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ARAŞTIRMA MAKALESİ | RESEARCH ARTICLE

Process Noise Source Localization Using Kalman Filter

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Kalman Filtresi Kullanılarak Sistem Gürültüsünün Kaynağının Tespiti

Özet *–Kontrolör ve gözlemci tasarımında ihtiyaç duyulan sistem gürültüsünü kaynaklarının dağılımı sistemin kompleks olması nedeniyle nadiren bilinmektedir. Bu çalışmada Kalman Filtresi teorisine dayanan filtre kalıntısı korelasyon yöntemi kullanılarak proses gürültülerinin kaynağı ölçüm verisi ile hesaplanmıştır. Bu yöntemde rastgele bir filtre kazancı ile elde edilen filtre kalıntıları kovaryans matrisleri hesaplanır. Bu makale filtre kalıntıları korelasyonları yaklaşımını irdeler ve sistem gürültülerinin kaynaklarının hesaplanmasındaki performansını değerlendirir. Sayısal sonuçlar, bu yöntemin proses gürültüsünün kaynağının tespiti ve gürültü kovaryans matrislerinin tahmini için etkili bir şekilde kullanılabileceğini göstermektedir.*

Anahtar Kelimeler- *Gürültü kaynağının tespiti, Kalman filtresi, Ölçüm gürültüsü, Proses gürültüsü*

Abstract— Due to complexity in the systems, spatial distribution of unmeasured processnoise that is required for the controller and observer design are often unknown. In this study an innovations correlations approach developed in Kalman Filter theory is used to localize the process noise from output measurements. The approach calculates covariance matrices from analysis of resulting innovations from an arbitrary filter gain. Aim of this paper is to review the innovation correlations approach and to evaluate its performance for localization of the process noise.Numerical results suggest that the method can be effectively used for source localization of process noise as well as estimation of noise covariance matrices.

Index Terms - Disturbance Localization, Kalman Filter, Measurement Noise, Process Noise, Process Noise Localization.

I. INTRODUCTION

THE basic idea in estimation theory is to obtain approximations of the true response by using information from a model and from any available measurements. The mathematical structure used to perform estimation is known as an observer. The optimal observer for linear systems subjected to broad band disturbances is the Kalman Filter (KF), [1]. In the classical Kalman filter theory, one of the key assumptions is that a priori knowledge of the spatial distribution of process noise and noise covariance matrices are known without uncertainty. In reality, due to the complexity in the systems, this information is seldom known a priori. The objective of this study is to estimate the spatial distribution of process noise and the noise covariance matrices using correlations approaches. The two correlations approaches that have received most attention in the noise covariance estimation problem are based on: 1) correlations of the innovation sequence and 2) correlations of the output. In the innovations approach one begins by "guessing" a filter gain and then the approach calculates the noise covariance matrices from analysis of the resulting innovations. The correlations approaches to estimate the covariance matrices of process and measurement noise for Kalman Filtering from the measured data began soon after introduction of the filter. One of the most widely quoted strategies to carry out the estimation of noise covariance matrices are due to Mehra [2] and the subsequent paper by Carew and Bellanger [3]. A noteworthy contribution from this early work is the contributions by Neethling and Young [4], who suggested some computational adjustments that could be used to improve accuracy. Recently, some other contributions to the Mehra's approach on the estimation of noise covariance matrices are presented. Odelson, Rajamani and Rawlings applied the suggestions of Neethling and Young's on Mehra's approach and used the vector operator solution for state error covariance Riccati equation of suboptimal filter, [5]. Akesson et al. extended their work for mutually correlated process and measurement noise case, [6]. Bulut, Vines-Cavanaugh and Bernal compared the performance of the output and innovations correlations approaches to estimate noise covariance matrices, [7].In their study Bulut and Bayat [8] estimated a set of covariance matrices in order to characterize the uncertainty in the erroneous models using Kalman Filter based correlations approach. In this paper the same approach is utilized to estimate process and measurement noise covariances and to localize process noise using a model that is known without uncertainty.

The paper is organized as follows: the next section provides a brief summary of the KF particularized to a time invariant linear system with stationary disturbances (which is a condition we have implicitly assumed throughout the previous discussion). The following section reviews the innovations correlations approach for disturbance localization and the paper concludes with a numerical example.

II. THE KALMAN FILTER

Consider a time invariant linear system with unmeasured disturbances $w(t)$ and available measurements $y(t)$ that are linearly related to the state vector $x(t)$. The system has the following description in sampled time

$$x_{k+1} = Ax_k + Bw_k \quad (1)$$

$$y_k = Cx_k + v_k \quad (2)$$

where $A\varepsilon R^{nxn}$, $B\varepsilon R^{nxr}$ and $C\varepsilon R^{mxn}$ are the transition, input to state, and state to output matrices, $y_k \varepsilon R^{mx1}$ is the measurement vector and $x_k \varepsilon R^{nx1}$ is the state. The sequence $w_k \varepsilon R^{rx1}$ is the disturbance known as the process noise and $v_k \varepsilon R^{mx1}$ is the measurement noise. In the treatment here, it is assumed that these are mutually correlated Gaussian stationary white noise sequences with zero mean and known covariance matrices, namely

$$E(w_k) = 0 \quad (3)$$

$$E(v_k) = 0 \quad (4)$$

and

$$E\left(w_k w_j^T\right) = Q\delta_{kj} \quad (5)$$

$$E\left(v_k v_j^T\right) = R\delta_{kj} \quad (6)$$

$$E\left(w_k v_j^T\right) = S\delta_{kj} \quad (7)$$

where δ_{kj} denotes the Kronecker delta function, and$E(\cdot)$denotes expectation. Q and R are covariance matrices of the process and measurement noise and S is cross-covariance between them. For the system in Eqs.1 and 2, the KF estimate of the state can be computed from

$$\hat{x}_{k+1} = A\hat{x}_k + K(y_k - C\hat{x}_k) \quad (8)$$

where $\hat{x}_k$ is the estimate of x_k and K is the (steady state) KF gain that can be expressed in a number of alternative ways, a popular one is

$$K = (APC^T + BS)(CPC^T + R)^{-1} \quad (9)$$

where P, the steady state covariance of the state error, is the solution of the Riccati equation

$$P = APA^T - (APC^T + BS)(CPC^T + R)^{-1}(APC^T + BS)^T + BQB^T \quad (10)$$

The KF provides an estimate of the state for which trace of is minimal. The difference between measured and estimated output, namely $e_k = y_k - C\hat{x}_k$ in Eq.8 is known as innovations sequence of the filter which is a white process. The filter is initialized as follows

$$\hat{x}_0 = E(x_0) \quad (11)$$

III. INNOVATIONS CORRELATIONS APPROACH

We begin with the expression for the covariance function of the innovation process $(e_k)$for any stable observer with gain K_0. As initially shown by Mehra [2] this function is

$$L_j = C\bar{P}C^T + R \qquad j = 0 \quad (12)$$

$$L_j = C\bar{A}^j\bar{P}C^T + C\bar{A}^{j-1}B\bar{S} - C\bar{A}^{j-1}K_0R \qquad j > 0 \quad (13)$$

where $\bar{P}$ the covariance of the state error in the steady state, is the solution of the Riccati equation

$$\bar{P} = \bar{A}\bar{P}\bar{A} + K_0RK_0^T + BQB^T - K_0SB^T - BS^TK_0^T \tag{14}$$

and

$$\bar{A} = A - K_0C \tag{15}$$

Applying vecoperator to both sides of the auto-correlation function of the innovations in Eqs.12-13 one obtains

$$vec(L_j) = (C \otimes C)vec(\bar{P}) + vec(R) \;\; j = 0 \tag{16}$$

$$vec(L_j) = (C\bar{A}^j \otimes C)vec(\bar{P}) + (B^T \otimes C\bar{A}^{j-1})vec(S) - (I \otimes C\bar{A}^{j-1}K_0)vec(R) \quad j > 0 \tag{17}$$

and applying vec operator to error covariance equation in Eq.14, one has

$$vec(\bar{P}) = [I - (\bar{A} \otimes \bar{A})]^{-1}[(K_0 \otimes K_0)vec(R) + B \otimes Bvec(Q) - (B \otimes K_0)vec(S) - (K_0 \otimes B)vec(S^T)] \tag{18}$$

Substituting Eq.25 into Eqs.23 and 24 , and adding the terms related to S^T to the terms related to S and canceling S^T, one finds

$$vec(L_j) = [h_j^Q h_j^S h_j^R]\begin{bmatrix} vec(Q) \\ vec(S) \\ vec(R) \end{bmatrix} \tag{19}$$

where

$$h_j^Q = (C \otimes C)[I - (\bar{A} \otimes \bar{A})]^{-1}(B \otimes B) \quad j = 0 \tag{20}$$

$$h_j^Q = (C \otimes C\bar{A}^j)[I - (\bar{A} \otimes \bar{A})]^{-1}(B \otimes B) \quad j > 0 \tag{21}$$

$$h_j^S = -2I(C \otimes C)[I - (\bar{A} \otimes \bar{A})]^{-1}(B \otimes K_0) \quad j = 0 \tag{22}$$

$$h_j^S = (B^T \otimes C\bar{A}^{j-1}) - 2I[(C \otimes C\bar{A}^j)[I - (\bar{A} \otimes \bar{A})]^{-1}(B \otimes K_0) j > 0 \tag{23}$$

$$h_j^R = (C \otimes C)[I - (\bar{A} \otimes \bar{A})]^{-1}(K_0 \otimes K_0) + I \quad j = 0 \tag{24}$$

$$h_j^R = (C \otimes C\bar{A}^j)[I - (\bar{A} \otimes \bar{A})]^{-1}(K_0 \otimes K_0) - (I \otimes C\bar{A}^{j-1}K_0) j > 0 \tag{25}$$

Listing explicitly the correlation functions in Eq.26 for lags$j = 1, 2, .. p$ and writing in matrix form one has

$$HX = L \tag{26}$$

where

$$H = \begin{bmatrix} h_0^Q & h_0^S & h_0^R \\ h_1^Q & h_1^S & h_1^R \\ h_2^Q & h_2^S & h_2^R \\ \vdots & \vdots & \vdots \\ h_p^Q & h_p^S & h_p^R \end{bmatrix}, \mathrm{L} = \begin{bmatrix} vec(L_0) \\ vec(L_1) \\ vec(L_2) \\ \vdots \\ vec(L_p) \end{bmatrix}, \quad \mathrm{X} = \begin{bmatrix} vec(Q) \\ vec(S) \\ vec(R) \end{bmatrix} \tag{27}$$

Estimates of$Q$, $S$and R can be obtained from Eq.26. From its inspection, one finds that H has dimensions $m^2px(r^2 + m^2 + mr$. The sufficient condition for the uniqueness of the solution of Eq.26 is defined as follows in the general case; the number of unknown parameters in Q and S have to be smaller than the product of number of measurements and the state. The error in solving Eq.26 for X is entirely connected to the fact that the L is approximate since it is constructed from sample correlation functions of the innovations which are estimated from finite duration signals, namely

$$\hat{L}_j \stackrel{\text{def}}{=} E(e_k e_{k-j}^T) = \frac{1}{N-j}\sum_{k=1}^{N-j} e_k e_{k-j}^T \tag{28}$$

where N is the number of time steps. Substituting $\hat{L}$ as the estimate of L, the solution of Eq.26 can be presented as in the following.

$$\text{Case \#1 } mn \geq (r^2 + mr) \tag{29}$$

In this case H is full rank and there exists a unique minimum norm solution for a weighting matrix I given in the following,

$$\hat{X} = (H^TH)^{-1}H^T\hat{L} \tag{30}$$

$$\text{Case\#2 } mn < (r^2 + mr) \tag{31}$$

In this case the matrix is rank deficient, and the size of null space of H can be calculated from $t = r^2 - mn$. The solution is written as follows,

$$\hat{X} = \hat{X}_0 + null(H)Y \tag{32}$$

where $\hat{X}_0$ is the minimum norm solution given in Eq.30 and $Y\varepsilon R^{tx1}$ is an arbitrary vector. Therefore, we conclude Eq.26 has infinite solution when $mn < (r^2 + mr)$. We note that the innovations correlations approach allows to enforce the positive semi-definiteness when solving for Q, $S$and R from Eq.26.

IV. Numerical Experiment: Five-DOF Spring Mass System

In this numerical experiment we use the five-DOF spring mass system depicted in Fig.1 in order to examine the innovations correlations approach for the estimation of spatial distribution of process noise and noise covariance matrices. We assume that true stiffness and mass values of the spring-mass system are given in consistent units as$k_1, k_3, k_5, k_7 = 100$, $k_2, k_4, k_6 = 120$ and $m_i = 0.05$, respectively. The un-damped frequencies of the system are depicted in Table I. We obtain results for output sensors at the third masses, which are recording velocity data at 100Hz sampling.

Case I: (Q *and* R *are unknown*, B *is known*)
The unmeasured disturbances are acting on the masses #2 and #4. The measurement noise is prescribed to have a root-mean-square (RMS) equal to approximately 10% of the RMS of the response measured (R=0.090). Unmeasured disturbances and measurement noise are assumed to be mutually uncorrelated, with the covariance matrices,

$$Q = \begin{bmatrix} 15 & 0 \\ 0 & 30 \end{bmatrix} \quad R = 0.090 \quad S = 0$$

The arbitrary filter gain K_0, that is chosen such that eigenvalues of the matrix$(A - K_0 C)$ are assumed to have the same phase as those of A but with a 20% smaller radius. 80 lags of correlation functions of innovations process is taken into consideration and the sample innovation correlations functions are calculated using 200 seconds of data. 100 simulations are carried out and the disturbance covariance matrices are calculated from innovations correlations approach based on the assumption that the distribution of the unmeasured disturbances, namely input to state matrix (B) is known. The process noise covariance estimates $(\hat{Q}_{11}, and\ \hat{Q}_{22})$ obtained from the innovations correlations approach are presented in Fig.2. The mean value ofthe process noise covariance and measurement matrices obtained from 100 simulations are

$$\hat{Q} = \begin{bmatrix} 14.82 & 0 \\ 0 & 30.14 \end{bmatrix} \hat{R} = 0.084$$

Case II:(Q, R *and* B *are unknown*)
The unmeasured disturbances are acting on the masses #2 and #4, however, it's assumed that the locations of the disturbancesare unknown.In this case, as the distribution of process noise is unknown the input to state matrix is used as identity (I_{5x5}).

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R = 0.090 \quad S = 0$$

The mean value of the process and measurementnoise covariance matrices obtained from 100 simulations are

$$\hat{Q} = \begin{bmatrix} 0.23 & 0 & 0 & 0 & 0 \\ 0 & 14.73 & 0 & 0 & 0 \\ 0 & 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 29.74 & 0 \\ 0 & 0 & 0 & 0 & 0.16 \end{bmatrix} \hat{R} = 0.085$$

As seen in Fig.2. an identity of input to state matrix (B) lead to additional unknowns in the process noise covariance matrix $\hat{Q}(\hat{Q}_{11}, \hat{Q}_{22}, \hat{Q}_{33}, \hat{Q}_{44}, \hat{Q}_{55})$. However, the large diagonal elements corresponding to the masses #2 and #4 show that the process noise source are located at these positions.

V. Conclusions

In this paper innovations correlation approach based on Kalman Filter for estimation of noise covariance matrices is described. The classical innovations correlationsmethod to estimate the noise covariance matrices from output measurements is reviewed. This method produces a linear system of equations, based on innovations correlation function which are calculated from measurements. The method assumes that the system is subjected to unmeasured Gaussian stationary process and measurement noise, which are mutually correlated and the system is linear time invariant. The numerical examinations showed that when the duration is 300 times the fundamental period the mean of 100 simulations proved in good agreement of estimates with the noise covariance matrices. Numerical results suggest that the method can be effectively used for source localization of process noiseas well as estimation of noise covariance matrices.

Appendix

Fig. 1. Five-DOF spring mass system, $m_i = 0.05$, $k_1, k_3, k_5, k_7 = 100$,$k_2, k_4, k_6 = 120$ (inconsistent units). Damping is 2% in all modes.
Fig. 2. Process Noiseand Measurement Noise Covariance Estimates for 100 Simulations
Table I: The un-damped frequencies of the spring mass system.

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APPENDIX

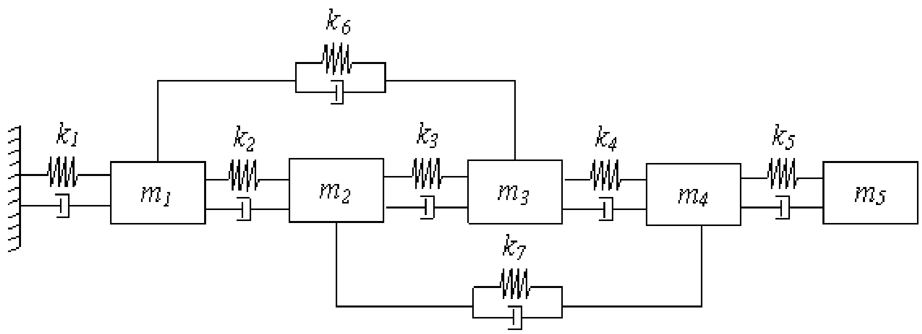

Fig. 1. Five-DOF spring mass system, $m_i = 0.05$, $k_1, k_3, k_5, k_7 = 100$,$k_2, k_4, k_6 = 120$ (inconsistent units). Damping is 2% in all modes.

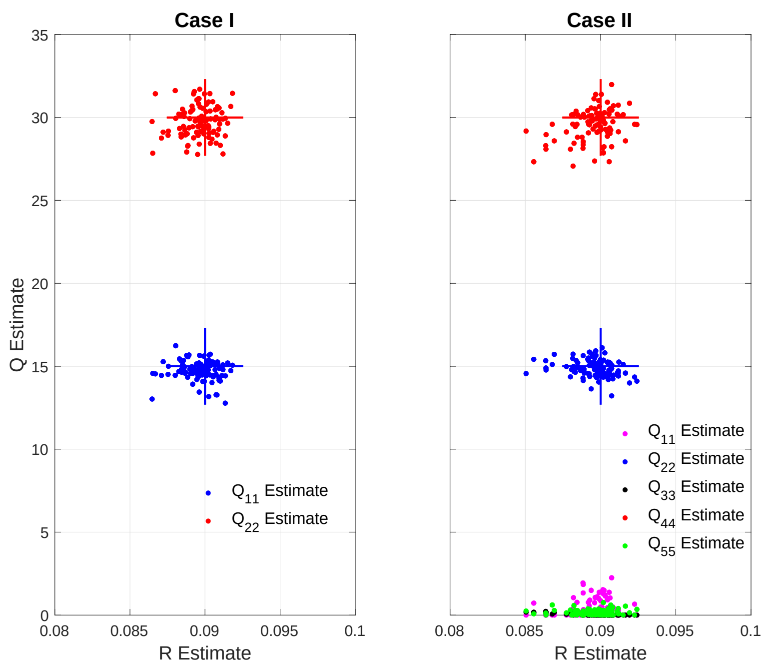

Fig. 2. Process Noiseand Measurement Noise Covariance Estimates for 100 Simulations

Table I: The un-damped frequencies of the spring mass system.

Frequency No.	Frequency (Hz)
1	2.666
2	7.208
3	13.306
4	14.745
5	16.304