The New Travelling Wave Solutions of Time Fractional Fitzhugh-Nagumo Equation with Sine-Gordon Expansion Method

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Abstract

Authors aimed to employ the sine-Gordon expansion method to acquire the new exact solutions of fractional Fitzhugh-Nagumo equation which is a stripped type of the Hodgkin-Huxley model that expresses in extensive way activation and deactivation dynamics of neuron spiking. By using the wave transformations, by the practicality of chain rule and applicability of the conformable fractional derivative, the fractional nonlinear partial differential equation (FNPDE) changes to a nonlinear ordinary differential equation. So the exact solution of the considered equation can be obtained correctly with the aid of efficient and reliable analytical techniques.

Keywords: Sine-Gordon expansion method; Fitzhugh-Nagumo Equation; Conformable derivative.

Zaman Kesirli Fitzhugh-Nagumo Denkleminin Sine-Gordon Açılım Yöntemi İle Yeni Yürüyen Dalga Çözümleri

Öz

Yazarlar, nöron artışının, geniş bir aktivasyon ve deaktivasyon dinamiğini ifade eden Hodgkin-Huxley modelinin sade bir tipi olan kesirli Fitzhugh-Nagumo denkleminin yeni tam çözümlerini elde etmek için sine-Gordon açılım yöntemini kullanmayı amaçladılar. Dalga dönüşümleri, zincir kuralının pratikliği ve conformable kesirli türevin uygulanabilirliği kullanılarak, lineer olmayan kesirli mertebeden kısmi diferansiyel denklem, lineer olmayan adi diferansiyel denkleme dönüşür. Böylece, ele alınan denklemin tam çözümü etkin ve güvenli analitik tekniklerin yardımı ile doğru bir biçimde elde edilebilir.

Anahtar Kelimeler: Sine-Gordon açılım yöntemi; Fitzhugh-Nagumo denklemi; Conformable türev.

1. Introduction

Fractional calculus that is gone by the name of as arbitrary order differentiation and integration is as old as known calculus. At the beginning fractional derivation was seen as a great paradox in L'Hospital's letter to Liouville who asked firstly about arbitrary order derivation. So the survey of the fractional calculus has been started. In the last decades many scientists dealt with this subject to explain complexity and nonlinearity of the natural events with more accuracy. But they need some tools for modeling these natural phenomenons. So they expressed some definitions as a tool for modeling physical, engineering and social problems mathematically. Especially Grunwald-Letnikov, Riemann-Liouville and Caputo type fractional order derivation and integration definitions are the most popular ones as modeling tools [1-3]. But recently scientists recognized some deficiencies of these definitions. For instance Riemann-Liouville fractional order derivative of a constant is not zero. Both Riemann-Liouville and Caputo derivative definitions do not satisfy the chain rule. Also these definitions do not satisfy the rule for derivative of quotient of two functions and derivative of product of two functions. To annihilate these deficiencies recently Khalil et al. [4] expressed a new definition called "conformable fractional derivative and integral".

Definition 1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a function. The αth order *"conformable fractional derivative"* (CFD) of f is expressed by,

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \alpha \in (0,1)$ [4].

Definition 2. Let f be α-differentiable over some $(0,a), a > 0$ and $\lim\limits_{t \to 0^+} f^{(\alpha)}(t)$ subsists then $f^{(\alpha)}(0) = \lim\limits_{t \to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a \geq 0$ is defined as:

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where considered integral is by means of Riemann improper integral, also, $\alpha \in (0,1]$ [4].

Basic properties of CFD are declared in [4, 5]. The conformable fractional calculus aroused scientist's great interest and huge amount of implementation of this derivative is visible in the literature [6-9]. Korkmaz [10] constructed the solutions of the conformable fractional RLW-Burgers, potential KdV, and CRWP equations by using the modified Kudryashov method. Rosales et al. [11] used the conformable Laplace transform to get the solution of the classical Drude model. Srivastava et al. [12] employed the conformable DTM to evaluate an numerical solution of the SIR epidemic model of childhood disease. Sabiu et al. [13] evaluated analytical solution for the conformable fractional (3+1) dimensional mKdV equation.

In this article authors established the traveling wave solutions of time fractional Fitzhugh-Nagumo Equation

$$D_t^\alpha u - D_x^2 u = u(u-s)(1-u), \tag{1}$$

which arises in nonlinear reaction-diffusion, the transmission of nerve impulses models [14, 15], circuit theory and the area of population genetics [16] as mathematical models. In Eqn. (1) $\alpha \in (0,1)$ and $D_t^\alpha u$ means CFD of function $u(x,t)$.

2. A Brief Description of Implemented Method

The sine-Gordon expansion method is represented briefly in this section. Regarded method relying on the sine-Gordon equation [17] and complex wave transformation [18].

Regard the sine-Gordon equation [19];

$$u_{xx} - u_{tt} = v^2 \sin(u), \tag{2}$$

considering $u = u(x,t)$ and v is constant. Supposing the transformation $u = u(x,t) = U(\varrho)$, $\varrho = \mu(x - ct)$ in the Eqn. (2) led to,

$$U'' = \frac{v^2}{\eta^2(1-c^2)} \sin(U).$$ (3)

By regulating the Eqn. (3), it turns into

$$\left[\left(\frac{U}{2} \right)' \right]^2 = \frac{v^2}{\eta^2(1-c^2)} \sin^2 \left(\frac{U}{2} \right) + W,$$ (4)

where W is a constant. Choosing $W = 0$, $\varphi(\varrho) = \dfrac{U}{2}$, $o^2 = \dfrac{v^2}{\eta^2(1-c^2)}$ and replacing in Eqn. (4), we get,

$$\varphi' = o \sin(\varphi),$$ (5)

also establishing $o = 1$ in Eqn. (5), yields

$$\varphi' = \sin(\varphi).$$ (6)

The solution of (6) can be evaluated by the method of separation of variables, we have,

$$\sin(\varphi) = \sin(\varphi(\varrho)) = \frac{2\vartheta e^{\varrho}}{\vartheta^2 e^{2\varrho} + 1} \big|_{\vartheta=1} = \operatorname{sech}(\varrho),$$ (7)

$$\cos(\varphi) = \sin(\varphi(\varrho)) = \frac{\vartheta^2 e^{2\varrho} - 1}{\vartheta^2 e^{2\varrho} + 1} \big|_{\vartheta=1} = \tanh(\varrho)$$ (8)

regarding ϑ is constant. For obtaining following conformable PDE's solution;

$$G(u, D_t^\alpha u, D_x u, D_{xx} u, D_t^\alpha D_t^\alpha u, ...),$$ (9)

we use the wave transform $\varrho = kx + w\dfrac{t^\alpha}{\alpha}$. So the time fractional nonlinear PDE turns into an integer order ODE

$$G(U, U'(\varrho), U''(\varrho), U'''(\varrho), ...),$$ (10)

Suppose that the solution of Eqn. (10)

$$U(\varrho) = \sum_{i=1}^{n} \tanh^{i-1}(\varrho)\left[B_i \sec h(\varrho) + A_i \tanh(\varrho)\right] + A_0, \tag{11}$$

then due to Eqn. (7) and Eqn. (8), Eqn. (11) can be reorganized as

$$U(\varphi) = \sum_{i=1}^{n} \cos^{i-1}(\varphi)\left[B_i \sin(\varphi) + A_i \cos(\varphi)\right] + A_0. \tag{12}$$

In the Eqn. (10) using the balancing principle n can be examined. Equating the coefficients of $\cos^i(\varphi)$ and $\sin^i(\varphi)$ to zero yields a system. Solving obtained system by using software Mathematica the values of A_i, B_i, k and w can be derived. At the end substituting all the obtained the results for A_i, B_i, η and c in Eqn. (11), the solutions can be derived.

3. Wave Solutions of Time Fractional Fitzhugh-Nagumo Equation

We employ above expressed method to acquire the traveling wave solutions of fractional Fitzhugh-Nagumo equation (1). Regarding the wave transform and chain rule [5] to the Eqn. (1) led to

$$wU' - k^2 U'' = U(U - s)(1 - U), \tag{13}$$

where the prime defines derivative of function U with respect to ϱ. Applying the balancing procedure between the terms U'' and U^3 gives $n = 1$. Suppose that the solution of Eqn. (12) is as follows

$$U = B\sin(\varrho) + A\cos(\varrho) + C. \tag{14}$$

Subrogating Eqn. (14) into Eqn. (13) and equating all the coefficients of $\cos^i(\varrho)$ and $\sin^i(\varrho)$ to be zero, we get the following solution sets

Set 1:

$$A = \frac{s-1}{2}, B = \frac{1}{2}\sqrt{-s^2 + 2s - 1}, C = \frac{s+1}{2}, k = \frac{s-1}{\sqrt{2}}, w = \frac{1}{2}(s-1)(s+1), \tag{15}$$

Set 2:

$$A = -\frac{1}{2}, B = 0, C = \frac{1}{2}, k = -\frac{1}{2\sqrt{2}}, w = -\frac{1}{4}(1 - 2s), \tag{16}$$

Set 3:

$$A = -\frac{s}{2}, B = 0, C = \frac{s}{2}, k = -\frac{s}{2\sqrt{2}}, w = -\frac{1}{4}(s-2)s, \tag{17}$$

Set 4:

$$A = \frac{1-s}{2}, B = 0, C = \frac{s+1}{2}, k = -\frac{s-1}{2\sqrt{2}}, w = -\frac{1}{4}(s^2-1). \tag{18}$$

Based on these solution sets, we acquire the following solutions respectively,

$$u_1(x,t) = \frac{1}{2}(s-1)\tanh\left(\frac{(s^2-1)t^\alpha}{2\alpha} + \frac{(s-1)x}{\sqrt{2}}\right)$$
$$+ \frac{1}{2}\sqrt{-s^2+2s-1}\,\text{sech}\left(\frac{(s^2-1)t^\alpha}{2\alpha} + \frac{(s-1)x}{\sqrt{2}}\right) + \frac{s+1}{2},$$

$$u_2(x,t) = \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}} - \frac{(2s-1)t^\alpha}{4\alpha}\right) + \frac{1}{2},$$

$$u_3(x,t) = \frac{1}{2}s\tanh\left(\frac{(s-2)st^\alpha}{4\alpha} + \frac{sx}{2\sqrt{2}}\right) + \frac{s}{2},$$

$$u_4(x,t) = \frac{s+1}{2} - \frac{1}{2}(1-s)\tanh\left(\frac{(s-1)x}{2\sqrt{2}} - \frac{(1-s^2)t^\alpha}{4\alpha}\right).$$

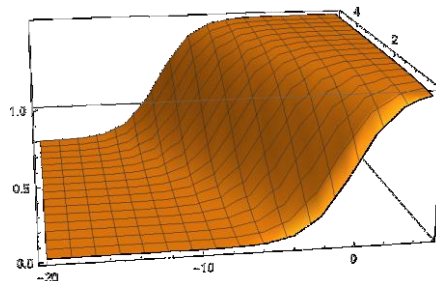The 3D graphics of the obtained solutions are given as follows

Figure 1: The 3D graphical illustration of $u_2(x,t)$ for $s = -1$, $\alpha = 0.9$
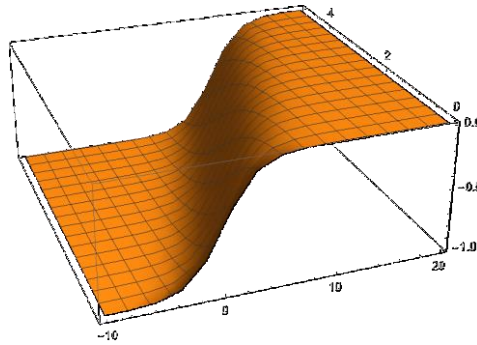
Figure 2: The 3D graphical representation of $u_3(x,t)$ for $s = -1$, $\alpha = 0.9$
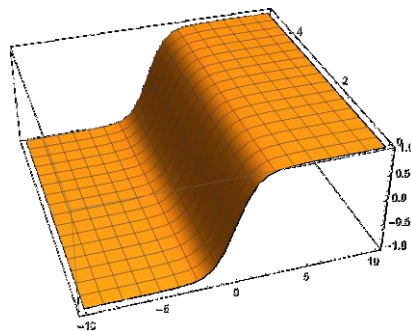

Figure 3: The 3D graphics of $u_4(x,t)$ for $s = -1$, $\alpha = 0.9$

4. Conclusions

In this study authors obtained the new traveling wave solutions of time fractional Fitzhugh-Nagumo equation with the aid of sine-Gordon expansion method. Obtained results show that the method is applicable, accurate and efficient for acquiring the analytical solutions of FPDEs in conformable sense. Also 3D graphical representations are given for the obtained results. Although authors used the advantages of CFD. In other fractional derivative definitions, the considered equations can't be turned into ODE with integer order by using wave transform. Also all the solutions are verified by putting them into the considered equation.

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