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# On The Jacobsthal Numbers By Matrix Method

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**Abstract:** In this paper we consider the usual Jacobsthal numbers. We investigate the identities between the Jacobsthal numbers and matrices, which are introduced for the first time in this paper. We also present a new complex sum formula.

***Key words:*** Jacobsthal number, matrix, permanent

# Matris Metoduyla Jacobsthal Sayılar Üzerine

**Özet:** Bu çalışmada, alışılmış Jacobsthal sayılarını göz önüne aldık. Jacobsthal sayıları ve bu çalışmada ilk kez tanıtılan matrisler arasındaki özdeşlikleri inceledik. Birde yeni bir karmaşık toplam formülü sunduk.

***Anahtar kelimeler:*** Jacobsthal sayı, matris, permanent

## 1. Introduction

The Fibonacci sequence is an inexhaustible source of many interesting identities. It is one of the most famous numerical sequences in mathematics and constitutes an integer sequence. If certain fruits are looked at, the number of little bumps around each ring are counted or the sand on the beach and how waves hit is watched out, the Fibonacci sequence is seen there. For more details about the Fibonacci sequence, see [1]. The same statements can easily be said for the Jacobsthal sequences. For instance, it is well-known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits,…, which are exactly the Jacobsthal numbers. At first, it is studied by Horadam in [2]. The usual Jacobsthal sequence is represented with $\{J_n\}$ and defined by the following recurrence:

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n \geq 2 \tag{1}$$

with initial conditions $J_0 = 0$ and $J_1 = 1$. Similarly, the usual Jacobsthal-Lucas sequence is represented with $\{j_n\}$ and defined by the same recurrence but initial conditions $j_0 = 2$ and $j_1 = 1$. Then the Jacobsthal and Jacobsthal-Lucas sequences are written as

$$\{J_n\}_{n\geq 0} = \{0,1,1,3,5,11,21,43,85,171,\ldots\}$$

and

$$\{j_n\}_{n\geq 0} = \{2,1,5,7,17,31,65,127,\ldots\},$$

respectively. The members of these integer sequences can also be obtained in different ways. It appears that this can be done in either of two ways: the Binet formulas or matrix method. In [2], the explicit Binet formulas of these numbers are given by Horadam as follows:

$$J_n = \frac{2^n - (-1)^n}{3} \tag{2}$$

and

$$j_n = 2^n + (-1)^n, \tag{3}$$

respectively. As a second way, they can be obtained by a generating matrix, which is called the matrix method. In [8] and [9], Koken and Bozkurt showed:

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \tag{4}$$

and

$$E^n = \begin{cases} 3^n \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}, & \textit{if } n \textit{ even} \\ 3^{n-1} \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix}, & \textit{if } n \textit{ odd} \end{cases} \tag{5}$$

where $F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$, respectively. The matrices $F$ and $E$ are called the Jacobsthal $F$-matrix and Jacobsthal-Lucas $E$-matrix, respectively. Two relationships between these matrices and these sequences are given by Köken and Bozkurt as follows [8]:

$$\begin{bmatrix} J_{n+1} \\ J_n \end{bmatrix} = F \begin{bmatrix} J_n \\ J_{n-1} \end{bmatrix} \text{ and } \begin{bmatrix} j_{n+1} \\ j_n \end{bmatrix} = F \begin{bmatrix} j_n \\ j_{n-1} \end{bmatrix}$$

More information about the above generating matrices can be found in [8-9].

There exist very miscellaneous properties of the usual Jacobsthal and Jacobsthal-Lucas numbers. In particular, their Cassini-like and sum formulas consisting of consecutive terms are very nice and quite important. The Cassini-like formulas for these numbers are given by Horadam as follows [2]:

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1} \tag{6}$$

and

$$j_{n+1}j_{n-1} - j_n^2 = 3^2(-2)^{n-1}. \tag{7}$$

Further, sums of their consecutive terms are given by Horadam as follows [2]:

$$\sum_{i=1}^{n} J_i = \frac{1}{2}(J_{n+2} - 1) \tag{8}$$

and

$$\sum_{i=1}^{n} j_i = \frac{1}{2}(j_{n+2} - 5). \tag{9}$$

The sums of odd and even terms of the Jacobsthal and Jacobsthal-Lucas sequences are investigated by Köken and Bozkurt [10].

$$\sum_{i=0}^{n} J_{2i+1} = \frac{1}{3}(2J_{2n+2} + n + 1), \tag{10}$$

$$\sum_{i=0}^{n} J_{2i} = \frac{1}{3}(J_{2n+2} - n - 1), \tag{11}$$

$$\sum_{i=0}^{n} j_{2i+1} = 2J_{2n+2} - n - 1, \tag{12}$$

and

$$\sum_{i=0}^{n} j_{2i} = J_{2n+2} + n + 1. \tag{13}$$

It is well-known that the permanent of an $n \times n$ matrix $A$ is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{14}$$

where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$. The most important applications of permanents are in the areas of physics and chemistry. The permanent of a matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

It is the aim of this article to investigate the corresponding new elementary identities associated with the classical Jacobsthal numbers by matrix method.

## 2. Main Results

In this section the Jacobsthal numbers and certain matrices are considered. Recall that in [8], authors give the Jacobsthal *F*-matrix as follows:

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}. \tag{15}$$

Let us define two new matrices such that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \tag{16}$$

and

$$D_n = \begin{bmatrix} 1 & 0 & 0 \\ J_n^+ & J_{n+1} & 2J_n \\ J_{n-1}^+ & J_n & 2J_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}(J_{n+2}-1) & J_{n+1} & 2J_n \\ \frac{1}{2}(J_{n+1}-1) & J_n & 2J_{n-1} \end{bmatrix}, \tag{17}$$

where $J_n^+$ is defined such that $J_n^+ = \sum_{i=1}^{n} J_i$. Then, we can write the following interesting result.

*Theorem 2.1*. Let $A$ and $D_n$ be $3\times 3$ matrices as in (16) and (17), respectively. Then

$$D_n = A^n. \tag{18}$$

*Proof*. Let us prove the Theorem by induction on *n*. For $n=1$, the proof is clear. For $n=2$,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

or equivalently

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}(J_4 - 1) & J_3 & 2J_2 \\ \frac{1}{2}(J_3 - 1) & J_2 & 2J_1 \end{bmatrix} = D_2$$

is obtained. Assume that for $n = k$, $A^k = D_k$. Now we must show that for $n = k+1$, $A^{k+1} = D_{k+1}$ is correct. Then

$$A^{k+1} = A^k A = \begin{bmatrix} 1 & 0 & 0 \\ J_k^+ & J_{k+1} & 2J_k \\ J_{k-1}^+ & J_k & 2J_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ J_k^+ + J_{k+1} & J_{k+2} & 2J_{k+1} \\ J_{k-1}^+ + J_k & J_{k+1} & 2J_k \end{bmatrix},$$

which completes the proof.

*Theorem 2.2.* If $J_n$ is the *n*th Jacobsthal number, then

$$J_{m+n}^+ = J_m^+ + J_{m+2} J_{n-1}^+ + J_{m+1} J_n. \tag{19}$$

*Proof.* $a_{21}$ entry of $D_{m+n}$ is equal to $J_{m+n}^+$. Computing $a_{21}$ entry of $D_m D_n$,

$$\begin{aligned} J_{m+n}^+ = J_m^+ + J_{m+1} J_n^+ + 2J_m J_{n-1}^+ &= J_m^+ + J_{m+1}(J_{n-1}^+ + J_n) + 2J_m J_{n-1}^+ \\ &= J_m^+ + J_{n-1}^+ \underbrace{(J_{m+1} + 2J_m)}_{J_{m+2}} + J_{m+1} J_n, \end{aligned}$$

as desired.

*Theorem 2.3.* For all $n \in \mathbb{Z}^+$, the following equality holds:

$$F^n = J_n F + 2J_{n-1} I, \tag{20}$$

where $F$ is defined as in (15) and $I$ is an $2 \times 2$ identity matrix.

*Proof.* By the definition of the usual Jacobsthal number, we can write

$$J_n F + 2J_{n-1} I = \begin{bmatrix} J_n & 2J_n \\ J_n & 0 \end{bmatrix} + \begin{bmatrix} 2J_{n-1} & 0 \\ 0 & 2J_{n-1} \end{bmatrix} = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

Thus the proof is completed.

*Theorem 2.4.* Let $F$ be a matrix as in (15). Then

$$perF^n = \frac{2}{3}(J_{2n+1} + (-1)^n J_{n-1}) . \tag{21}$$

*Proof.* From [8], we know that

$$F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

Using the definition of the permanent of a matrix and considering the Binet form of the usual Jacobsthal numbers, one can write

$$\begin{aligned} perF^n &= 2(J_{n-1}J_{n+1} + J_n^2) \\ &= \frac{2}{9}(2^{2n} + (-1)^n 2^{n+1} + (-1)^n 2^{n-1} + 2 + 2^{2n} - (-1)^n 2.2^n) \\ &= \frac{2}{9}(2^{2n+1} + (-1)^n 2^{n-1} + 2) \\ &= \frac{2}{3}\left[\frac{2^{2n+1} - (-1)^{2n+1}}{3} + (-1)^n \frac{2^{n-1} - (-1)^{n-1}}{3}\right] \\ &= \frac{2}{3}(J_{2n+1} + (-1)^n J_{n-1}) \end{aligned}$$

Thus the proof is completed.

*Theorem 2.5.* Let $F$ and $D_n$ be a matrices as in (15) and (17). Then

$$perD_n = perF^n . \tag{22}$$

*Proof.* From (17),

$$D_n = \begin{bmatrix} 1 & 0 & 0 \\ J_n^+ & J_{n+1} & 2J_n \\ J_{n-1}^+ & J_n & 2J_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2}(J_{n+2} - 1) & J_{n+1} & 2J_n \\ \frac{1}{2}(J_{n+1} - 1) & J_n & 2J_{n-1} \end{bmatrix}$$

is known. Then computing $perD_n$ by the Laplace expansion of the permanent with respect to the first row, we obtain

$$perD_n = perF^n + \underbrace{0 \cdot per\begin{bmatrix} J_n^+ & 2J_n \\ J_{n-1}^+ & 2J_{n-1} \end{bmatrix}}_{0} + \underbrace{0 \cdot per\begin{bmatrix} J_n^+ & J_{n+1} \\ J_{n-1}^+ & J_n \end{bmatrix}}_{0} = perF^n.$$

Thus the proof is completed.

## 3. Discussion and Conclusion

In this paper, we consider the amazing relationships between the usual Jacobsthal numbers and matrices. Five results are essentially obtained. Sum formula involving the terms of Jacobsthal numbers is one of the most important results obtained in this study.

However, we do not investigate the following research areas:

- There exists no relationship involving the Jacobsthal-Lucas numbers and matrices. One can derive similar properties with respect to them. For instance, we can define a new matrix such that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 2 \\ 0 & 1 & 4 \end{bmatrix}. \tag{23}$$

Then what is the result for consecutive power of the matrix $B$? Are there any identities for the matrix $B$ similar to identities derived for the matrix $A$?

- We do not search determinantal identities of the matrices $A$ and $D_n$. Also the following statements can be investigated:

$$\det D_n = ?, \tag{24}$$

$$\det B^n = ?, \tag{25}$$

and

$$perB^n = ?. \tag{26}$$

- Also, determinantal and permanental identities between the matrices $B^n$ and $E$ can be investigated.

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