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Kuratowski Theorems in Soft Topology

Müge ÇERÇİ[1], Gültekin SOYLU[2, *]

[1]*Haliç University, Department of Mathematics, İstanbul, Turkey*
mugecerci@halic.edu.tr, ORCID: 0000-0001-9018-463X
[2]*Akdeniz University, Department of Mathematics, Antalya, Turkey*
gsoylu@akdeniz.edu.tr, ORCID: 0000-0003-2640-5690

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Abstract

This paper deals with the soft topological counterparts of concepts introduced by Kuratowski. First the closure operator is investigated in the soft topological setting and afterwards the Kuratowski Closure-Complement Theorem is stated and proved.

Keywords: Soft topology; Kuratowski closure operator; Kuratowski Closure-Complement Theorem.

Soft Topolojideki Kuratowski Teoremleri

Öz

Bu çalışmada Kuratowski tarafından ortaya konan bazı topolojik kavramların Soft topolojideki karşılıkları ele alınmıştır. Öncelikle kapanış operatörü soft topolojide tanımlanmış ve incelemesi yapılmış daha sonra Kuratowski Kapanış-Tümleyen Teoremi ifade edilmiş ve kanıtlanmıştır.

Anahtar Kelimeler: Soft topoloji; Kuratowski kapanış operatörü; Kuratowski Kapanış-Tümleyen Teoremi.

* Corresponding Author
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1. Introduction

Various theories have been proposed with the purpose of dealing with different types of uncertainties. Besides to probability theory the most known ones are the theory of fuzzy sets [1], the theory of vague sets [2], the theory of rough sets [3]. Nevertheless not suprisingly all these theories have their own drawbacks. In 1999, Molodtsov [4] introduced the notion of soft set theory claiming to overcome the drawbacks of the theories mentioned above. Molodtsov proposed applications of this new tool in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement. After the introduction of soft sets, several researchers started to extend the theory in different paths. In 2003, Maji et al [5] defined and studied several fundamental notions of soft set theory. The outcome of soft set theory in algebraic structures was introduced by Aktaş and Çağman [6]. They not only defined the notion of soft groups but also obtained most of their basic properties. In 2011, Shabir and Naz [7] brought to light the idea of soft topological spaces. They interestingly observed that a soft topological space is actually a parameterized family of topological spaces.

This paper participates to all those discussions in the direction of soft topological spaces. First the concept of Kuratowski closure operator is introduced in the soft topological setting. The correspondence of a closure operator with a soft topology is given. In addition the well-known Kuratowski Closure-Complement Theorem is stated and proved for soft topological spaces.

2. Preliminaries

In this section, the fundamental definitions and results of soft set theory and its topology are presented. They may be found in earlier studies [4 , 5, 7-10].

Definition 1. Let U denote the universe of discourse and E be a set of parameters. Let $\wp(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over $U,$ where F is a mapping given by $F: A \longrightarrow \wp(U)$.

Basically, a soft set over U is a parametrized family of subsets of the universe U. In the sequel U will allways be the universe of discourse and E the set of parameters unless stated otherwise.

For the sake of simplicity we hereafter will suppose any given soft set (F, A) over U with parameters set E is extended as following:

$F': E \to \wp(U), F'(e) = \begin{cases} F(e) & , if e \in A \\ \varnothing & , if e \in E - A \end{cases}$. By this extension we will denote any given soft set (F, A) by (F, E) just by replacing F' with F.

Definition 2. Let U be an initial universe set and E be an universe set of parameters. Let (F, E) and (G, E) be soft sets over a common universe set U. Then (F, E) is a soft subset of (G, E), denoted by $(F, E) \tilde{\subset} (G, E)$, if for all $e \in E$, $F(e) \subset G(e)$.

(F, E) is called a soft super set of (G, E), if $(G, E)$is a soft subset of (F, E). We denote it by $(F, E) \tilde{\supset} (G, E)$.

Definition 3. Two soft set (F, E) and (G, E) over a common universe U are said to be soft equal if, (F, E) is a soft subset of (G, E) and (G, E) is a soft subset of (F, E).

Definition 4. A soft set (F, E) over U is said to be the empty soft set denoted by Φ_E if for all $e \in E$, $F(e) = \emptyset$.

Definition 5. A soft set (F, E) over U is said to be an absolute soft set denoted by U_E if for all $e \in E$, $F(e) = U$.

Clearly $U_E^c = \Phi_E$ and $\Phi_E^c = U_E$.

Definition 6. The union (H, E) of two soft sets (F, E) and (G, E) over the common universe U, denoted $(F, E) \tilde{\cup} (G, E)$, is defined as $H(e) = F(e) \cup G(e)$, for all $e \in E$.

Definition 7. The intersection (H, E) of two soft sets (F, E) and (G, E) over the common universe U, denoted $(F, E) \tilde{\cap} (G, E)$, is defined as $H(e) = F(e) \cap G(e)$, for all $e \in E$.

Definition 8. The complement of a soft set (F, E) is denoted by $(F, E)^c$ and is defined by $(F, E)^c := (F^c, E)$, where $F^c: E \longrightarrow \wp(U)$ is a mapping given by $F^c(e) = U - F(e)$, for all $e \in E$.

Proposition 9. Let (F, E) and (G, E) be the soft sets over U. Then

i) $\left((F, E) \tilde{\cup} (G, E)\right)^c = (F, E)^c \tilde{\cap} (G, E)^c$,

ii) $\left((F, E) \tilde{\cap} (G, E)\right)^c = (F, E)^c \tilde{\cup} (G, E)^c$.

Definition 10. Let U be an initial universe and E be the non-empty set of parameters. The difference (H, E) of two soft sets (F, E) and (G, E) over U, denoted by $(F, E)\tilde{\setminus}(G, E)$, is defined as $H(e) = F(e)\setminus G(e)$, for all $e \in E$.

Definition 11. Let (F, E) be a soft set over U and $u \in U$. We say that $u \tilde{\in} (F, E)$ read as u belongs to the soft set (F, E) whenever $u \in F(e)$, for all $u \in E$.

Note that for any $u \in U$, $u \tilde{\notin} (F, E)$ if $u \notin F(e)$, for some $e \in E$.

Definition 12. Let Y be a non-empty subset of U, then Y_E denotes the soft set (Y, E) over U for which $Y(e) = Y$, for all $e \in E$.

Definition 13. Let $u \in U$, then (u, E) denotes the soft set over U for which $u(e) = \{u\}$, for all $e \in E$.

Definition 14. Let τ be a collection of soft sets over U, then τ is said to be a soft topology on U if

T1) Φ_E,U_E belong to τ,

T2) The union of any number of soft sets in τ belongs to τ,

T3) The intersection of any two soft sets in τ belongs to τ.

The triplet (U, τ, E) is called a soft topological space over U.

Definition 15. Let (U, τ, E) be a soft topological space over U, then the members of τ are said to be soft open sets in U.

Definition 16. Let (U, τ, E) be a soft topological space over U. A soft set (F, E) over U is said to be a soft closed set in U, if its complement $(F, E)^c$ belongs to τ.

Proposition 17. Let (U, τ, E) be a soft topological space over U and E be the non-empty set of parameters. Then

i) Φ_E,U_E are closed soft sets over U,

ii) The intersection of any number of soft closed sets is a soft closed set over U,

iii) The union of any two soft closed sets is a soft closed set over U.

Definition 18. Let (U, τ, E) be a soft space over U and (F, E) be a soft set over U. Then the soft closure of (F, E), denoted by $\overline{(F, E)}$ is the intersection of all soft closed super sets of (F, E).

Clearly $\overline{(F, E)}$ is the smallest soft closed set over U which contains (F, E).

Theorem 19. Let (U, τ, E) be soft topological space over U and (F, E), (G, E) are soft sets over U. Then the following hold:

i) $\Phi_E = \overline{\Phi_E}$ and $U_E = \overline{U_E}$,

ii) $(F, E) \tilde{\subseteq} \overline{(F, E)}$,

iii) (F, E) is a soft closed set if and only if $(F, E) = \overline{(F, E)}$,

iv) $\overline{\overline{(F, E)}} = \overline{(F, E)}$,

v) $(F, E) \tilde{\subseteq} (G, E)$ implies $\overline{(F, E)} \tilde{\subseteq} \overline{(G, E)}$,

vi) $\overline{(F, E) \tilde{\cup} (G, E)} = \overline{(F, E)} \tilde{\cup} \overline{(G, E)}$,

vii) $\overline{(F, E) \tilde{\cap} (G, E)} \tilde{\subseteq} \overline{(F, E)} \tilde{\cap} \overline{(G, E)}$.

Definition 20. Let (U, τ, E) be a soft topological space over U then soft interior of soft set (F, E) over U is denoted by $(F, E)^{\circ}$ and is defined as the union of all soft open sets contained in (F, E).

Thus $(F, E)^{\circ}$ is the largest soft open set contained in (F, E).

Theorem 21. Let (U, τ, E) be a soft topological space over U and (F, E), (G, E) are soft sets over U. Then the followings hold:

i) $\Phi_E^{\circ} = \Phi_E$ and $U_E^{\circ} = U_E$,

ii) $(F, E)^{\circ} \tilde{\subseteq} (F, E)$,

iii) $((F, E)^{\circ})^{\circ} = (F, E)^{\circ}$,

iv) (F, E) is a soft open set if and only if $(F, E)^{\circ} = (F, E)$,

v) $(F, E) \tilde{\subseteq} (G, E)$ implies $(F, E)^{\circ} \tilde{\subseteq} (G, E)^{\circ}$,

vi) $((F,E)\,\tilde{\cap}\,(G,E))^{\circ} = (F,E)^{\circ}\,\tilde{\cap}\,(G,E)^{\circ}$,

vii) $(F,E)^{\circ}\,\tilde{\cup}\,(G,E)^{\circ} \tilde{\subseteq} ((F,E)\,\tilde{\cup}\,(G,E))^{\circ}$.

Theorem 22. Let (F,E) be a soft set of soft topological space over U. Then

i) $((F,E)^{c})^{\circ} = \left(\overline{(F,E)}\right)^{c}$,

ii) $\overline{((F,E)^{c})} = ((F,E)^{\circ})^{c}$,

iii) $(F,E)^{\circ} = \left(\overline{((F,E)^{c})}\right)^{c}$.

Definition 23. Let (U,τ,E) be a soft topological space over U then the soft boundary of soft set (F,E) over U is denoted by $\underline{(F,E)}$ and defined as $\underline{(F,E)} = \overline{(F,E)}\,\tilde{\cap}\,\overline{((F,E)^{c})}$.

Remark 24. From the above definition it follows directly that the soft sets (F,E) and $(F,E)^{c}$ have same soft boundary.

Theorem 25. Let (F,E) be a soft set of soft topological space over U. Then the followings hold:

(1) $(\underline{(F,E)})^{c} = (F,E)^{\circ}\,\tilde{\cup}\,((F,E)^{c})^{\circ} = (F,E)^{\circ}\,\tilde{\cup}\,(F,E)_{\circ}$

(2) $\overline{(F,E)} = (F,E)^{\circ}\,\tilde{\cup}\,\underline{(F,E)}$

(3) $\underline{(F,E)} = \overline{(F,E)}\,\tilde{\cap}\,\overline{((F,E)^{c})} = \overline{(F,E)}\tilde{\setminus}(F,E)^{\circ}$

(4) $(F,E)^{\circ} = (F,E)\tilde{\setminus}\underline{(F,E)}$.

3. Soft Kuratowski Closure Operator

Theorem 26. Let (U,τ,E) be a soft topological space. The operator $\varphi: \wp(U_E) \to \wp(U_E)$, defined by $\tilde{\varphi}((F,E)) = \overline{(F,E)}$ satisfies following properties:

K1) $(F,E) \tilde{\subseteq} \tilde{\varphi}((F,E))$,

K2) $\tilde{\varphi}(\tilde{\varphi}((F,E))) = \tilde{\varphi}((F,E))$,

K3) $\tilde{\varphi}((F,E)\,\tilde{\cup}\,(G,E)) = \tilde{\varphi}((F,E))\,\tilde{\cup}\,\tilde{\varphi}((G,E))$,

K4) $\tilde{\varphi}(\Phi_E) = \Phi_E$.

Conversely for any operator $\tilde{\varphi}: \wp(U_E) \rightarrow \wp(U_E)$ satisfying these four conditions there exists a unique soft topology $\tau_{\tilde{\varphi}}$ on U such that for all $(F,E) \tilde{\subseteq} U_E$ the soft closure of $(F,E)$is just $\tilde{\varphi}((F,E))$.

Proof. (K1)-(K4) are obvious from Theorem 19.

For the second part of the theorem let us define $\tau_{\tilde{\varphi}}$ as following:

$\tau_{\tilde{\varphi}} = \{(F,E) \tilde{\subseteq} U_E: \tilde{\varphi}((F,E)^c) = (F,E)^c\}$

We will show that the family $\tau_{\tilde{\varphi}}$ is a soft topology on U_E.

T1) $\tilde{\varphi}(U_E^c) = \tilde{\varphi}(\Phi_E)$ and by (K4) $\tilde{\varphi}(\Phi_E) = \Phi_E$ therefore $U_E \in \tau_{\tilde{\varphi}}$. $\tilde{\varphi}(\Phi_E^c) = \tilde{\varphi}(U_E)$ and since by (K1) $U_E \subset \tilde{\varphi}(U_E)$ we obeserve that $\tilde{\varphi}(U_E) = U_E$ and $\Phi_E \in \tau_{\tilde{\varphi}}$.

T2) Let $(F,E), (G,E) \in \tau_{\tilde{\varphi}}$ then we have $\tilde{\varphi}((F,E)^c) = (F,E)^c$ and $\tilde{\varphi}((G,E)^c) = (G,E)^c$. Now by the Morgan's $\tilde{\varphi}(((F,E) \tilde{\cap} (G,E))^c) = \tilde{\varphi}((F,E)^c \tilde{\cup} (G,E)^c)$ and by (K3) $\tilde{\varphi}((F,E)^c \tilde{\cup} (G,E)^c) = \tilde{\varphi}((F,E)^c) \tilde{\cup} \tilde{\varphi}((G,E)^c) = (F,E)^c \tilde{\cup} (G,E)^c = ((F,E) \tilde{\cap} (G,E))^c$ which means that $(F,E) \tilde{\cap} (G,E) \in \tau_{\tilde{\varphi}}$.

T3) Let $(F_i,E) \in \tau_{\tilde{\varphi}}$ for $\forall i \in I$. For $\forall i \in I$ we have $(F_i,E) \tilde{\subseteq} \tilde{\bigcup}_{i\in I} (F_i,E)$ therefore,

$U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E) \tilde{\subseteq} U_E \tilde{\setminus} (F_i,E)$ and with help of (K3) it can be seen that for $\forall i \in I$, $\tilde{\varphi}\left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right) \tilde{\subseteq} \tilde{\varphi}(U_E \tilde{\setminus} (F_i,E))$. Since each (F_i,E) is a member of $\tau_{\tilde{\varphi}}$ we have $\tilde{\varphi}(U_E \tilde{\setminus} (F_i,E)) = U_E \tilde{\setminus} (F_i,E)$thus, $\tilde{\varphi}\left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right) \tilde{\subseteq} \tilde{\bigcap} \left(U_E \tilde{\setminus} (F_i,E)\right)$ and finally by De Morgan's

$$\tilde{\varphi}\left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right) \tilde{\subseteq} \left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right). \tag{1}$$

On the other hand by (K1),

$$U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E) \tilde{\subseteq} \tilde{\varphi}\left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right). \tag{2}$$

Combining Eqn. (1) and Eqn. (2) we get the equality $\tilde{\varphi}\left(U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)\right) = U_E \tilde{\setminus} \tilde{\bigcup}_{i\in I} (F_i,E)$ hence

$\tilde{\bigcup}_{i\in I} (F_i,E) \in \tau_{\tilde{\varphi}}$.

Once we have seen that $\tau_{\tilde{\varphi}}$ is a soft topolgy the property (K2) will help us showing that $\tilde{\varphi}((F,E)) = \overline{(F,E)}$:

Since by (K2) $\tilde{\varphi}(\tilde{\varphi}((F,E))) = \tilde{\varphi}((F,E))$ we have by definition of $\tau_{\tilde{\varphi}}$, $\tilde{\varphi}((F,E))^c \in \tau_{\tilde{\varphi}}$ and therefore $\tilde{\varphi}((F,E))$ is a soft closed set which means,

$$\overline{(F,E)} \tilde{\subseteq} \tilde{\varphi}((F,E)). \tag{3}$$

For the reverse inclusion we first observe that since $\overline{(F,E)}$ is a closed set, by definition of $\tau_{\tilde{\varphi}}$, $\tilde{\varphi}((\overline{(F,E)})) = \overline{(F,E)}$. Additionally we have $(F,E) \tilde{\subseteq} \overline{(F,E)}$ and by (K1) $\tilde{\varphi}((F,E)) \tilde{\subseteq} \tilde{\varphi}(\overline{(F,E)}) = \overline{(F,E)}$ which is the required inclusion,

$$\tilde{\varphi}((F,E)) \tilde{\subseteq} \overline{(F,E)}. \tag{4}$$

Thus by Eqn. (3) and Eqn. (4) $\tilde{\varphi}((F,E)) = \overline{(F,E)}$. The operator $\tilde{\varphi}$ in this theorem is called to be a Kuratowski soft closure operator.

Example 27. Let $U = \mathbb{R}$, $E = \{e_1, e_2, \dots, e_n\}$ and $\tilde{\varphi}: \wp(\mathbb{R}_E) \to \wp(\mathbb{R}_E)$ be defined as following:

$$\tilde{\varphi}((F,E)) = \begin{cases} \Phi_E, & \text{if } (F,E) = \Phi_E; \\ (F,E) \tilde{\cup} \left(\sqrt{2}, E\right), & \text{if } (F,E) \neq \Phi_E; \end{cases}$$

$\tilde{\varphi}$ is a soft Kuratowski closure operator.

It can be easily verified that $\tilde{\varphi}$ satisfies (K1)-(K4). The topology generated by $\tilde{\varphi}$ is

$$\tau_{\tilde{\varphi}} = \left\{(F,E): (\sqrt{2}, E) \tilde{\subseteq} (F,E)^c\right\} \tilde{\cup}\, U_E.$$

For sure the duallity between closednes and opennes of soft sets in soft topological spaces reflects a dual concept to Soft Closure Operators: In the sequel we introduce the Soft Interior Operator.

Theorem 28. In a soft topological space (X, τ, E) the operator $\tilde{\Psi}: \wp(U_E) \to \wp(U_E)$, $\tilde{\Psi}((F,E)) = (F,E)^\circ$ satisfies following properties:

I1) $\tilde{\Psi}((F,E)) \tilde{\subseteq} (F,E)$,

I2) $\tilde{\Psi}\left(\tilde{\Psi}((F,E))\right) = \tilde{\Psi}((F,E))$,

I3) $\tilde{\Psi}((F,E) \tilde{\cap} (G,E)) = \tilde{\Psi}\left((F,E)\right) \tilde{\cap} \tilde{\Psi}((G,E))$, I4) $\widetilde{\Psi}(U_E) = U_E$.

Conversely, suppose that the operator $\widetilde{\Psi}: \wp(U_E) \to \wp(U_E)$ satisfies the 4 conditions given above then there exists a soft topology $\tau_{\widetilde{\Psi}}$ on U such that for each $(F, E) \tilde{\subseteq} U_E$ the soft interior of (F, E) is just $\widetilde{\Psi}((F, E))$.

Proof. The first part of the theorem is obvious from Theorem 21. For the converse part it can be first varified that $\tilde{\varphi}((F, E)) = U_E \tilde{\setminus} \widetilde{\Psi}((F, E)^c)$ is a soft Kuratowski closure. Afterwards it can be seen that for the topology $\tau_{\widetilde{\Psi}} = \{(F, E) \tilde{\subseteq} U_E : \widetilde{\Psi}((F, E)) = (F, E)\}$ we have the equality $(F, E)^\circ = \widetilde{\Psi}((F, E))$.

Example 29. Let $U = \mathbb{R}$, $E = \{e_1, e_2, \dots, e_n\}$ and $\widetilde{\Psi}: \wp(\mathbb{R}_E) \to \wp(\mathbb{R}_E)$ be defined as following:

$$\widetilde{\Psi}((F, E)) = \begin{cases} \mathbb{R}_E, & \text{if } (F, E) = \mathbb{R}; \\ (F, E) \tilde{\setminus} (\sqrt{2}, E), & \text{if } (F, E) \neq \mathbb{R}. \end{cases}$$

$\widetilde{\Psi}$ is a soft interior operator and the corresponding soft topology is the family

$$\tau_{\widetilde{\Psi}} = \{(F, E) \in \wp(\mathbb{R}_E) : (F, E)) \tilde{\cap} (\sqrt{2}, E) = \Phi_E\}.$$

4. Soft Kuratowski Closure-Complement Theorem

The Kuratowski Closure-Complement Theorem which was first proved by the Polish mathematician Kazimierz Kuratowski in 1922 can be given by using soft sets as follows:

Theorem 30. Let (U, τ, E) be a soft topological space and (F, E) be a soft set over U. The number of different soft sets obtained by soft complementing and soft closing the set (F, E) can not exceed 14. Moreover, this number can be attained for a soft set in the soft standard topology.

Several proofs of the classical version of this theorem are demonstrated by different mathematicians. The way we have choosen is analoguous to the proof of Strabel [11] but before going to the proof of the theorem, we will review some algebraic notions that are in connection with the operators used in the proof. Given a soft topological space (U, τ, E) define the soft complement operator a and the soft closure operator b on soft subsets $(F, E) \tilde{\subseteq} U_E$ by $a(F, E) = U_E \tilde{\setminus} (F, E)$ and $b(F, E) = \overline{(F, E)}$, respectively. Obviously $aa(F, E) = (F, E)$. Starting with any soft topological space (U, τ, E), possible distinct operators on (U, τ, E) that can be obtained by composing the elements of the set $\{a, b\}$ yield to a monoid with the identity element aa. This monoid is called the Kuratowski monoid on (U, τ, E).

For any soft topological space (U, τ, E), a natural partial order on the Kuratowski monoid exists.(U, τ, E) If $\circ_1$ and $\circ_2$ are elements of the Kuratowski monoid on (U, τ, E), we define the partial order $\leq$ as $\circ_1 \leq \circ_2$ if for every $(F, E) \tilde{\subseteq} U_E$, $\circ_1 (F, E) \leq \circ_2 (F, E)$.

After this short introduction we are ready for the proof of the theorem.

Proof. Let (U, τ, E) be a soft topological space. We mentioned above that $aa = id$. At the same time $bb = b$. Therefore it can be easily observed that a member of the Kuratowski monoid on (U, τ, E) has to be equivalent to one of the operators: id, a, b, ab, ba, aba, bab, $abab$, $baba$, $ababa$, $babab$, $abab \ldots ab$, $baba \ldots ba$, $abab \ldots aba$ or $baba \ldots bab$.

We will now obtain that $bab = bababab$. Firstly,

$$aba\big((F,E)\big) = ab\big(U_E \tilde{\setminus} (F,E)\big)$$

$$= a\Big(\big(U_E \tilde{\setminus} (F,E)\big)^{\circ} \tilde{\cup} \underline{(F,E)}\Big)$$

$$= (F,E)^{\circ}$$

Now $ababab \leq bab$ since $ababab(F,E)$ is the interior of $bab(F,E)$. By the fact that $bb = b$, it follows that $bababab \leq bbab = bab$. Also, $abab \leq b$ since $abab(F,E)$ is the interior of $b(F,E)$. Hence, $babab \leq bb = b$. Then $ababab \geq ab$, and therefore $bababab \geq bab$. We conclude that $bab = bababab$. From this it can be deduced that any word on $\{a, b\}$ that is longer than 7 places has to be equivalent to a word of length at most 7. Thus, for any soft topological space (U, τ, E), each operator in the Kuratowski monoid on (U, τ, E) is equivalent to at least one of the following:

id, a, b, ab, ba, aba, bab, $abab$, $baba$, $ababa$, $babab$, $ababab$, $bababa$, $abababa$

Therefore, for a soft topological space (U, τ, E), the Kuratowski monoid on (U, τ, E) can have order at most 14 and hence for any $(F, E) \tilde{\subseteq} U_E$, there are at most 14 distinct soft sets that can be obtained via soft closures and soft complements of (F, E).

To complete the proof of the theorem we need to show that this bound of 14 can be attained for a soft set. The following set will serve to this objective. $(F, E) \tilde{\subseteq} \mathbb{R}_E$ given by

$$(F,E) = \big\{\big(e_1, (0,1)\big), \big(e_2, (1,2)\big), (e_3, \{3\}), (e_4, [4,5] \cap \mathbb{Q})\big\}$$

attains the bound of 14; that is, we can produce 14 distinct soft sets from (F, E) by taking soft complements and soft closures. These soft sets are:

$$id(F,E) = \{(e_1,(0,1)),(e_2,(1,2)),(e_3,\{3\}),(e_4,[4,5]\cap\mathbb{Q})\}, \quad \text{(i)}$$

$$a(F,E) = \begin{Bmatrix} (e_1,(-\infty,0]\cup[1,+\infty)),(e_2,(-\infty,1]\cup[2,+\infty)), \\ (e_3,(-\infty,3]\cup(3,+\infty)),(e_4,(-\infty,4)\cup(5,+\infty)\cup([4,5]\cap I)) \end{Bmatrix}, \quad \text{(ii)}$$

$$b(F,E) = \{(e_1,[0,1]),(e_2,[1,2]),(e_3,\{3\}),(e_4,[4,5])\}, \quad \text{(iii)}$$

$$ab(F,E) = \begin{Bmatrix} (e_1,(-\infty,0)\cup(1,+\infty)),(e_2,(-\infty,1)\cup(2,+\infty)), \\ (e_3,(-\infty,3)\cup(3,+\infty)),(e_4,(-\infty,4)\cup(5,+\infty)) \end{Bmatrix}, \quad \text{(iv)}$$

$$ba(F,E) = \begin{Bmatrix} (e_1,(-\infty,0]\cup[1,+\infty)),(e_2,(-\infty,1]\cup[2,+\infty)), \\ (e_3,\mathbb{R}),(e_4,\mathbb{R}) \end{Bmatrix}, \quad \text{(v)}$$

$$aba(F,E) = \{(e_1,(0,1)),(e_2,(1,2))\}, \quad \text{(vi)}$$

$$bab(F,E) = \begin{Bmatrix} (e_1,(-\infty,0]\cup[1,+\infty)),(e_2,(-\infty,1]\cup[2,+\infty)), \\ (e_3,\mathbb{R}),(e_4,(-\infty,4]\cup[5,+\infty)) \end{Bmatrix}, \quad \text{(vii)}$$

$$abab(F,E) = \{(e_1,(0,1)),(e_2,(1,2)),(e_4,(4,5))\}, \quad \text{(viii)}$$

$$baba(F,E) = \{(e_1,[0,1]),(e_2,[1,2])\}, \quad \text{(ix)}$$

$$ababa(F,E) = \begin{Bmatrix} (e_1,(-\infty,0)\cup(1,+\infty)),(e_2,(-\infty,1)\cup(2,+\infty)), \\ (e_3,\mathbb{R}),(e_4,\mathbb{R}) \end{Bmatrix}, \quad \text{(x)}$$

$$babab(F,E) = \{(e_1,[0,1]),(e_2,[1,2]),(e_4,[4,5])\}, \quad \text{(xi)}$$

$$ababab(F,E) = \begin{Bmatrix} (e_1,(-\infty,0)\cup(1,+\infty)),(e_2,(-\infty,1)\cup(2,+\infty)), \\ (e_3,\mathbb{R}),(e_4,(-\infty,4)\cup(5,+\infty)) \end{Bmatrix}, \quad \text{(xii)}$$

$$bababa(F,E) = \begin{Bmatrix} (e_1,(-\infty,0]\cup[1,+\infty)),(e_2,(-\infty,1]\cup[2,+\infty)), \\ (e_3,\mathbb{R}),(e_4,\mathbb{R}) \end{Bmatrix}, \quad \text{(xiii)}$$

$$abababa(F,E) = \{(e_1,(0,1)),(e_2,(1,2))\}. \quad \text{(xiv)}$$

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