SDU Journal of Science (E-Journal), 2011, 6 (2): 149-159

Step Size Strategies Based On Error Analysis For The Linear Systems

Gülnur Çelik Kızılkan[1,*], Kemal Aydın[2]

[1]*Kahramanmaraş Sütçü İmam University, Faculty of Education, Department of Primary Mathematics Education, 46100, Merkez, Kahramanmaraş, Turkey*
[2]*Selçuk University, Faculty of Science, Department of Mathematics, 42075, Selçuklu, Konya, Turkey*
[*]*Corresponding author e-mail: gckizilkan@ksu.edu.tr*

Received: 25 August 2009, Accepted: 03 June 2011

Abstract: In this paper, we have obtained the step size strategies for numerical integration of the linear differential equation systems. We have given the algorithms which calculate step sizes based on the given strategies and numerical solutions. These strategies and algorithms are generalized to systems by modifying the algorithm and strategy in [1]. We have applied our strategies to Cauchy problem with order *m*. We have also give the numerical examples.

Key words: Variable step size, error analysis, linear systems, numerical integration, step size strategy

Lineer Sistemler İçin Hata Analizi Tabanlı Adım Genişliği Stratejileri

Özet: Bu çalışmada, lineer diferensiyel denklem sistemlerinin nümerik integrasyonu için adım genişliği stratejileri elde edilmiştir. Verilen stratejilere uygun olarak adım genişlikleri ve nümerik çözümler hesaplayan algoritmalar verilmiştir. Bu strateji ve algoritmalar [1] de verilen strateji ve algoritmanın değiştirilerek sistemlere genişletilmesidir. Verilen stratejiler *m*. mertebeden Cauchy problemine uygulanmıştır. Ayrıca, sonuçların doğruluğunu göstermek için nümerik örnekler de verilmiştir.

Anahtar kelimeler: Değişken adım genişliği, hata analizi, lineer sistemler, nümerik integrasyon, adım genişliği stratejisi

1. Introduction

Choosing the step size is one of the most important concepts in numerical integration of the Cauchy problem

$$x' = f(t, x) \ , \ x(t_0) = x_0 . \qquad (1.1)$$

The use of constant step size is not practical in numerical integration. If the step size used is large in numerical integration, it provides fast convergence but also may lead to error. And the computed solution may diverge from the exact solution. On the other hand, if the step size used is small, it may give the opposite performance, *i.e.* the calculation time, number of the arithmetic operations, the calculation errors start to increase [2]. So, if the solution changes rapidly, the step size should be chosen small. Inversely, if the solution changes slowly, then step size should be chosen larger.

In [1,3], the step size strategies based on *error analysis* were given for numerical integration of the Cauchy problem (1.1) on the region $D = \{(t, x) : t \in [t_0, T], |x - x_0| \le b\}$ and there was also given an algorithm which calculates the step size based on error analysis and numerical solution in each step is given. For Euler method the step sizes are given by the following inequality

$$h_i \le \left(\frac{2\delta_L}{M_{t_i}}\right)^{\frac{1}{2}}, \tag{1.2}$$

where $\max_{t_{i-1}\le\tau_i<t_i} |z''(\tau_i)| \le M_{t_i}$ and for second order Runge-Kutta method as follows,

$$h_i \le \left(\frac{12\delta_L}{M_{t_i}}\right)^{\frac{1}{3}},$$

where $\max_{\tau\in[t_{i-1},t_i)} |(f_{tt} + 2f.f_{tx} + f_x.f_t + f.f_x^2 + f^2 f_{xx})(\tau)| \le M_{t_i}$ such as local error is smaller than the required error level δ_L in each step of the integration. Here δ_L is the error level that is determined by the user and $z(t)$ is the solution of the Cauchy problem

$$z' = f(t, z), \ z(t_{i-1}) = y_{i-1}, \ t \in [t_{i-1}, t_i), \ (y_0 = x_0), \tag{1.3}$$

where y_i is the numerical solution taken from numerical method the i-th step.

If the existence of the solution of Cauchy problem given by equation (1.1) on region $D = \{(t, x): |t - t_0| \le a, |x - x_0| \le b\}$ is unknown; the step size has been given by

$$h_i = \min\{a, b_{0i-1}/M_i\},$$

where y_i is the numerical solution obtained in the i-th step, $z(t)$ is the solution of the Cauchyproblem (1.3), b_{i-1} is the upper bound of $|z - y_{i-1}|$ error, $b_{0i-1} = \min\{b_{0i-2}, b_{i-1}\}$, $D_{i-1} = \{(t,z): |t - t_{i-1}| \le a, |z - y_{i-1}| \le b_{0i-1}\}$ and M_i is the upper bound of $f(t,z)$ on region D_{i-1} [1, 3, 4].

For detailed knowledge on the numerical integration of the Cauchy problem (1.1) the references [5-7] can be examined.

In this paper, we want to investigate a step size strategy for the Cauchy problem

$$X' = F(t, X), \ X(t_0) = X_0 \tag{1.4}$$

on the region

$$D = \{(t, X): |t - t_0| \le T, |x_j - x_{j0}| \le b_j\} \tag{1.5}$$

by generalizing the step size strategy given in [1, 3] for Cauchy problem (1.1). Here, $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $F(t, X) = (f_j)$; $f_j = f_j(t, x_1, x_2, \ldots, x_N)$, $F(t, X) \in C^m([t_0, T] \times \mathbb{R}^N)$ and $X(t)$, X_0, $b = (b_j) \in \mathbb{R}^N$.

We suppose that $F(t, X) = AX(t)$, where $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ and consider the Cauchy problem given by

$$X'(t) = AX(t) = F(t, X), \ X(t_0) = X_0. \tag{1.6}$$

The aim of this paper is to generalize the algorithm and strategy given in [1, 3] for the Cauchy problem (1.6).

In our study, we have used the Euler's method for simplicity. In section 2; the concept of local error given in [1, 3, 8] as being defined for systems of differential equations and local error analysis has been examined. In section 3, the step size strategies based on error analysis have been applied to systems and algorithms which calculate step size and numerical solution in each step have been given. In section 4; the step size strategies have been given for *m*-th order Cauchy problem. Finally, in Section 5 numerical examples have been given as applications of the algorithms.

2. Preliminaries

In this study, as a norm in $\mathbb{R}^N$ we use Euclidean norm, which is defined as follows

$$\| y \| = \sqrt{\sum_{j=1}^{N} y_j^2}\ ,\ y = (y_j) \in \mathbb{R}^N .$$

For every $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, we use the Frobenius norm, *i.e.*

$$\| A \| = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2} .$$

2.1. Local Error

We give the concept of local error, which is given for Cauchy problem (1.1) in [1, 3, 8], for Cauchy problem (1.4) as follows.

Let us construct the Cauchy problem given as follows

$$Z' = F(t\ , Z)\ ,\ Z(t_{i-1}) = Y_{i-1}\ ,\ Y_0 = X_0\ \ t \in [t_{i-1}\ , t_i)\ , \tag{2.1}$$

where $Y_i = (y_{ij}) \in \mathbb{R}^N$ is the numerical solution taken from numerical method the *i*-th step.

The vector of local error LE_i of a numerical method is given by

$$LE_i = Y_i - Z(t_i) = \begin{pmatrix} y_{i1} - z_1(t_i) \\ y_{i2} - z_2(t_i) \\ \vdots \\ y_{iN} - z_N(t_i) \end{pmatrix} = \begin{pmatrix} LE_{i1} \\ LE_{i2} \\ \vdots \\ LE_{iN} \end{pmatrix} \tag{2.2}$$

and $\| LE_i \|$ is the local error of the numerical method.

2.2. Euler's Method

Euler's method for Cauchy problem (1.4) is defined in [9] by

$$Y_{i+1} = Y_i + h_{i+1} F_i .$$

Here $F_i = (f_{ij}) \in \mathbb{R}^N$, $Y_i = (y_{ij}) \in \mathbb{R}^N$ and $h_{i+1} = t_{i+1} - t_i$.

2.3. Error Analysis For Systems

Let us apply error analysis in [1] to Cauchy problem (1.3). The component LE_{ij} of (2.2) is given as follows,

$$
\begin{aligned}
LE_{ij} &= y_{ij} - z_j(t_i) \\
&= y_{(i-1)j} + h_i f_{(i-1)j} - (z_j(t_{i-1}) + z_j'(t_{i-1})(t_i - t_{i-1}) + \frac{1}{2!} z''(\tau_{ij})(t_i - t_{i-1})^2) \\
&= y_{(i-1)j} + h_i f_{(i-1)j} - (y_{(i-1)j} + h_i f_{(i-1)j} + \frac{1}{2!} z''(\tau_{ij}) h_i^2) \\
LE_{ij} &= -\frac{1}{2!} z''(\tau_{ij}) h_i^2 \ , j = 1, 2, ..., \cdots N \ , \tau_{ij} \in (t_{i-1} \ , t_i).
\end{aligned}
$$

The vector of local error on interval $[t_{i-1} \ , t_i)$ is as follows,

$$
LE_i = \begin{pmatrix} LE_{i1} \\ LE_{i2} \\ \vdots \\ LE_{iN} \end{pmatrix} = -\frac{1}{2!} \begin{pmatrix} z_1''(\tau_{i1}) \\ z_2''(\tau_{i2}) \\ \vdots \\ z_N''(\tau_{iN}) \end{pmatrix} = -\frac{1}{2!} h_i^2 Z''(\tau_{ij}) \ . \tag{2.3}
$$

3. The Step Size Strategies For The Linear Systems

Let us give the theorem which gives the upper bound of local error of the system

$$X' = AX(t) = F(t \ , X) \ , \ X(t_0) = X_0 \, .$$

Theorem 1. The vector of local error of Cauchy problem (1.6) is

$$LE_i = -\frac{h_i^2}{2!} A^2 Z(\tau_{ij}) \, , \tau_{ij} \in [t_{i-1} \ , t_i) \, . \tag{3.1}$$

Proof. If we take $F(t \ , Z) = AZ(t)$ in (2.1), then we find

$$Z'(t) = AZ(t) \, ,$$

where $A = (a_{ij}) \in \mathbb{R}^{N \times N}$. So it is clear that

$$Z''(t) = AZ'(t) = A(AZ(t)) = A^2 Z(t) \, . \tag{3.2}$$

If we substitute this result into the equation (2.3), then the vector of local error will be as follows:

$$LE_i = -\frac{h_i^2}{2!} A^2 Z(\tau_{ij}) \, , \tau_{ij} \in [t_{i-1} \ , t_i) \, .$$

Theorem 2. The upper bound of local error for the system (1.6) is

$$\| LE_i \| \leq (\frac{1}{2} \alpha^2 \beta_{i-1}) \sqrt{N^5} h_i^2 \, , \tag{3.3}$$

where $\alpha = \max\limits_{1 \leq i, j \leq N} | a_{ij} |$ and $\max\limits_{1 \leq j \leq N} (\sup\limits_{t_{i-1} \leq \tau_i < t_i} | z_j(\tau_i) |) \leq \beta_{i-1}$.

Proof. If we take the norm of the equation (3.1), then we obtain

$$\| LE_i \| = \frac{1}{2} h_i^2 \| A^2 Z(\tau_i) \| \leq \frac{1}{2} h_i^2 \| A^2 \| \| Z(\tau_i) \| \, . \tag{3.4}$$

Since the inequalities

$$\| A \| \leq N \max_{1 \leq i, j \leq N} | a_{ij} | \ , \ \| Z \| \leq \sqrt{N} \max_{1 \leq j \leq N} | z_j |$$

are valid for every $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ and $Z = (z_j) \in \mathbb{R}^N$ [10], taking $\alpha = \max\limits_{1 \leq i, j \leq N} | a_{ij} |$ and $\max\limits_{1 \leq j \leq N} (\sup\limits_{t_{i-1} \leq \tau_i < t_i} | z_j(\tau_i) |) \leq \beta_{i-1}$, the inequality (3.4) can be written as

$$\| LE_i \| \le (\frac{1}{2}\alpha^2 \beta_{i-1})\sqrt{N^5}\, h_i^2 .$$

Here, it is clear that $\max_{1\le j\le N} | z_j | \le \beta_{i-1}$.

3.1. Step Size Strategy

From (3.3), the step size is computed using the inequality

$$h_k \le \frac{1}{\alpha\sqrt[4]{N^5}} (\frac{2\delta_L}{\beta_{k-1}})^{\frac{1}{2}} \tag{3.5}$$

in the k-th step such that local error $\| LE_k \| < \delta_L$, where δ_L is the error level determined by the user.

Since we have considered the problem (1.6) on the region

$$D = \{(t\ , X) : | t - t_0 | \le T\ , | x_j - x_{j0} | \le b_j\ , j = 1, 2, ..., N\}\ ,$$

it is clear that

$$\sup_{t_{k-1}\le \tau_k < t_k} \ | z_j(\tau_k) |) \le b_j + | z_j(t_{k-1}) | .$$

So, the calculation of β_k can be as follows in practice:

$$\beta_k = \max_{1\le j\le N}\{b_j + | z_j(t_k) |\} = \max_{1\le j\le N}\{b_j + | y_{kj} |\} .$$

Corollary 1. For N =1, the inequality (3.5) is equivalent to the inequality (1.2), which is the step size strategy given in [1] for the first order nonlinear nonhomogeneous ordinary differential equations.

Proof. Let us take $f(t\ , x) = ax$ in the Cauchy problem (1.1). Since $z'(t) = az$ and $\max_{t_{k-1}\le \tau_k < t_k} | z''(\tau_k) | \le M_{t_k}$ are valid, we obtain

$$\begin{aligned}\max_{t_{k-1}\le \tau_k < t_k} | z''(\tau_k) | &\le \sup_{t_{k-1}\le \tau_k < t_k} | z''(\tau_k) | = \sup_{t_{k-1}\le \tau_k < t_k} | a^2 z(\tau_k) | \\ &= a^2 \sup_{t_{k-1}\le \tau_k < t_k} | z(\tau_k) | \le \alpha^2 \beta_{k-1} = M_{t_k}\end{aligned}$$

where M_{t_k} is the number in (1.2). It is clear that $\alpha = | a |$ and

$$\sup_{t_{k-1}\le \tau_k < t_k} | z_j(\tau_k) | = \sup_{t_{k-1}\le \tau_k < t_k} | z(\tau_k) | = \max_{j=1}(\sup_{t_{k-1}\le \tau_k < t_k} | z_j(\tau_k) |) \le \beta_{k-1} .$$

If we substitute the result above into the equation (3.5), then the step size is computed as follows:

$$h_k \le \frac{1}{\alpha} (\frac{2\delta_L}{\beta_{k-1}})^{\frac{1}{2}} = (\frac{2\delta_L}{\alpha^2 \beta_{k-1}})^{\frac{1}{2}} = (\frac{2\delta_L}{M_{t_k}})^{\frac{1}{2}} . \tag{3.6}$$

So, it is shown that the inequality (3.5) is equivalent to (1.2) for N = 1.

Remark 1. In accordance with our goal, the step size can be chosen from the equation (3.4) in the k-th step. Theoretically, when step sizes are computed by (3.4), local error of the problem may be very close to the number δ_L. Since all numerical computations on the computer are performed using floating point arithmetic, the round-off errors occur. So it may occur that $LE_k > \delta_L$ for some k. Therefore, we have given the step size strategy in (3.5) to avoid possible effects of errors in floating point arithmetic. It is clear that the step sizes computed by (3.5) are smaller than those obtained by (3.4).

3.2. Algorithm 1

Now, let us give the algorithm which calculates step sizes using the equation (3.5) and numerical solution in each step. This algorithm is a modification of the algorithm in [1].

Step 0: Give the $t_0, T, b, h^*, \delta_L, X_0, A$ data.

Step 1: Calculate β_0 and α .

Step 2: Calculate β_{k-1} and $\hat{h}_k$; $\hat{h}_k \le \frac{1}{\alpha\sqrt[4]{N^5}}(\frac{2\delta_L}{\beta_{k-1}})^{\frac{1}{2}}$

Step 3: Control $\hat{h}_k$ with K;

K: **1.** If $t_{k-1} + \hat{h}_k \le T$; then

1.1. If $\hat{h}_k > h^*$ then $h_k = \hat{h}_k$.

1.2. If $\hat{h}_k < h^*$ then h_k=0 and the process stops.

2. If $t_{k-1} + \hat{h}_k > T$ then $\hat{\hat{h}}_k = T - t_{k-1}$.

2.1. If $\hat{\hat{h}}_k > h^*$ then $h_k = \hat{\hat{h}}_k$.

2.2. If $\hat{\hat{h}}_k < h^*$ then h_k=0 and the process stops.

Step 4: Calculate $t_k = t_{k-1} + h_k$ and $Y_k = (I + h_k A)Y_{k-1}$. Replace k by k+1 and go to step 2.

Here; k is the step number, T is the number given on D, $\hat{h}_k$ is the proposed step size by the step size strategy, h^* is the practical parameter for step size, $t_{k-1} = t_0 + \sum_{i=1}^{k-1} h_i$, where $\sum_{i=1}^{0} h_i = 0$.

In Algorithm 1, Algorithm K which is called *Step Size Control Algorithm* in [1] concludes the computation procedure.

Remark 2. For the reason mentioned in Remark 1, (3.5) is obtained by increasing the inequality (3.4) and step sizes obtained by (3.5) become smaller. Therefore, if the step sizes are computed by the inequality (3.5), their values increase and the local errors occur much smaller than the number δ_L.

To compute the numerical solutions in error level δ_L with sufficiently large step sizes, let us give the following modified step size strategy.

3.3. Modified Step Size Strategy

Initially, the step size is chosen by the inequality (3.5) in the k-th step. For any $\gamma > 1$ ($\gamma \in \mathbb{R}$); $h_k^i = \gamma^{i-1}\hat{h}_k$ and LE_k^i are calculated from i = 2 to $i = p$ such that $\| LE_k^p \| > \delta_L$ and $\| LE_k^{p-1} \| < \delta_L$, where $\| LE_k^1 \| < \delta_L$ in the k-th step. Then, the step size is computed by

$$h_k = \gamma^{p-2}\hat{h}_k \ , \ \hat{h}_k \le \frac{1}{\alpha\sqrt[4]{N^5}}\left(\frac{2\delta_L}{\beta_{k-1}}\right)^{\frac{1}{2}} \tag{3.7}$$

in the k-th step for numerical integration of the linear system (1.6). The step size (3.7) is sufficiently large to calculate the numerical solution in error level δ_L. Although $\gamma > 1$, $\gamma \in \mathbb{R}$ is chosen by the user, we suggest the user take $1 < \gamma < 2$. The following diagram represents the modified strategy:

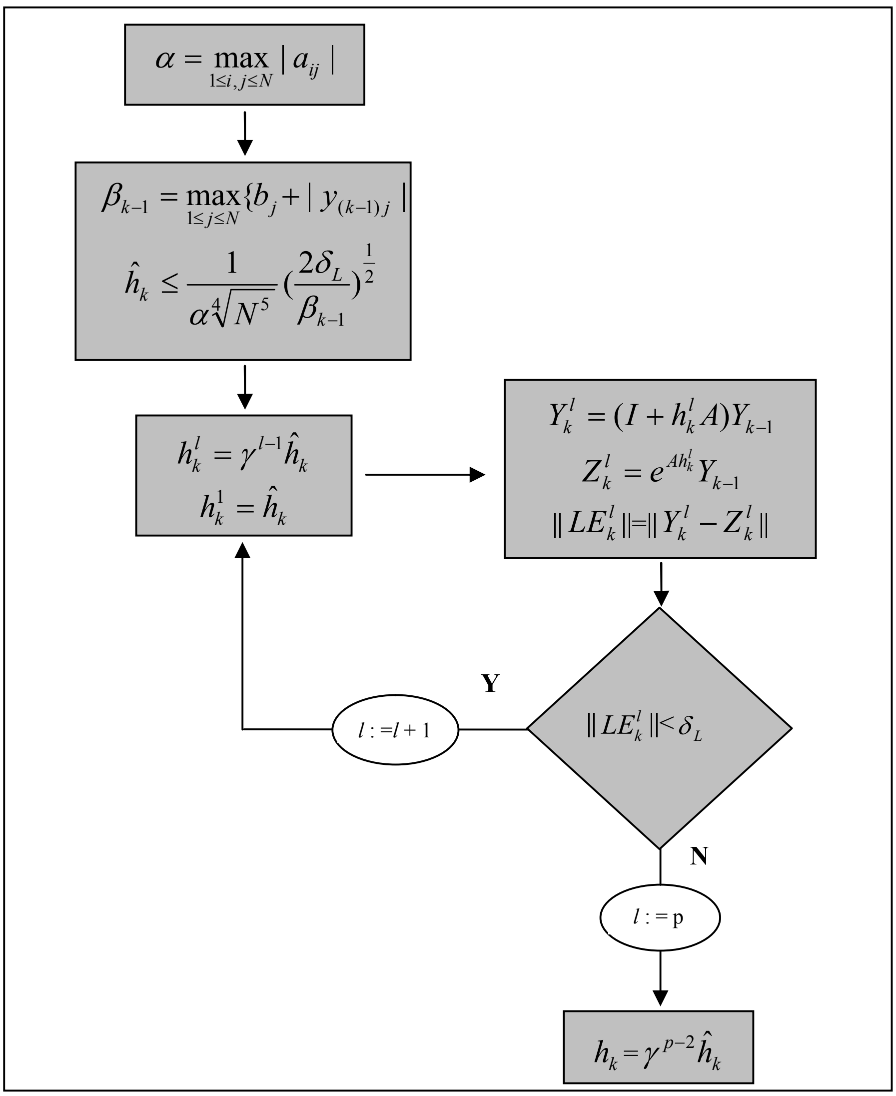

Diagram 1. Calculation of the step size with respect to modified strategy in the k-th step.

3.4. Algorithm 2

Step 0: Give the $t_0, T, b, h^*, \delta_L, X_0, A$ data.

Step 1: Calculate β_0 and α.

Step 2: Calculate β_{k-1} and $\hat{h}_{ki}$; $\hat{h}_{ki} \leq \frac{1}{\alpha\sqrt[4]{N^5}}(\frac{2\delta_L}{\beta_{k-1}})^{\frac{1}{2}}$;

Step 3: Calculate Y_{ki}, Z_{ki} and LE_{ki} for $i \geq 1$, $i \in N$;

$Y_{ki} = (1+\hat{h}_{ki}A)Y_{k-1}$, $Z_{ki} = e^{A\hat{h}_{ki}}\, Y_{k-1}$, $LE_{ki} = \| Y_{ki} - Z_{ki} \|$

Step 4: If $LE_{ki} < \delta_L$ then replace i by i+1. Calculate $h_{ki} = \gamma^{i-1}\,\hat{h}_{ki}$ and go to step 3.

Step 5: Calculate $\hat{h}_k = \gamma^{i-2}\,\hat{h}_{ki}$,

Step 6: Control $\hat{h}_k$ with K.

Step 7: Calculate $t_k = t_{k-1} + h_k$ and $Y_k = (I + h_k A)Y_{k-1}$. Replace k by k+1 and go to step 2.

Algorithm 2 is obtained from Algorithm 1 by replacing step 4 by steps 4, 5 and 6.

4. Application of Step Size Strategies to The *m*-th Order Cauchy Problem

Consider the Cauchy problem as follows:

$$x^{(m)} - a_{m-1}x^{(m-1)} - \cdots - a_1 x' - a_0 x = 0\,. \tag{4.1}$$

By taking $x = x_1$, $x' = x_2$, $x^{(m-1)} = x_m$ the equation (4.1) can be written as

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_m' \end{pmatrix} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \dots & a_{m-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_m(t_0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{m0} \end{pmatrix}$$

with initial condition $x_j(t_0) = x_{j0}$ (j = 1, 2, ..., m). Shortly, it can be given as

$$X' = CX = F(t\,,X)\,,\ X(t_0) = X_0\,. \tag{4.2}$$

Here, it is clear that the matrix C is the companion matrix. Then, the step size strategies and algorithms which are given for the numeric integration of the Cauchy problem (1.6) can be easily used for the numeric integration of m-th order Cauchy problem (4.2).

5. Numerical Results

Example 1. Consider the Cauchy problem

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.1}$$

on the region $D = \{(t\,,X) : t \in [0\,,5], | x_i - x_{i0} | \leq 5\}$. Let $h^*=10^{-12}$ and $\delta_L=10^{-1}$.

To calculate the step sizes and the numerical solutions of the Cauchy problem (5.1), the Maple procedure has been used. The results obtained from the procedure for the solution of Example 1 have been summarized in Table 1 and Table2.

Table 1. The values of h_k and $\| LE_k \|$ calculated by Algorithm 1

k	h[k]	||LE[k]||
1	0.7676298925e-1	0.48621353329606599 8e-2
2	0.7627660496e-1	0.52349220589724193 3e-2
3	0.7576630534e-1	0.56263693510811216 3e-2
4	0.7523192579e-1	0.60363494583954837 8e-2
5	0.7467341901e-1	0.64646124589766898 0e-2
⋮	⋮	⋮
151	0.1245476253e-1	0.21851532238038797 0e-1
152	0.1237374845e-1	0.21839921154790210 2e-1
153	0.7630850e-2	0.83967442631867703 8e-2

Table 2. The values of h_k and $\| LE_k \|$ calculated by Algorithm 2

k	h[k]	||LE[k]||
1	0.3206580563	0.936760210176839220e-1
2	0.2840167462	0.996658237505510110e-1
3	0.2280553189	0.833971827410726896e-1
4	0.2217416715	0.989712685239365620e-1
5	0.1953166375	0.948779191352420310e-1
⋮	⋮	⋮
66	0.2700564562e-1	0.845638340427681824e-1
67	0.2662635074e-1	0.844494222141915913e-1
68	0.2354630e-2	0.672714172473744743e-3

Figure 1 and Figure 2 illustrate the values of h_k and $\| LE_k \|$ calculated by Algorithm 1 and Algorithm 2. For the same example, when Figure 1 and Figure 2 are examined it is seen that the step sizes in Figure 2 are larger than those in Figure 1 and as a result of this, the local errors in Figure 2 are closer to the error level δ_L than those in Figure 1.

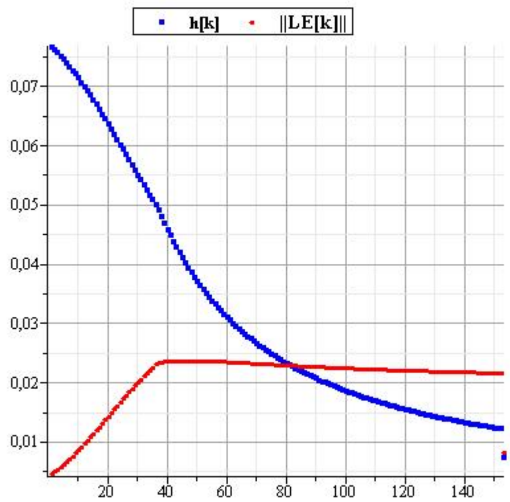

Figure 1. h_k and $\| LE_k \|$ values calculated by Algorithm 1

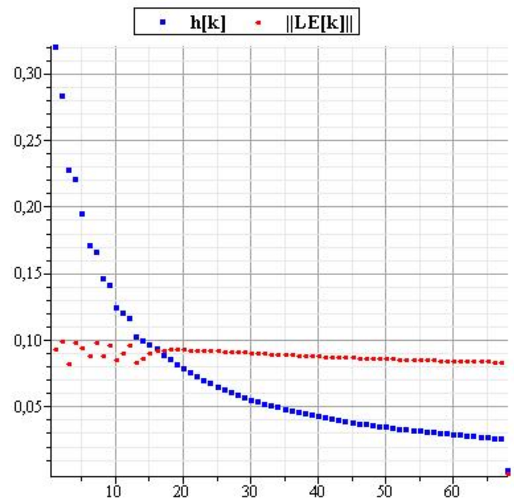

Figure 2. h_k and $\| LE_k \|$ values calculated by Algorithm 2, γ=1.1

Example 2. Consider the Cauchy problem

$$x'' - x' + 2x = 0\,,\ x(0) = 1\,,\ x'(0) = 2 \qquad (5.2)$$

on the region $D = \{(t\,, x) : t \in [0\,, 5], | x - x_0 | \le 5\}$. We can write the Cauchy problem (5.2) as the first order Cauchy problem

$$X'(t) = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} X(t)\,,\ X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \qquad (5.3)$$

Calculate the step sizes of Cauchy problem (5.2) by using Algorithm 1 and Algorithm 2 (take h^*=10^{-12} and δ_L=10^{-1}). The results have been summarized in Table 3 and Table 4.

Table 3. The values of h_k and $\| LE_k \|$ calculated by Algorithm 1

k	h[k]	||LE[k]||
1	0.3553435919e-1	0.255520075611192014e-2
2	0.3553435919e-1	0.264643518554205822e-2
3	0.3554718578e-1	0.273944928167041153e-2
4	0.3557334725e-1	0.283423858472207566e-2
5	0.3561340429e-1	0.293081941938916607e-2
⋮	⋮	⋮
187	0.1741910786e-1	0.856607155253301564e-2
188	0.1745794177e-1	0.873206199579816372e-2
189	0.3211990e-2	0.298731866088589842e-3

Table 4. The values of h_k and $\| LE_k \|$ calculated by Algorithm 2

k	h[k]	||LE[k]||
1	0.2154091358	0.993073650329621138e-1
2	0.1951026910	0.986083761942254668e-1
3	0.1824488304	0.992032623554028881e-1
4	0.1713558100	0.967589437811161440e-1
5	0.1683553928	0.998941563771873488e-1
⋮	⋮	⋮
46	0.5828613353e-1	0.990812977891800395e-1
47	0.5652879068e-1	0.997574431122475736e-1
48	0.49608525e-1	0.817337742041106264e-1

The tables above have been obtained by using Maple procedure. The results obtained from the procedure for the solution of Example 2 have been summarized in Figure 3 and Figure 4. Figure 3 and Figure 4 illustrate the values of h_k and $\| LE_k \|$ calculated by Algorithm 1 and Algorithm 2.

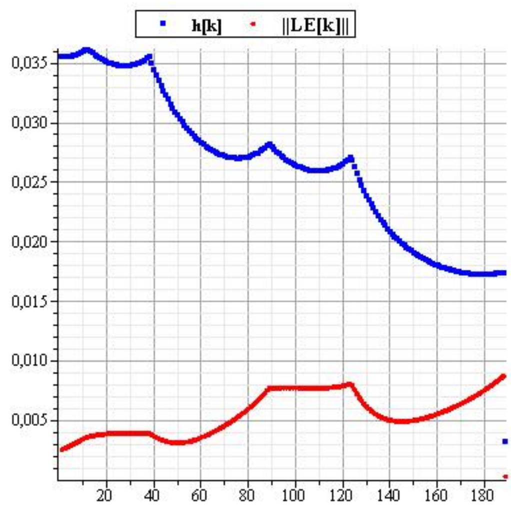

Figure 3 . h_k and $\| LE_k \|$ values calculated by Algorithm 1

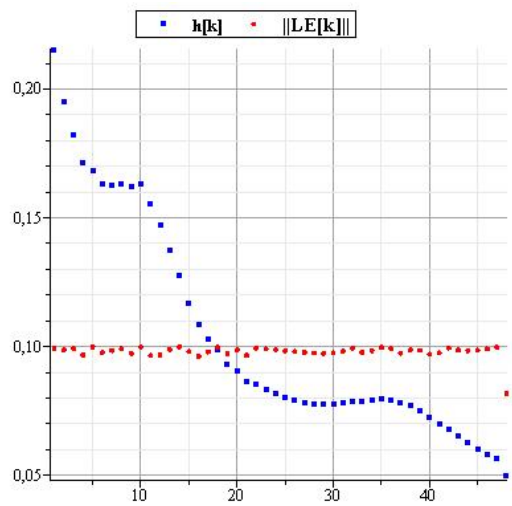

Figure 4 . h_k and $\| LE_k \|$ values calculated by Algorithm 2, γ =1.02

6. Conclusion

In this work, the new step size strategies and the new algorithms have been given for the Cauchy problem (1.6). Cauchy problem (1.6) arises in many applications such as spring-mass systems, LRC circuits and the simple pendulum. First order series and parallel chemical reactions and process control models are also usually represented by Cauchy problem (1.6). The strategies and algorithms given in this work are the generalization to systems by modifying the algorithm and strategy in [1]. The algorithms calculate the step sizes based on our strategies and the numerical solution of the Cauchy problem (1.6) such that local error $\| LE_k \| < \delta_L$ in the step k-th step, where

δ_L is the error level determined by the user. The strategies and algorithms have been applied to *m*-th order Cauchy problem. The numerical examples have also been constructed using the algorithms. The algorithms are suitable to write the computer procedure. To compute the step sizes and the numerical solutions, the Maple procedure has been used.

As it can be seen from the examples, local errors occurred in numerical solutions computed by Algorithm 1 are smaller than δ_L, so the number of the step sizes occurs more and numerical solutions are quite close to the exact solution. Local errors in numerical solutions computed by Algorithm 2 are quite close to δ_L, so the number of the step sizes occur less and numerical solutions are close enough to the exact solution. Algorithm 2 is an adaptive algorithm in this aspect.

So, if the number of the step sizes is desired to be less, Algorithm 2 is preferable. Otherwise, both Algorithm 1 and 2 can be also used.

References

[1] Çelik Kızılkan G., 2004. On the finding of step size in the numerical integration of initial value problem, Master thesis, Graduate Natural and Applied Sciences, Selcuk University, Konya (in Turkish).

[2] Ban S.J., Lee C.L., Cho H., Kim S.W., 2010. A variable step-size adaptive algorithm for direct frequency estimation, *Signal Processing*, 90: 2800-2805.

[3] Çelik Kızılkan G., Aydın K., 2005. Step Size Strategy Based on Error Analysis, *Selcuk University Science and Art Faculty Journal of Science*, 25: 79-86 (in Turkish).

[4] Çelik Kızılkan G., Aydın K., 2006. A new variable step size algorithm for Cauchy problem, *Applied Mathematics and Computation*, 183: 878-884.

[5] Gear C.W., 1971. Numerical initial value problems in ordinary differential equations, *Prentice- Hall*, New Jersey, 1971.

[6] Miranker W.L., 1981. Numerical methods for stiff equations and singular perturbation problems, *D. Rediel Publishing Company*, Holland.

[7] Shampine L.F., Allen R.C., Pruess S.,1996. Fundamentals of numerical computing, *John Wiley&Sons*, INC, New York.

[8] Heath M.T., 2002. Scientific Computing an Introductory Survey, Second edition, *McGraw-Hill*, New York.

[9] Loan C.F.V., 2000. Introduction to Scientific Computing, *Prentice Hill*, United States of America.

[10] Gu M., 1998. Stable and Efficient Algorithms for Structured Systems of Linear Equations, *SIMAX*, 19(2), 279-306.

Kemal Aydın e-mail: kaydin@selcuk.edu.tr