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On (p,q)-Rogers-Szegő Matrices

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Abstract

In the present article, we have discussed the (p,q)-numbers, the Rogers-Szegő polynomial and the (p,q)-Rogers-Szegő polynomial and have defined the (p,q)-matrices and the (p,q)-Rogers-Szegő matrices. We have presented some algebraic properties of these matrices and have proved them. In particular, we have obtained the factorization of these matrices, their inverse matrices, as well as the matrix representations of the (p,q)-numbers, the Rogers-Szegő polynomials and the (p,q)-Rogers-Szegő polynomials.

Keywords: (p,q)-analogue; (p,q)-Rogers-Szegő polynomials; (p,q)-Rogers-Szegő matrix.

(p,q)-Rogers-Szegő Matrisleri Üzerine

Öz

Bu çalışmada, (p,q)-sayılarını, Rogers-Szegő polinomunu ve (p,q)-Rogers-Szegő polinomunu ele aldık ve (p,q)-matrislerini ve (p,q)-Rogers-Szegő matrislerini tanımladık. Bu matrislere ait bazı özellikleri verdik ve bunların ispatlarını yaptık. Özellikle, bu matrislerin

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çarpanlara ayrılışını, bunların ters matrislerini ve (p,q)-sayılarının, Rogers-Szegő polinomlarının ve (p,q)-Rogers-Szegő polinomlarının matris temsillerini elde ettik.

Anahtar Kelimeler: (p,q)-analoji; (p,q)-Rogers-Szegő polinomu; (p,q)-Rogers-Szegő matrisi.

1. Introduction

The q-binomial is defined as;

$$\begin{bmatrix} m \\ s \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_s (q;q)_{m-s}}, \text{ and } (a;q)_m = \prod_{j=0}^{m-1} (1-aq^j). \tag{1}$$

Another way to present the q-binomial is;

$$\begin{bmatrix} m \\ s \end{bmatrix}_q = \frac{[m]_q!}{[s]_q![m-s]_q!}$$

where $[m]_q = \frac{1-q^m}{1-q}$ and $[m]_q! = [1]_q [2]_q \dots [m]_q$.

The q-oscillator algebra is an important part of the quantum groups [1- 3]. Accordingly, the (p,q)-oscillator algebra was presented in [4] and studied in [5-6]. The (p,q)-number $[m]_{p,q}$ was introduced as a result of the studies on the (p,q)-oscillator. The (p,q)-number is defined as;

$$[m]_{p,q} = \frac{p^m - q^m}{p-q}.$$

And it is obvious that;

$$\lim_{p\to 1} [m]_{p,q} = [m]_q.$$

The (p,q)-binomial coefficient is defined as;

$$\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} = \frac{(p,q;p,q)_m}{(p,q;p,q)_s (p,q;p,q)_{m-s}} = \frac{[m]_{p,q}!}{[s]_{p,q}![m-s]_{p,q}!}, \text{ and } (a,b;p,q)_m = \prod_{s=0}^{m-1} (ap^s - bq^s)$$

where $[0]_{p,q}! = 1$ and $[m]_{p,q}! = [1]_{p,q} [2]_{p,q} \dots [m]_{p,q}$.

After the presentation of the (p,q)-number, the (p,q)-calculus studied in [1, 4, 7-9].

Recently, many researchers have studied the q analogue of number sequences and polynomials, see [10-13]. The Rogers-Szegő polynomials were shown up in the studies of Rogers [14, 15] and were discussed by Szegő [16]. The single variable Rogers-Szegő polynomial is defined as;

$$H_m(x)=\sum_{k=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_q x^s .$$

The Rogers-Szegő polynomials satisfy the recursion in [17, p. 49] given as;

$$H_{m+1}(x)=(1+x)H_m(x)+x(q^m-1)H_{m-1}(x) .$$

Jagannathan and Sridhar [18] defined the (p,q)-Rogers-Szegő polynomial using the (p,q)-number and showed that it is related to the (p,q)-oscillator. The (p,q)-Rogers-Szegő polynomial is defined as;

$$H_m(x,p,q)=\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} x^s .$$

This polynomial is a natural generalization of the Rogers-Szegő polynomial as;

$$\lim_{p\to 1} H_m(x,p,q)=H_m(x).$$

Additionally, some matrices associated with the number sequences and the polynomials were studied. For example, in [19] the authors studied the *k*-Fibonacci matrix and the symmetric *k*-Fibonacci matrix, in [20] the authors studied the Pell matrix. Lee et al. [21] gave the factorization of the Fibonacci matrix. In [10] the authors studied the $(q;x;s)$-Fibonacci and Lucas matrices. Fonseca and Petronilho [22] gave explicit inverses of some tridiagonal matrices. Some of the important works on the number sequences are determinant and permanent representations [23-28].

2. $(\boldsymbol{p},\boldsymbol{q})$-Matrices and Rogers-Szegő Polynomial

Definition 1. The $n\times n$ lower triangular (p,q)-matrices $\mathbf{N}_n^{p,q}=[a_{rs}]$ are defined by

$$a_{rs}=\begin{cases}[r-s+1]_{p,q}, & r-s\geq 0\\ 0, & \textit{otherwise.}\end{cases}$$

Theorem 2. Let $\mathbf{N}_n^{p,q}$ are (p,q)-matrices, then

$$(\mathbf{N}_n^{p,q})^{-1}=\left[b_{rs}\right]=\begin{cases}1, & r=s\\ -(p+q), & r-s=1\\ pq, & r-s=2\\ 0, & \text{otherwise.}\end{cases}$$

Proof. It suffices to prove that $\mathbf{N}_n^{p,q}(\mathbf{N}_n^{p,q})^{-1}=I_n$. It is obvious, for $r<s$,

$\sum_{k=0}^{n}a_{rk}b_{ks}=0$ and for $r=s$, $\sum_{k=0}^{n}a_{rk}b_{ks}=a_{rr}b_{rr}=1$. For $r>s>0$ we have

$$\sum_{k=0}^{n}a_{rk}b_{ks}=\sum_{k=s}^{s+2}a_{rk}b_{kr}=[r-s+1]_{p,q}+[r-s]_{p,q}(-p-q)+[r-s-1]_{p,q}pq$$
$$=[r-s+1]_{p,q}-([r-s]_{p,q}(p+q)+[r-s-1]_{p,q}(-pq))=0$$

which implies that $\mathbf{N}_n^{p,q}(\mathbf{N}_n^{p,q})^{-1}=I_n$.

Definition 3. The $n\times n$ tridiagonal matrices $\mathbf{H}_n^q=[h_{rs}]$ are defined by

$$h_{rs}=\begin{cases}-1, & s-r=1\\ x+1, & r=s\\ x(q^{r-1}-1), & r-s=1\\ 0, & \textit{otherwise.}\end{cases}\tag{3}$$

Theorem 4. Let $\widehat{\mathbf{H}}_n^q$ be a lower triangular matrix defined as

$$\widehat{\mathbf{H}}_n^q=\begin{bmatrix}1 & 0 & 0\\ & & 0\\ & -\mathbf{H}_n^q & \\ & & 1\end{bmatrix}.\tag{4}$$

Then, $(\widehat{\mathbf{H}}_n^q)^{-1}=\left[\bar{h}_{ij}\right]=\begin{cases}H_{i-j}(x), & i\geq j\\ 0, & \text{otherwise.}\end{cases}$

Proof. It is obvious by using matrix product.

3. (p,q)Rogers-Szego Matrices

Definition 5. The $n\times n$ lower triangular (*p,q*)-Rogers-Szegő matrices $\mathbf{R}_n^{p,q}=[r_{ij}]$ are defined by

$$r_{ij}=\begin{cases} H_{i-j}(x,p,q), & i-j\geq 0 \\ 0, & otherwise. \end{cases}$$

The sequence $h_n(x,p,q)$ is defined by $h_0(x,p,q)=1,\ h_1(x,p,q)=x+1$ and

$$h_n(x,p,q)=(x+1)h_{n-1}(x,p,q)+\sum_{s=1}^{n-1}(-1)^{n-s}H_{n-s+1}(x,p,q)h_{s-1}(x,p,q).$$

Definition 6. The $n\times n$ lower Hessenberg matrices $\mathbf{S}_n^{p,q}=[a_{rs}]$ are defined by

$$a_{rs}=\begin{cases} H_{r-s+1}(x,p,q), & r-s\geq -1 \\ 0, & otherwise. \end{cases}$$

Lemma 7. [28] Let H_n be an $n\times n$ lower Hessenberg matrix for all $n\geq 1$ and define det(H_0) = 1. Then, $\det(H_1)=a_{11}$ and for $n\geq 2$

$$\det(H_n)=a_{s+1,s+1}\det(H_{s-1})+\sum_{k=1}^{s-1}[(-1)^{s-k}a_{s,k}\prod_{j=k}^{s}a_{j,j+1}\det(H_{k-1})].$$

Lemma 8. For $n\geq 1$, $\det(\mathbf{S}_n^{p,q})=h_n(x,p,q)$.

Proof. $\det(\mathbf{S}_1^{p,q})=h_1(x,p,q)=x+1$. Suppose that the result is true for all $m\leq n$. We prove it for $m=n+1$. Actually, by using Lemma 7 we have

$$\begin{aligned}\det(\mathbf{S}_{n+1}^{p,q})&=s_{n+1,n+1}\det(\mathbf{S}_n^{p,q})+\sum_{i=1}^{n}[(-1)^{n+1-i}s_{n+1,i}\prod_{j=i}^{n}s_{j,j+1}\det(\mathbf{S}_{i-1}^{p,q})]\\ &=(x+1)\det(\mathbf{S}_n^{p,q})+\sum_{i=1}^{n}[(-1)^{n+1-i}H_{n+2-i+1}(x,p,q)\det(\mathbf{S}_{i-1}^{p,q})]\\ &=(x+1)\det(\mathbf{S}_n^{p,q})+\sum_{i=1}^{n}[(-1)^{n+1-i}H_{n+2-i}(x,p,q)h_{i-1}(x,p,q)]\\ &=h_{n+1}(x,p,q).\end{aligned}$$

Lemma 9. Let $n\geq 1$ then,

$$H_n(x,p,q)=\sum_{k=1}^{n}(-1)^{k+1}h_k(x,p,q)H_{n-k}(x,p,q).$$

Proof. From the previous definition of $h_n(x,p,q)$ we know $h_0(x,p,q)=1$ and

$$h_n(x,p,q)=(x+1)h_{n-1}(x,p,q)+\sum_{k=1}^{n-1}(-1)^{n-k}H_{n-k+1}(x,p,q)h_{k-1}(x,p,q).$$

Using this equation, we have;

$$\begin{aligned}&h_n(x,p,q)-(x+1)h_{n-1}(x,p,q)+H_2(x,p,q)h_{n-2}(x,p,q)-...+H_{n-1}(x,p,q)h_1(x,p,q)\\&-H_n(x,p,q)h_0(x,p,q)=0\\&\Rightarrow H_n(x,p,q)=h_n(x,p,q)-(x+1)h_{n-1}(x,p,q)+H_2(x,p,q)h_{n-2}(x,p,q)-...\\&+H_{n-1}(x,p,q)h_1(x,p,q).\end{aligned}$$

Theorem 10. Let $\mathbf{R}_n^{p,q}$ be the (*p,q*)-Rogers-Szegő matrices, then

$$(\mathbf{R}_n^{p,q})^{-1}=\left[t_{ij}\right]=\begin{cases}(-1)^{i-j}h_{i-j}(x,p,q), & i-j\geq 0\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The proof runs like in Theorem 2 using Lemma 9.

Corollary 11. Let $\mathbf{R}_n^{p,q}$ be the (*p,q*)-Rogers-Szegő matrices and $\widehat{\mathbf{H}}_n^q$ be the lower triangular matrices in Eqn. (4). Then, $\mathbf{R}_n^{1,q}=(\widehat{\mathbf{H}}_n^q)^{-1}$.

Theorem 12. Let $\mathbf{D}_n^{p,q}=[d_{rs}]_{n\times n}$ be an $n\times n$ lower Hessenberg matrix defined as

$$d_{rs}=\begin{cases}-1, & r+1=s\\ (-1)^{r-s}h_{r-s+1}(x,p,q), & r-s\geq 0\\ 0, & \text{otherwise.}\end{cases}$$

Then,

$$\det\mathbf{D}_n^{p,q}=H_n(x,p,q).$$

Proof. The proof runs like in Lemma 8 using Lemma 7.

The following two corollaries are easy consequences of known results in the literature.

Corollary 13. Let $\mathrm{N}_n = [n_{ij}]$ be an $n \times n$ tridiagonal matrix defined as

$$n_{ij} = \begin{cases} -1, & j-i=1 \\ p+q, & i=j \\ -pq, & i-j=1 \\ 0, & otherwise. \end{cases}$$

Then, $\det \mathrm{N}_n = [n+1]_{p,q}$, where $i = \sqrt{-1}$.

Proof. $\det \mathrm{N}_1 = [2]_{p,q} = p+q$. Suppose that the result is true for all the $m \le n-1$. We prove it for $m = n$.

$$\mathrm{N}_n([1]_{p,q}, ..., [n]_{p,q})^T = (0, ..., 0, [n+1]_{p,q})^T.$$

In fact, using Cramer's rule we have

$$[n]_{p,q} = \frac{\det(N_{n-1})[n+1]_{p,q}}{\det(N_n)} \Rightarrow [n+1]_{p,q} = \frac{\det(N_n)[n]_{p,q}}{\det(N_{n-1})}.$$

We obtain $\det \mathrm{N}_n = [n+1]_{p,q}$.

Therefore, $\det \mathrm{N}_n = [n+1]_{p,q}$ holds for all positive integers *n*.

Corollary 14. Let $\mathbf{H}_n^q$ be the matrix in Eqn. (3). Then, $\det \mathbf{H}_n^q = H_n(x)$.

Proof. The proof runs like in Corollary 13.

If we multiply the *r*th row by $(-1)i^r$ and the *s*th column by $(-i)^{s+2}$ of the matrices N_n, $\mathbf{H}_n^p$ and $\mathbf{D}_n^{p,q}$, then the determinant is not altered [23]. In addition, there is a connection between the determinant and permanent of the Hessenberg matrix [26, 29]. Then, it is clear that the following corollaries holds.

Corollary 15. Let $\mathbf{E}_n^{p,q} = [e_{ij}]$be an $n \times n$ Hessenberg matrix defined as

$$e_{st} = \begin{cases} i, & t-s=1 \\ i^{3(s-t)} h_{s-t+1}(x,p,q), & s-t \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\det \mathbf{E}_n^{p,q} = H_n(x,p,q)$, where $i=\sqrt{-1}$.

Corollary 16. Let $H_n(x,p,q)$ be (*p,q*)-Rogers-Szegő polynomial, $\overline{\mathbf{D}}_n^{p,q} = [\overline{d}_{ij}]$ and $\overline{\mathbf{E}}_n^{p,q} = [\overline{e}_{ij}]$ be an $n \times n$ Hessenberg matrices defined as

$$\overline{d}_{ij} = \begin{cases} 1, & j-i=1 \\ (-1)^{i-j} h_{i-j+1}(x,p,q), & i-j \geq 0 \\ 0, & \text{otherwise}, \end{cases} \qquad \overline{e}_{st} = \begin{cases} -i, & t-s=1 \\ (i)^{3(s-t)} h_{s-t+1}(x,p,q), & s-t \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

Then, $per\overline{\mathbf{D}}_n^{p,q} = per\overline{\mathbf{E}}_n^{p,q} = H_n(x,p,q)$, where $i=\sqrt{-1}$.

Corollary 17. Let $H_n(x)$ be Rogers-Szegő polynomial, then

$$\det \mathbf{D}_n^{1,q} = \det \mathbf{E}_n^{1,q} = per\overline{\mathbf{D}}_n^{1,q} = per\overline{\mathbf{E}}_n^{1,q} = H_n(x).$$

Proof. Proof is obvious from equation $\lim_{p \to 1} H_n(x,p,q) = H_n(x)$.

Corollary 18. Let $[n]_{p,q}$ be a (*p,q*)-number and $M_n = [m_{st}]$ be an $n \times n$ tridiagonal matrix defined as

$$m_{st} = \begin{cases} -i, & t-s=1 \\ p+q, & s=t \\ ipq, & s-t=1 \\ 0, & \textit{otherwise}. \end{cases}$$

Then, $\det M_n = [n+1]_{p,q}$, where $i=\sqrt{-1}$.

Corollary 19. Let $\mathbf{K}_n^q = [k_{st}]$ be an $n \times n$ tridiagonal matrix defined as

$$k_{st} = \begin{cases} -i, & t-s=1 \\ 1+x, & t=s \\ ix(q^{s-1}-1), & s-t=1 \\ 0, & \textit{otherwise}. \end{cases}$$

Then, $\det \mathbf{K}_n^q = H_n(x)$.

4. Factorization of (p,q)-Rogers-Szegő Matrix

The lower triangular matrix $K_n = [k_{ij}]_{n\times n}$, is defined by

$$k_{ij} = H_{i-1}(x,p,q) - pH_{i-2}(x,p,q) - qH_{i-2}(x,p,q) + pqH_{i-3}(x,p,q).$$

Theorem 20. Let $\mathbf{R}_n^{p,q}$ be (*p,q*)-Rogers-Szegő matrices and $\mathbf{N}_n^{p,q}$ be (*p,q*)-matrices. Then,

$$\mathbf{R}_n^{p,q} = K_n \mathbf{N}_n^{p,q}.$$

Proof. It suffices to prove that $\mathbf{R}_n^{p,q}(\mathbf{N}_n^{p,q})^{-1} = K_n$. For $i > j$, $k_{ij} = 0$. For $i \geq j \geq 0$,

$$\sum_{s=1}^{n} r_{ik}b_{kj} = \sum_{s=j}^{j+2} r_{ik}b_{kj} = \sum_{s=1}^{3} r_{ik}b_{k1} = H_{i-1}(x,p,q) - H_{i-2}(x,p,q)(p+q) + H_{i-3}(x,p,q)pq = k_{ij}$$

which implies desired result.

References

[1] Burban, I.M., Klimyk, A.U., *P,Q-differentiation, P,Q-integration, and P,Q-hypergeometric functions related to quantum groups*, Integral Transforms and Special Functions, 2, 15-36, 1994.

[2] Chaichian, M., Demichev, A., Introduction to Quantum Groups, World Scientific, Singapore, 1996.

[3] Chari, V., Pressley, A., A Guide to Quantum Groups, Cambridge Univ. Press, Cambridge, 1994.

[4] Chakrabarti, R., Jagannathan, R.A., *(p, q)-oscillator realization of two-parameter quantum algebras,* Journal of Physics A: Mathematical and General, 24, L711, 1991.

[5] Jannussis, A., Brodimas, G., Mignani, L., *Quantum groups and Lie-admissible time evolution*, Journal of Physics A: Mathematical and General, 24(14), L775, 1991.

[6] Arik, M., Demircan, E., Turgut, E., Ekinci, L., Mungan, M., *Fibonacci oscillators*, Zeitschrift für Physik C Particles and Fields, 55(1), 89-95, 1992.

[7] Katriel, J., Kibler, M., *Normal ordering for deformed boson operators and operator-valued deformed Stirling numbers*, Journal of Physics A: Mathematical and General, 25(9), 2683, 1992.

[8] Jagannathan, R., Srinivasa Rao, K., *Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series,* arXiv:math/0602613, 2006.

[9] Smirnov, Y.F., Wehrhahn, R.F., *The Clebsch-Gordan coefficients for the two-parameter quantum algebra $SU_{p,q}(2)$ in the Lowdin-Shapiro approach*, Journal of Physics A: Mathematical and General, 25, 5563, 1992.

[10] Şahin, A. *On the Q analogue of Fibonacci and Lucas matrices and Fibonacci polynomials*, Utilitas Mathematica, 100, 2016.

[11] Cigler, J., *q-Fibonacci polynomials*, Fibonacci Quarterly, 41, 31-40, 2003.

[12] Cigler, J., *Einige q-Analoga der Lucas- und Fibonacci-Polynome*, Sitzungsberichte Abt.II*.*, 211, 3-20, 2002.

[13] Cigler, J., *A new class of q-Fibonacci polynomials*, The Electronic Journal of Combinatorics, 10, R19, 2003.

[14] Rogers, L. J., *On a three-fold symmetry in the elements of Heine's series*, Proceedings of the London Mathematical Society, 24, 171–179, 1893.

[15] Rogers, L. J., *On the expansion of some infinite products*, Proceedings of the London Mathematical Society*,* 24, 337–352, 1893.

[16] Szegő, G., *Ein Beitrag zur Theorie der Thetafunktionen*, S. B. Preuss. Akad. Wiss. Phys.- Math. Kl, 242–252, 1926.

[17] Andrews, G., The Theory of Partitions, Encyclopedia of Mathematics and Its Applications, Addison-Wesley, Reading, Mass.-London-Amsterdam, 1976.

[18] Jagannathan, R., Sridhar, R., *(p, q)-Rogers-Szegő Polynomial and the (p, q)-Oscillator*, arXiv:1005.4309v1[math.QA], 2010.

[19] Lee, G.Y., Kim, J.S., *The linear algebra of the k-Fibonacci matrix,* Linear Algebra and Its Applications, 373, 75-87, 2003.

[20] Kılıç, E., Taşcı, D., *The linear algebra of the Pell matrix*, Boletin de la Sociedad Matematica Mexicana, 2(11), 163-174, 2005.

[21] Lee, G.Y., Kim, J.S., Lee, S.G., *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices*, Fibonacci Quarterly, 40(3), 203-211, 2002.

[22] Fonseca, C.M.da.,Petronilho, J., *Explicit inverses of some tridiagonal matrices*, Linear Algebra and Its Applications, 325,7–21, 2001.

[23] Şahin, A., Ramirez, J.L., *Determinantal and permanental representations of convolved Lucas polynomials*, Applied Mathematics and Computation, 281, 314-322, 2016.

[24] Şahin, A. *On the generalized Perrin and Cordonnier matrices*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 66(1), 242-253, 2017.

[25] Fonseca, C.M. da., *Unifying some Pell and Fibonacci identities*, Applied Mathematics and Computation, 236, 41-42, 2014.

[26] Kaygısız, K., Şahin, *A., Determinant and permanent of Hessenberg matrix and generalized Lucas polynomials*, Bulletin of the Iranian Mathematical Society*.,* 39(6), 1065-1078, 2013.

[27] Anđelić, M., Du, Z., C.M. da Fonseca, Kılıç, E., *A matrix approach to some second-order difference equations with sign-alternating coefficients*, Journal of Difference Equations and Applications, 26:2, 149-162, 2020.

[28] Cahill, N.D., D'Errico, J.R., Narayan, D.A., Narayan, J.Y., *Fibonacci determinants*, College Mathematics Journal, 33, 221-225, 2002.

[29] Gibson, P.M., *An identity between permanents and determinants*, The American Mathematical Monthly, 76, 270-271, 1969.