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A Note On The Quasi-Conformal And M-Projective Curvature Tensor Of A Semi-Symmetric Recurrent Metric Connection On A Riemannian Manifold

Rajesh Kumar[1,*] and Jagannath Chowdhury[1]

[1]*Department of Mathematics, Pachhuanga University college, Aizawl-796001, Mizoram, India.*
[*]*Correspending author e-mail: rajesh_mzu@yahoo.com*

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Abstract: In the present note we have considered M^n to be a Riemannian manifold admitting a semi-symmetric recurrent metric connection. The aim of the present paper is to obtain the conditions under which the quasi-conformal curvature tensor and M-projective curvature tensor of semi-symmetric recurrent metric connection and the Riemannian connection to be equal.

Key words: Semi-symmetric recurrent metric connection, quasi-conformal curvature tensor, M-projective curvature tensor.

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1. Introduction

Let M^n be an n-dimensional Riemannian manifold with Riemannian metric g and Levi-Civita connection ∇. An affine connection $\overline{\nabla}$ on a Riemannian manifold is called a recurrent metric connection [5] if there exist a differentiable 1-form μ on M^n such that

$$(\overline{\nabla}_X g)(Y,Z) = \mu(X)g(Y,Z)$$

holds for all differentiable vector fields $X, Y, Z\ldots$ on M^n, μ is called the 1-form of recurrence.

If, further, the torsion tensor $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$ of the connection $\overline{\nabla}$ is of the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y$$

where π is a differential 1-form on M^n, then $\overline{\nabla}$ is called a semi-symmetric recurrent metric connection on M^n.

[4] have defined a connection of the form

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(Y)X - g(X,Y) - \tfrac{1}{2}\mu(X)Y \tag{1.1}$$

where α is a differentiable 1-form given by

$$\alpha(X) = \pi(X) - \tfrac{1}{2}\mu(X), \tag{1.2}$$

and A is a differentiable vector field satisfying

$$g(A,X) = \alpha(X). \tag{1.3}$$

For the connection (1.1) it has proved that

$$(\overline{\nabla}_X g)(Y,Z) = \mu(X)g(Y,Z) \tag{1.4}$$

so, the connection $\overline{\nabla}$ on a Riemannian manifold is called a recurrent metric connection [5].

Further the torsion tensor $\overline{T}(X,Y)$ for the connection $\overline{\nabla}$ gives

$$\bar{T}(X,Y) = \pi(Y)X - \pi(X)Y$$

then $\overline{\nabla}$ defined in (1.1) is called a semi-symmetric recurrent metric connection.

2. Curvature Tensor.

The curvature tensor $\bar{R}(X,Y)Z$ of M^n with respect to the semi-symmetric recurrent metric connection $\overline{\nabla}$ is defined as

$$\bar{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z \tag{2.1}$$

From (1.1) and (2.1), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z - \lambda(Y,Z)X + \lambda(X,Z)Y - g(Y,Z)LX$$
$$+g(X,Z)LY - d\mu(X,Y)Z \tag{2.2}$$

where

$$\lambda(Y,Z) = (\nabla_Y\alpha)(Z) - \alpha(Y)\alpha(Z) + \tfrac{1}{2}\alpha(A)g(Y,Z) \tag{2.3}$$

$$LY = \nabla_Y A - \alpha(Y)A + \tfrac{1}{2}\alpha(A) \tag{2.4}$$

and $R(X,Y)Z$ is the Riemannian curvature tensor for the connection ∇ [2].

Again, if $\bar{S}$ is the Ricci tensor of M^n with respect to the semi-symmetric recurrent metric connection $\overline{\nabla}$ and $S(Y,Z)$ is the Ricci tensor of connection ∇,

then from (2.2) ,we have

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)\lambda(Y,Z) - \text{trace }(\lambda)\, g(Y,Z) + d\mu(Y,Z) \tag{2.5}$$

and
$$\bar{Q}Y = QY - (n-2)LY - trace(\lambda)\,Y\ , \tag{2.6}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric recurrent metric connection defined by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$.

Also the scalar curvature is given by

$$\bar{r} = r - 2(n-1)trace(\lambda) \tag{2.7}$$

where r is the scalar curvature of the manifold and $\bar{r}$ is the scalar curvature of the manifold with respect to the semi-symmetric recurrent metric connection.

3. Quasi-Conformal Curvature Tensor

For an n-dimensional Riemannian manifold, the quasi-conformal curvature tensor $C(X,Y)Z$ is given by

$$C(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X.Z)Y] \tag{3.1}$$

where a and b are non zero arbitrary constants and r is the scalar curvature of the manifold. The notion of quasi-conformal curvature tensor was introduced by [6]. If $a = 1$ and $b = -\frac{1}{n-2}$, then quasi-conformal curvature tensor reduces to conformal curvature tensor [1].

A quasi-conformal curvature tensor $\bar{C}(X,Y)Z$ with respect to a semi-symmetric recurrent metric connection $\overline{\nabla}$ in an n-dimensional Riemannian manifold is defined by

$$\bar{C}(X,Y)Z = a\bar{R}(X,Y)Z + b[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] - \frac{\bar{r}}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y]. \tag{3.2}$$

By using (2.2), (2.5), (2.6), (2.7) and (3.1) in (3.2) , we have

$$\bar{C}(X,Y)Z = C(X,Y)Z - \{a + b(n-2)\}[\lambda(y,Z)X - \lambda(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] - a\, d\mu(X,Y)Z - \left[2b - \frac{2(n-1)}{n}\left(\frac{a}{n-1} + 2b\right)\right] trace(\lambda)[g(Y,Z)X - g(X,Z)Y] + b[d\mu(Y,Z)X - d\mu(X,Z)Y]. \tag{3.3}$$

If $\bar{C}(X,Y)Z = C(X,Y)Z$, then from (3.3), we have

$$\{a + b(n-2)\}[\lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] + a\, d\mu(X,Y)Z + \left[2b - \frac{2(n-1)}{n}\left(\frac{a}{n-1} + 2b\right)\right] trace(\lambda) [g(Y,Z)X - g(X,Z)Y] - b[d\mu(Y,Z)X - d\mu(X,Z)Y] = 0. \tag{3.4}$$

Taking scalar product with respect to Z in (3.4), we have

$$(an + 2b)\, d\mu(X,Y) = 0,$$

which gives $d\mu(X,Y) = 0$, provided that $(an + 2b) \neq 0$.

that is, μ is closed 1-form.

Hence we can state the following :

Theorem 3.1. A necessary condition for the quasi-conformal curvature tensor of a semi-symmetric recurrent metric connection be equal to the quasi-conformal curvature tensor of the Riemannian manifold is that the differential 1-form μ defining the recurrence is closed, that is $d\mu = 0$, *provided that* $(an + 2b) \neq 0$.

Again, let μ is given by

$$\mu = 2\pi \text{ and 1-form } \pi \text{ is closed,} \tag{3.5}$$

then, from (1.2), (2.2), (2.4), (2.5), (2.6) and (3.5), we find that

$$\bar{R}(X,Y)Z = R(X,Y)Z \quad \text{and} \quad \bar{S}(Y,Z) = S(Y,Z)$$

and consequently from (3.3), we have

$$\bar{C}(X,Y,Z) = C(X,Y,Z).$$

Hence, we have the following theorem:

Theorem 3.2. The sufficient condition for the equality of the quasi-conformal curvature tensor of a semi-symmetric recurrent metric connection and the Riemannian connection on a Riemannian manifold are that the relation

$\mu = 2\pi$ *and* *1-form* π *is closed*

hold good.

4. M-Projective Curvature Tensor

In the paper [3] defined a tensor M-projective curvature tensor $M(X,Y)Z$ on a Riemannian manifold as

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)} \;[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{4.1}$$

M-projective curvature tensor for the connection $\bar{\nabla}$ is given by

$$\bar{M}(X,Y,Z) = \bar{R}(X,Y,Z) - \frac{1}{2(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \tag{4.2}$$

By using (2.2), (2.5), (2.6), (2.7) and (4.1) in (4.2), we get

$$\bar{M}(X,Y)Z = M(X,Y)Z - \frac{n}{2(n-1)}[\lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] - \frac{trace(\lambda)}{n-1}[g(X,Z)Y - g(Y,Z)X] - \frac{1}{2(n-1)}[d\mu(Y,Z)X - d\mu(X,Z)Y] \tag{4.3}$$

If $\bar{M}(X,Y,Z) = M(X,Y,Z)$, then from (4.3), we get

$$n[\lambda(Y,Z)X - \lambda(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] + 2\,trace(\lambda)[g(X,Z)Y - g(Y,Z)X] + [d\mu(Y,Z)X - d\mu(X,Z)Y] = 0. \tag{4.4}$$

Contracting with respect to Z, we have

$$n[\lambda(Y,X) - \lambda(X,Y) + g(Y,LX) - g(X,LY)] + 2\,trace(\lambda)[g(X,Y) - g(Y,X)] + [d\mu(Y,X) - d\mu(X,Y)] = 0$$

$$\Rightarrow d\mu(X,Y) = d\mu(Y,X). \tag{4.5}$$

Hence, we have the following:

Theorem 4.1. A necessary condition for the M-projective curvature tensor of a semi-symmetric recurrent metric connection be equal to M-projective curvature tensor of the Riemannian manifold is that the differential 1-form μ *defining the recurrence is symmetric.*

Again, let μ is given by

$$\mu = 2\pi \quad \text{and} \quad \text{1-form } \pi \text{ is closed,} \tag{4.6}$$

then from (1.2) , (2.2), (2.4), (2.5), (2.6) and (4.6), we find that

$$\bar{R}(X,Y)Z = R(X,Y)Z \quad \text{and} \quad \bar{S}(Y,Z) = S(Y,Z)$$

and consequently from (4.3),we have

$$\bar{M}(X,Y,Z) = M(X,Y,Z).$$

Hence, we have the following theorem:

Theorem 4.2. The sufficient condition for the equality of the M-projective curvature tensor of a semi-symmetric recurrent metric connection and the Riemannian connection on a Riemannian manifold are that the relation

$$\mu = 2\pi \quad \textit{and} \quad \textit{1-form } \pi \textit{ is closed}$$

hold good.

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Jagannath Chowdhury email: jagai_76@yahoo.com