and it represents a module on the D. Thus, the set D^3 is called a D- module. Furthermore, the ordered dual triples which are the elements of the module-D are called dual vectors.

Definition 2.3. Let $\mathbf{A}=\mathbf{a}+\varepsilon\mathbf{a}^*$ and $\mathbf{B}=\mathbf{b}+\varepsilon\mathbf{b}^*$ be the dual numbers where $\mathbf{a},\mathbf{a}^*,\mathbf{b},\mathbf{b}^*\in\mathbb{R}^3$ and $\varepsilon^2=0$. The inner and the cross product of the dual vectors are respectively described by

$$\langle\mathbf{A},\mathbf{B}\rangle=\langle\mathbf{a},\mathbf{b}\rangle+\varepsilon[\langle\mathbf{a}^*,\mathbf{b}\rangle+\langle\mathbf{a},\mathbf{b}^*\rangle] \text{ and } \mathbf{A}\times\mathbf{B}=\mathbf{a}\times\mathbf{b}+\varepsilon\left(\mathbf{a}\times\mathbf{b}^*+\mathbf{a}^*\times\mathbf{b}\right).$$

Definition 2.4. Let $\mathbf{A}=\mathbf{a}+\varepsilon\mathbf{a}^*$ be a dual number where $\mathbf{a},\mathbf{a}^*,\mathbf{b},\mathbf{b}^*\in\mathbb{R}^3$ and $\varepsilon^2=0$. The norm of the dual vector is defined by $\left\|\mathbf{A}\right\|=\sqrt{\left\langle\mathbf{A},\mathbf{A}\right\rangle}=\|\mathbf{a}\|+\varepsilon\dfrac{\langle\mathbf{a},\mathbf{a}^*\rangle}{\|\mathbf{a}\|}$. Since $\left\|\mathbf{A}\right\|=1$, the dual number $\mathbf{A}$ is called a dual vector. Additionally, the set $S^2=\left\{\mathbf{A}\in D^3:\left\langle\mathbf{A},\mathbf{A}\right\rangle=(1,0)\right\}$ is called a unit dual sphere, (Hacısalihoğlu(1983)).

Theorem 2.5. (E. Study Theorem) The dual points of the unit dual sphere which is proved the condition $\left\|\mathbf{A}\right\|=(1,0)$ where $\mathbf{A}\neq(\mathbf{0},\mathbf{a})\in D$ corresponds to the directed lines in the Euclid-3 space, (Hacısalihoğlu(1983)).

Definition 2.6. Let the curve $X(t)=x(t)+\varepsilon x^*(t)$ be a dual curve on the dual unit sphere. The Blaschke frame of the dual curve $X(t)$ is described by $\mathbf{A}_1=X(t)$, $\mathbf{A}_2=\dfrac{\mathbf{A}_1'}{\|\mathbf{A}_1'\|}$, $\mathbf{A}_3=\mathbf{A}_1\times\mathbf{A}_2$ where $\langle\mathbf{A}_1,\mathbf{A}_1\rangle=\langle\mathbf{A}_2,\mathbf{A}_2\rangle=\langle\mathbf{A}_3,\mathbf{A}_3\rangle=1$, and $\langle\mathbf{A}_1,\mathbf{A}_2\rangle=\langle\mathbf{A}_1,\mathbf{A}_3\rangle=\langle\mathbf{A}_3,\mathbf{A}_2\rangle=0.$ Since these inner product properties of the dual numbers are provided for the Blaschke frame, the lines $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3$ show the perpendicular lines in the Euclidean-3 space, simultaneously and reciprocally. The center of the intersection point is the point $\mathbf{A}_1$, and this point is called the tangent of the dual curve. Moreover, the line $\mathbf{A}_2=\mathbf{A}(t)$ is called the normal mean of $\mathbf{A}(t)$ which is in the central point. The derivative formulas of the Blaschke frame are given by $\mathbf{A}_1'=P\mathbf{A}_2$, $\mathbf{A}_2'=-P\mathbf{A}_1+Q\mathbf{A}_3$, $\mathbf{A}_3'=-Q\mathbf{A}_2$. The statements $P=p+\varepsilon p^*=\|\mathbf{A}_1'\|$ and $Q=q+\varepsilon q^*=\dfrac{\det(\mathbf{A},\mathbf{A}',\mathbf{A}'')}{p^2}$ are called the Blaschke integral invariants. The derivatives of $\int Pd(t)$ and $\int Qd(t)$ are the dual arc length between the dual curve $\mathbf{A}_1(t)$ and $\mathbf{A}_3(t)$, respectively, (Baky (2002)).

Let the set of the quaternion and the dual quaternions be denoted H and H_D, respectively.

Definition 2.7. The statement $Q=D+iX+jY+kZ$ is called a dual quaternion where $D=d+\varepsilon d_0^*$, $X=x+\varepsilon x_1^*$, $Y=y_2+\varepsilon y_2^*$ and $Z=z_3+\varepsilon z_3^*$ are the dual number coefficients and $i^2=j^2=k^2=-1$, $jk=i$, $ij=k$, $ki=j$, $ji=-k$, $kj=-i$, $ik=-j$ are the dual quaternion bases. Namely, the components of the dual quaternion Q are composed of the dual numbers. When their scalar and vectorial parts are respectively denoted by S_{H_D} and $\mathbf{V}_{\mathbf{H_D}}$, the statements become $S_{H_D}=s_q+\varepsilon s_{q^*}=D$ and $\mathbf{V}_{H_D}=\mathbf{v_q}+\varepsilon\mathbf{v}_{q^*}=iX+jY+kZ$. Besides, the other representation of the quaternion is in the form of $Q=q+\varepsilon q^*$ where $\varepsilon^2=0$, and $q,\ q^*\in H$ (Hacısalihoğlu(1983)).

Definition 2.8. Let $M=M_0+M_1i+M_2j+M_3k$, $P=P_0+P_1i+P_2j+P_3k$ be two dual quaternions where $P,M\in H_D$. Thus, the two dual quaternions addition is defined by

$$+:H_D\to H_D,\qquad M+P=\left(M_0+P_0\right)+\left(M_1+P_1\right)i+\left(M_2+P_2\right)j+\left(M_3+P_3\right)k$$

The dual quaternions set $\left(H_D,+\right)$ is an abelian group due to the addition operator, (Hacısalihoğlu(1983)).

Definition 2.9. If all components of a dual quaternion are zero, individually, this dual quaternion is called zero dual quaternion, (Hacısalihoğlu(1983)).

Definition 2.10. If the reel and the dual parts of two dual quaternions are equal to each other, it is called equal two dual quaternions, (Hacısalihoğlu(1983)).

Definition 2.11. Let $M = M_0 + M_1 i + M_2 j + M_3 k$ be a dual quaternion where $\lambda \in R$ and $M_0, M_1, M_2, M_3 \in D.$ The scalar multiplication of the dual quaternions is defined by

$$\odot : R \times H_D \to H_D \,,$$
$$\left(\lambda, M\right) \to \odot\left(\lambda, M\right) = \lambda \odot M = \lambda M_0 + \lambda M_1 i + \lambda M_2 j + \lambda M_3 k,$$ (Hacısalihoğlu(1983)).

Definition 2.12. Let $M = M_0 + M_1 i + M_2 j + M_3 k$ be the dual quaternion where $M_0, M_1, M_2, M_3 \in D.$ The conjugate of the dual quaternion M is defined by $\overline{M} = M_0 - M_1 i - M_2 j - M_3 k,$ (Hacısalihoğlu(1983)).

Definition 2.13. Let $M = m + \varepsilon m^*,\ \hat{P} = p + \varepsilon p^*$ be two dual quaternions where $m, m^*, p, p^* \in H$. Then, the multiplication of the two dual quaternions is defined by $M \times P = m \times p + \varepsilon(m \times p^* + m^* \times p)$, (Hacısalihoğlu(1983)).

Definition 2.14. Let $M \in H_D$ be a dual quaternion. The norm of the dual quaternion M is defined by

$$\|\ \| : H_D \to R,\ \ M \to \left\|M\right\| = \sqrt{\left\langle M, \overline{M}\right\rangle} = \sqrt{M_0^2} = \left|M_0\right|,$$ (Hacısalihoğlu(1983)).

Definition 2.15 Let $M = m + \varepsilon m^* \in H_D$ be a dual quaternion where $m, m^* \in H$. The inverse of a dual quaternion M is $M^{-1} = \dfrac{1}{M} = \dfrac{\left(m - \varepsilon m^*\right)}{|m|^2} = \dfrac{\overline{M}}{\left\|M\right\|^2}$. If $m = 0$ in the dual quaternion M , the dual quaternion is called pure dual quaternion, and the inverse of the pure dual quaternion is not computed, (Hacısalihoğlu(1983)).

Definition 2.16. Let M and P be two dual quaternions. Divison of M by P is defined as follows:
If $M = Z_1 P\ \ (M = P Z_2),$ multiply each side by P^{-1} from the right (left), (Hacısalihoğlu(1983)).

Definition 2.17. Let $Q = D + iX + jY + kZ$ be a dual quaternion given by the dual number coefficients $D = d + \varepsilon d_0^*,$ $X = x + \varepsilon x_1^*,\ Y = y_2 + \varepsilon y_2^*\ ,\ Z = z_3 + \varepsilon z_3^*$. Since the norm of a dual quaternion is one, then the dual quaternion is called unit dual quaternion, and the set of the unit quaternions is denoted by H_{D1}. If the dual quaternion Q is a unit dual quaternion, the condition $d^2 + a^2 + b^2 + c^2 = 1$ and $dd^* + aa^* + bb^* + cc^* = 0$ are provided. Moreover, the unit dual quaternion can be shown by $\hat{Q} = \cos\theta + \hat{\mathbf{S}}_{\mathbf{0}} \sin\theta$, as well. Here, we get $\cos\theta = D,$ and $\sin\theta = \sqrt{X^2 + Y^2 + Z^2}$ where $\theta_1 = \theta + \varepsilon\theta^*$ is a dual-angle. Also, the axis of the unit dual vector

$$\hat{\mathbf{S}}_{\mathbf{0}} = S_{\mathbf{0}} + \varepsilon S_0^* = \frac{iX + jY + kZ}{\sqrt{X^2 + Y^2 + Z^2}}$$

yields a dual rotation. Therefore, the dual angle and the dual part become $\theta = \arccos(d)$ and $\theta^* = -\dfrac{d^*}{\sqrt{X^2+Y^2+Z^2}}$, respectively. The reel part and the dual part of the unit dual-axis are obtained as

$$S_{\mathbf{0}} = \frac{iX + jY + kZ}{\sqrt{X^2+Y^2+Z^2}} \text{ and } S_0^* = \frac{iX^* + jY^* + kZ^*}{\sqrt{X^2+Y^2+Z^2}} + \frac{dd^*\left(iX + jY + kZ\right)}{\sqrt{X^2+Y^2+Z^2}},$$

respectively, (Hacısalihoğlu(1983)).

Definition 2.18. Let $P = \cos\hat{\theta} + \hat{\mathbf{S}}\sin\hat{\theta}$ be a dual quaternion where $\hat{\theta}$ is a dual-angle and $\hat{\mathbf{S}}$ is a dual-axis. The De-Moivre rule of the dual quaternion P is defined by the formula $P^k = \cos\left(k\hat{\theta}\right) + \hat{\mathbf{S}}\sin\left(k\hat{\theta}\right)$, (Hacısalihoğlu(1983)).

Definition 2.19. (The Screw Linear (Sclerp) Interpolation) In 2006, Kavan studied the screw linear interpolation, which is known as the abbreviation *Sclerp*, as an expanded form of the SLERP. Since the dual quaternion is used in the Sclerp screw linear interpolation, it consists of both the rotation and the translation motion. Therefore, the Sclerp interpolation curve is defined by

$$Sclerp\left(D_1, D_2; t\right) = D_1 \times \left(\cos t\frac{\alpha}{2} + \sin t\frac{\alpha}{2}u\right)$$

where $t\alpha$ shows the rotation angle and u is the translation on the screw axis, (Sheomake (1985), Smitth (2013)).

Definition 2.20. (The Geometric Screw Linear Interpolation) The geometric screw linear interpolation is an interpolation between two unit dual vectors on the unit dual sphere, and this interpolation draws a dual curve on the dual big circle which is on the unit dual sphere. The geometric screw linear interpolation curve between the unit dual vectors B_0, and B_1 is defined by

$$\mathbf{B}(\mathbf{t}) = L\left(\mathbf{B}_0, \mathbf{B}_1, t\right) = \frac{\sin(1-t)\phi}{\sin\phi}\mathbf{B}_0 + \frac{\sin t\phi}{\sin\phi}\mathbf{B}_1,$$

where the parameters $t \in [0,1]$, (Ge et al.(1994)).

3.Main Results

In this study, the new definitions such as the sequential Sclerp interpolations, the fast dual interpolations, and the fast Sclerp are introduced in the dual spherical mechanisms located on a dual sphere. Besides, the computations of the invariants of the Blaschke frame of the dual spherical interpolation curves on the unit dual sphere are done.

Definition 3.1. (The Sclerp Interpolation) Let H_{D1} be the set of the dual quaternions and $Q_{(i-1)}, Q_i \in H_{D1}$ be two unit dual quaternions. The Sclerp interpolation of the unit dual quaternions $Q_{(i-1)}, Q_i$ for $t \in [0,1]$ is yielded by

$$Q_{(i-1)i} = Q_{(i-1)}\frac{\sin\left(\left(1-t\right)\phi\right)}{\sin\phi} + Q_i\frac{\sin\left(t\phi\right)}{\sin\phi}.$$

The angle ϕ between these two unit dual quaternions is a dual-angle, and this dual-angle is shown by $\phi = \phi + \varepsilon\phi^*$ where ϕ is the rotation angle and ϕ^* is the translation angle.

Definition 3.2. (The Sequential Sclerp Interpolation Method) Let $Q_{(i-1)}, Q_i \in H_{D1}$ be two unit dual quaternions, and the joints $\mathbf{R}_i$ be the unit dual-position vectors in a dual spherical mechanism. The Sclerp interpolation of the unit dual quaternions $Q_{(i-1)}, Q_i$ is computed by the equation $Sclerp\left(Q_{(i-1)}, Q_i, t\right) = Q_{(i-1)\,i}\left(t\right)$ for $t \in \left[0,1\right]$. Also, the Sclerp interpolation where $t = 0.5$ is found by the equation $Sclerp\left(Q_{(i-1)}, Q_i, 0.5\right) = Q_{(i-1)\,i}\left(0.5\right)$. Here, the dual-angle between these two dual quaternions is computed by the equation $\cos\phi = Q_{(i-1)i}\left(0.5\right).Q_{i(i+1)}\left(0.5\right)$. Thus, the position vectors of the joints $\mathbf{R}_i$ are obtained by the equation $R_{i+1} = Q_{(i-1)i}\left(R_i\right)Q_{(i-1)i}^{-1}$. When this operation is done sequentially, the sequential Sclerp interpolation

$$R_{n+1} = Q_{(n-1)n} \ldots Q_{23} Q_{12} R_1 Q_{12}^{-1} Q_{23}^{-1} \ldots Q_{(n-1)n}^{-1}$$

is obtained.

Definition 3.3. (An Alternative Sclerp Interpolation in the Dual Spherical Mechanisms) Let $Q_{(i-1)i}, Q_{i(i+1)} \in H_{D1}$ be two unit quaternions. A new interpolation motion can be defined by

$$Q_{(i-1)i(i+1)}(t) = Q_{(i-1)i}\frac{\sin\left(\left(1-t\right)\phi\right)}{\sin\phi} + Q_{i(i+1)}\frac{\sin\left(t\phi\right)}{\sin\phi}$$

of the sequential interpolations $Q_{(i-1)i}$ and $Q_{i(i+1)}$ which are obtained by the Sclerp interpolation defined by the unit dual quaternions which are yielding the motion of a dual sphere mechanism for the value $t = t_0 \in \left[0,1\right]$. Here, the dual-angle between the unit dual quaternions $Q_{(i-1)i}$ and $Q_{i(i+1)}$ is obtained by the equation $\cos\phi = Q_{(i-1)i}.Q_{i(i+1)}$ where $t = t_0 \in \left[0,1\right]$.

Definition 3.4. (The Design of the Links of the Dual Spherical Mechanisms Between the Joints by Sclerp) Let $\mathbf{A}_i$ and $\mathbf{A}_{i+1}$ be sequential unit dual-position vectors and ϕ be a dual angle between the dual vectors. The links between the two sequential unit dual-position vectors $\mathbf{A}_i$ and $\mathbf{A}_{i+1}$ in a dual spherical mechanism are computed by

$$Sclerp(\mathbf{A}_i, \mathbf{A}_{i+1}, t) = \mathbf{A}_i \frac{\sin\left(\left(1-t\right)\phi\right)}{\sin\phi} + \mathbf{A}_{i+1}\frac{\sin\left(t\phi\right)}{\sin\phi} \quad \text{where } t \in \left[0,1\right].$$

Theorem 3.5. (The Fast Dual Spherical Interpolation) Let $\mathbf{D_0}, \mathbf{D_1} \in \mathbb{D}$ be unit dual vectors and $\theta = \theta + \varepsilon\theta^*$ be a dual angle between these unit dual vectors. The interpolation

$$L_{fast}\left(\mathbf{D_0}, \mathbf{D}_1, t\right) = \mathbf{D_0}\cos\left(t\theta\right) + \mathbf{D}_R \sin\left(t\theta\right)$$

is called fast dual spherical interpolation where $t \in \left[0,1\right]$.

Proof. Using the dual spherical interpolation in Equation (9) for $t \in \left[0,1\right]$, we get

$$L_{fast}\left(\mathbf{D_0},\mathbf{D_1},t\right)=\mathbf{D_0}\frac{\sin\left(\left(1-t\right)\theta\right)}{\sin\theta}+\mathbf{D_1}\frac{\sin\left(t\theta\right)}{\sin\theta}=\mathbf{D_0}\cos\left(t\theta\right)-\mathbf{D_0}\frac{\cos\theta\sin\left(t\theta\right)}{\sin\theta}+\mathbf{D_1}\frac{\sin\left(t\theta\right)}{\sin\theta},$$

where $\mathbf{D_0},\mathbf{D_1}\in \mathbb{D}$ be unit dual vectors and $\theta=\theta+\varepsilon\theta^*$ be a dual angle between these unit dual vectors. Then, with the help of the dual trigonometric identities $\mathbf{D_0}.\mathbf{D_1}=\left\|\mathbf{D_0}\right\|\left\|\mathbf{D_1}\right\|\cos\theta=\cos\theta$, $\sin^2\theta+\cos^2\theta=1$ and $\sin\theta=\pm\sqrt{1-\cos^2\theta}$, the statement $\sin\theta=\sqrt{1-\left(\mathbf{D_0}.\mathbf{D_1}\right)^2}$ is obtained. Thus, the dual interpolation equation

$$L_{fast}\left(\mathbf{D_0},\mathbf{D_1},t\right)=\mathbf{D_0}\cos\left(t\theta\right)+\frac{\mathbf{D_1}\sin\left(t\theta\right)-\mathbf{D_0}\cos\theta\sin\left(t\theta\right)}{\sin\theta}=\mathbf{D_0}\cos\left(t\theta\right)+\frac{\mathbf{D_1}-\mathbf{D_0}\left(\mathbf{D_0}.\mathbf{D_1}\right)}{\sqrt{1-\left(\mathbf{D_0}.\mathbf{D_1}\right)^2}}\sin\left(t\theta\right)$$

is found. The statement

$$\mathbf{D}_r=\mathbf{D_1}-\frac{\left(\mathbf{D_0}\mathbf{D_1}\right)}{\left\|\mathbf{D_0}\right\|^2}\mathbf{D_0}=\mathbf{D_1}-\left(\mathbf{D_0}\mathbf{D_1}\right)\mathbf{D_0}$$

is obtained by the Gram-Schmidt orthogonalization method. In addition, when taking the norm of this vector, the equation

$$\left\|\mathbf{D}_r\right\|=\sqrt{\left(\mathbf{D_1}-\left(\mathbf{D_0}\mathbf{D_1}\right)\mathbf{D_0}\right)^2}=\sqrt{1-\left(\mathbf{D_0}\mathbf{D_1}\right)^2}$$

is obtained. Then, when the notation $\mathbf{D}_R=\dfrac{\mathbf{D}_r}{\left\|\mathbf{D}_r\right\|}$ is used, the interpolation $L_{fast}\left(\mathbf{D_0},\mathbf{D_1},t\right)$ becomes

$$L_{fast}\left(\mathbf{D_0},\mathbf{D_1},t\right)=\mathbf{D_0}\cos\left(t\theta\right)+\mathbf{D}_R\sin\left(t\theta\right).$$

This form of the dual spherical interpolation is called **fast dual spherical interpolation**. In this equation, when the angle $\theta=\theta+\varepsilon\theta^*$ is written by opening, the fast dual spherical interpolation can be defined by the equation

$$\begin{aligned}L_{fast}\left(\mathbf{D_0},\mathbf{D_1},t\right)&=\mathbf{D_0}\cos\left(t\theta\right)+\mathbf{D}_R\sin\left(t\theta\right)\\&=\mathbf{D_0}\cos\left(t\theta\right)+\mathbf{D}_R\sin\left(t\theta\right)+\varepsilon t\theta^*\left(\mathbf{D}_R\cos\left(-t\theta\right)+\mathbf{D_0}\sin\left(-t\theta\right)\right).\end{aligned}$$

Theorem 3.6. (The Fast Screw Linear Interpolation) Let $Q_0,Q_1\in H_{D1}$ be the unit dual quaternions and $\theta=\theta+\varepsilon\theta^*$ be the angle between these quaternions. where. The interpolation between the unit dual quaternions Q_0 and Q_1 **for** $t\in\left[0,1\right]$ is called **fast screw linear interpolation** by

$$Sclerp_{fast}=L_{fast}\left(Q_0,Q_1,t\right)=Q_0\cos\left(t\theta\right)+Q_R\sin\left(t\theta\right)$$

where $Q_R = \dfrac{Q_r}{\left\| Q_r \right\|}$ and $Q_r = Q_1 - \left(Q_0 . Q_1\right) Q_0$.

Proof. The Sclerp dual linear interpolation given by Ravani (1994) is written by the equation

$$L_{fast}\left(Q_0, Q_1, t\right) = Q_0 \frac{\sin\left((1-t)\theta\right)}{\sin\theta} + Q_1 \frac{\sin\left(t\theta\right)}{\sin\theta} = Q_0 \cos\left(t\theta\right) - Q_0 \frac{\cos\theta \sin\left(t\theta\right)}{\sin\theta} + Q_1 \frac{\sin\left(t\theta\right)}{\sin\theta},$$

where $t = \left[0,1\right]$. The dual-angle computation is done by the equation $Q_0 . Q_1 = \left\| Q_0 \right\| . \left\| Q_1 \right\| \cos\theta = \cos\theta$. By utilizing the dual trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ is obtained. Thus, when the statement $\sin\theta = \sqrt{1-\left(Q_0 . Q_1\right)^2}$ is written in the interpolation equation, the equation

$$\begin{aligned} L_{fast}\left(Q_0, Q_1, t\right) &= Q_0 \cos\left(t\theta\right) + \frac{Q_1 \sin\left(t\theta\right) - Q_0 \cos\theta \sin\left(t\theta\right)}{\sin\theta} \\ &= Q_0 \cos\left(t\theta\right) + \frac{Q_1 - Q_0 \left(Q_0 . Q_1\right)}{\sqrt{1-\left(Q_0 . Q_1\right)^2}} \sin\left(t\theta\right) \end{aligned}$$

is founded. From editing the dual-angle $\theta = \theta + \varepsilon\theta^*$, the fast screw linear interpolation ($Sclerp_{fast}$) is demonstrated by the equation

$$\begin{aligned} L_{fast}\left(Q_0, Q_1, t\right) &= \mathbf{Q_0} \cos\left(t\theta\right) + Q_R \sin\left(t\theta\right) \\ &= \mathbf{Q_0} \cos\left(t\theta\right) + Q_R \sin\left(t\theta\right) + \varepsilon t\theta^* \left(Q_R \cos\left(-t\theta\right) + \mathbf{Q_0} \sin\left(-t\theta\right)\right). \end{aligned}$$

Theorem 3.7. (The Blaschke Frame of the Dual Spherical Interpolation Curve) Let $\mathbf{D_0}, \mathbf{D_1} \in D$ be unit dual vectors. The dual points $\mathbf{D_0}$ and $\mathbf{D_1}$ which are taken from the unit dual sphere indicate two directed lines in the Euclidean 3-space due to the E. Study theorem. Let the dual-angle between the unit dual vectors $\mathbf{D_0}$ and $\mathbf{D_1}$ be $\phi = \phi + \varepsilon\phi^*$ where the angle ϕ yields the rotation, and ϕ^* yields translation. Taking the dual spherical interpolation of the unit dual vectors $\mathbf{D_0}$ and $\mathbf{D_1}$, the unit dual spherical curve is formed by

$$A(t) = \mathbf{D(t)} = L\left(\mathbf{D_0}, \mathbf{D_1}, t\right) = \frac{\sin\left((1-t)\phi\right)}{\sin\phi} \mathbf{D_0} + \frac{\sin\left(t\phi\right)}{\sin\phi} \mathbf{D_1},$$

where $t \in \left[0,1\right]$. While the interpolation curve $\mathbf{D(t)} = L\left(\mathbf{D_0}, \mathbf{D_1}, t\right)$ is a curve on the unit dual sphere, it corresponds to a ruled surface in the Euclidean 3-space due to E. Study theorem, as well. The Blaschke frame of the dual spherical interpolation curve $\mathbf{D(t)}$ is defined by

$$\mathbf{A_1} = \mathbf{D}\left(\mathbf{t}\right) = \frac{\sin\left((1-t)\phi\right)}{\sin\phi} \mathbf{D_0} + \frac{\sin\left(t\phi\right)}{\sin\phi} \mathbf{D_1},$$

$$\mathbf{A_2} = \frac{\mathbf{A}_1'}{\left\|\mathbf{A}_1'\right\|} = \pm\frac{1}{G}\left[-\cos\left(\left(1-t\right)\phi\right)\mathbf{D_0} + \cos\left(t\phi\right)\mathbf{D_1}\right],\ \mathbf{A_3} = \frac{1}{G}\mathbf{D_0}\wedge\mathbf{D_1},$$

where $G = \left[\cos^2\left(\left(1-t\right)\phi\right) + \cos^2\left(t\phi\right) - 2\cos\left(\left(1-t\right)\phi\right)\cos\left(t\phi\right)\cos\phi\right]^{1/2}$.

Proof. Let the first dual vector of the Blaschke frame be $\mathbf{A_1} = \mathbf{D}(\mathbf{t}) = \dfrac{\sin\left(\left(1-t\right)\phi\right)}{\sin\phi}\mathbf{D_0} + \dfrac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}$. The norm of the vector

$$\mathbf{D}'(t) = \frac{-\phi\cos\left(\left(1-t\right)\phi\right)}{\sin\phi}\mathbf{D_0} + \frac{\phi\cos\left(t\phi\right)}{\sin\phi}\mathbf{D_1}$$

is obtained by the equation

$$\begin{aligned}\left\|\mathbf{D}'(t)\right\|^2 &= \left\|\frac{-\phi\cos\left(\left(1-t\right)\phi\right)}{\sin\phi}\mathbf{D_0} + \frac{\phi\cos\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right\|^2 \\ &= \left|\frac{\phi^2}{\sin\phi}\right|\left[\cos^2\left(\left(1-t\right)\phi\right) + \cos^2\left(t\phi\right) - 2\cos\left(\left(1-t\right)\phi\right)\cos\left(t\phi\right)\cos\phi\right].\end{aligned}$$

To make it simpler, when the abbreviation $G = \left[\cos^2\left(\left(1-t\right)\phi\right) + \cos^2\left(t\phi\right) - 2\cos\left(\left(1-t\right)\phi\right)\cos\left(t\phi\right)\cos\phi\right]^{1/2}$

is used, the equality $\left\|\mathbf{D}'(t)\right\| = \left|\dfrac{\phi}{\sin\phi}\right|G$ is written. Thus, the second vector of the Blaschke frame is obtained by

$$\mathbf{A_2} = \frac{\mathbf{A}_1'}{\left\|\mathbf{A}_1'\right\|} = \pm\frac{1}{G}\left[-\cos\left(\left(1-t\right)\phi\right)\mathbf{D_0} + \cos\left(t\phi\right)\mathbf{D_1}\right].$$

The third vector of the frame is computed by the equation

$$\begin{aligned}\mathbf{A_3} &= \mathbf{A_1}\times\mathbf{A_2} = \mathbf{D}(t)\times\frac{\mathbf{D}'(t)}{\left\|\mathbf{D}'(t)\right\|} = \frac{1}{\left\|\mathbf{D}'(t)\right\|}\left[\mathbf{D}(t)\times\mathbf{D}'(t)\right] \\ &= \left[\frac{\sin\left(\left(1-t\right)\phi\right)}{\sin\phi}\mathbf{D_0} + \frac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right]\wedge\left[\pm\frac{1}{G}\left\{-\cos\left(\left(1-t\right)\phi\right)\mathbf{D_0} + \phi\cos\left(t\phi\right)\mathbf{D_1}\right\}\right] \\ &= \frac{1}{G}\mathbf{D_0}\wedge\mathbf{D_1},\end{aligned}$$

from the cross product of the vectors $\mathbf{A_1}$ and $\mathbf{A}_2$. One of the integral invariants of the Blaschke frame of the dual spherical interpolation is computed by $P = p + \varepsilon p^* = \left\|\mathbf{A_1}\right\| = \left\|\mathbf{D}_1'\right\| = \left|\frac{\phi}{\sin\phi}\right| G$. Then, the other one is computed by

$Q = q + \varepsilon q^* = \dfrac{\det\left(\mathbf{A}, \mathbf{A}', \mathbf{A}''\right)}{P^2}$ where the second-order derivative equation is

$$\det\left(\mathbf{A}, \mathbf{A}', \mathbf{A}''\right) = \langle \mathbf{A} \wedge \mathbf{A}', \mathbf{A}'' \rangle = \langle \mathbf{D}(t) \wedge \mathbf{D}'(t), \mathbf{D}''(t) \rangle$$

$$= \left\langle \begin{matrix} \left[\dfrac{\sin\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0} + \dfrac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right] \wedge \left[\dfrac{-\phi\cos\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0} + \dfrac{\phi\cos\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right], \\ -\phi^2.\left[\dfrac{\sin\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0} + \dfrac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right] \end{matrix} \right\rangle = 0,$$

where $\mathbf{D}''(t) = -\phi^2\mathbf{D_0} = -\phi^2.\left[\dfrac{\sin\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0} + \dfrac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}\right]$**.** Thus, the second invariant of the Blaschke frame is computed by $Q = \dfrac{\det\left(\mathbf{A}, \mathbf{A}', \mathbf{A}''\right)}{P^2} = 0$.

4.Numeric Examples

Example 4.1. The unit dual vector corresponding to the line with the direction vector $\mathbf{a_1} = (1,0,0)$ at the point $M_1(1,1,1)$ in the Euclid-3 space is demonstrated by $\mathbf{A}_1 = \mathbf{a_1} + \varepsilon\mathbf{a_1^*} = (1,0,0) + \varepsilon(0,1,-1)$. Here, the moment $\mathbf{a_1^*}$ is calculated by $\mathbf{a_1^*} = \mathbf{OM_1} \wedge \mathbf{a_1} = (0,1,-1)$. Then, when a line with the direction vector $\mathbf{a_2} = (0,1,0)$ at the point $M_2(1,0,1)$ in the Euclid-3 space corresponds to the unit dual vector in the dual space, the dual vector is shown by $\mathbf{A}_2 = \mathbf{a}_2 + \varepsilon\mathbf{a}_2^* = (0,1,0) + \varepsilon(-1,0,1)$. Here, the moment $\mathbf{a_2^*}$ is computed by $\mathbf{a}_2^* = OM_2 \wedge \mathbf{a}_2 = (-1,0,1)$. The dual-angle between the unit dual vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ is calculated by

$$\begin{aligned}\langle \mathbf{A}_1, \mathbf{A}_2 \rangle &= \langle \mathbf{a}_1, \mathbf{a}_2 \rangle + \varepsilon(\langle \mathbf{a}_1, \mathbf{a}_2^* \rangle + \langle \mathbf{a}_1^*, \mathbf{a}_2 \rangle) = \cos\theta = \cos(\theta + \varepsilon\theta^*) \\ &= \cos\theta \mp \varepsilon\theta^*\sin\theta = 0 + \varepsilon 0.\end{aligned}$$

Here, the angle $\theta = \dfrac{\pi}{2}$ is obtained by the equation of $\cos\theta = 0,$ and the angle $\theta^* = 0$ is obtained by the equation $\theta^*\sin\theta = 0.$

Thus, the dual-angle between the unit dual vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ is found by $\theta = \dfrac{\pi}{2} + \varepsilon 0$. The equation of the Sclerp interpolation between the unit dual vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ is obtained by

$$Sclerp(\mathbf{A}_1,\mathbf{A}_2,t)=\mathbf{A}_1\frac{\sin((1-t)\theta)}{\sin\theta}+\mathbf{A}_2\frac{\sin(t\theta)}{\sin\theta}$$
$$=(\sin((1-t)\frac{\pi}{2}),\sin(t\frac{\pi}{2}),0)+\varepsilon((-\sin(t\frac{\pi}{2}),\sin((1-t)\frac{\pi}{2}),-\sin((1-t)\frac{\pi}{2})+\sin(t\frac{\pi}{2})).$$

The interpolation value at the point $t=0.5$ is computed by

$$Sclerp(\mathbf{A}_1,\mathbf{A}_2,0.5)=(1,0,0)+\varepsilon(0,1,-1)\frac{\sin 45^\circ}{\sin 90^\circ}+(0,1,0)+\varepsilon(-1,0,1)\frac{\sin 45^\circ}{\sin 90^\circ}$$
$$=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)+\varepsilon(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0).$$

Example 4.2. Let's compute the Blaschke frame of the Sclerp interpolation

$$Sclerp(\mathbf{D_0},\mathbf{D_1},t)=\mathbf{D}(t)=\frac{\sin\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0}+\frac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}$$

which is defined by the unit dual vectors $\mathbf{D}_0=(1,0,0)+\varepsilon(0,1,-1)$ and $\mathbf{D}_1=(0,1,0)+\varepsilon(-1,0,1)$ at the point $t=0.5$. The dual-angle between the unit dual vectors $\mathbf{D}_0$ and $\mathbf{D}_1$ is $\theta=\frac{\pi}{2}+\varepsilon 0$. The first vector of the Blaschke frame is computed by

$$\mathbf{A}_1=\mathbf{D}(t)=\frac{\sin\left((1-t)\phi\right)}{\sin\phi}\mathbf{D_0}+\frac{\sin\left(t\phi\right)}{\sin\phi}\mathbf{D_1}=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)+\varepsilon(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},0).$$

The second dual vector of the Blaschke frame is obtained by

$$\mathbf{A_2}=\frac{\mathbf{A}_1'}{\left\|\mathbf{A}_1'\right\|}=\pm\frac{1}{G}\left[-\cos\left((1-t)\phi\right)\mathbf{D_0}+\cos\left(t\phi\right)\mathbf{D_1}\right]=(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0)+\varepsilon(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\frac{2\sqrt{2}}{2})$$

where $G=\left[\cos^2\left((1-t)\phi\right)+\cos^2\left(t\phi\right)-2\cos\left((1-t)\phi\right)\cos\left(t\phi\right)\cos\phi\right]^{1/2}=1.$ Finally, the third vector of the Blaschke frame is computed by the cross product of the dual vectors $\mathbf{A_1}$ and $\mathbf{A_2}$ as $\mathbf{A_3}=\mathbf{A_1}\times\mathbf{A_2}=\frac{1}{G}\mathbf{D_0}\wedge\mathbf{D_1}=(0,0,1)+\varepsilon(1,-2,1).$
The integral invariants of the Blaschke frame of the dual spherical interpolation are

$$P=p+\varepsilon p^*=\left|\frac{\phi}{\sin\phi}\right|G=90, \text{ and } Q=q+\varepsilon q^*=\frac{\det\left(\mathbf{A},\mathbf{A}',\mathbf{A}''\right)}{P^2}=0.$$

5.Conclusion

Quaternion interpolations have very important applications for many computer graphics in computer-aided geometric design (CAGD). Slerp interpolation, a curve fitting method between two quaternions and defined by Shoemake, is widely used in animation productions, and some other engineering fields. Thanks to the Slerp interpolation, a curve is formed from the geodesic curves of the sphere between the two quaternion points given on the greatest circles on the unit sphere. The Fast Slerp interpolation algorithm also has more advantages over Slerp interpolation. Similarly, interpolation of unit dual quaternions, called Sclerp, plays an important role in graphic design and the study of robotic motions. On the other hand, using quaternions in the spherical mechanism studies is one of the important topics studied in mechanical and robotic engineering. In this study, we theoretically generated a dual spherical mechanism and identified the links between the joints in the mechanism through Sclerp interpolations. Then, we defined for the first time the sequentially Sclerp and fast dual screw spherical interpolations as a special movement from these mechanism motions. The Blaschke frame of dual spherical interpolation curve was calculated using the E.Study theorem. In this way, Blaschke invariants at each point were calculated in the defined dual spherical mechanism. Since we bring a different perspective to the spherical mechanism, we think that this work will contribute to many scientific studies such as computer-aided geometric design, mechanism, and robotic studies.

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