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Dynamic output-feedback H_∞ control design for ball and plate system

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Abstract

Ball and plate system is a nonlinear and unstable system, thus introducing great challenges to control scientists and it resembles many complicated real-time systems in several perspectives. There has been a good number of efforts to design a stabilizing controller for this system. This paper presents a dynamic output-feedback H_∞ control strategy for the plate and ball system based on the solution of linear matrix inequalities (LMIs). The discussion involves deriving the equations of motion of the system by using the Lagrange method, linearizing the nonlinear equations, and designing an H_∞ controller to achieve required tracking specifications on the position of the ball. The intent is to show the specified trajectory tracking performance outcomes in time domain via simulation studies conducted using MATLAB/Simulink. A circular and square trajectory following of the designed controller is compared with a baseline PID controller. It is revealed that the proposed controller exhibits an improved tracking performance to following the reference trajectories.

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1. Introduction

The plate and ball system is an extension of beam and ball problem. While the latter has two degrees of freedom where a ball rolls on a beam, the first has four degrees of freedom where the ball can roll over a plate freely. Thus the actual system becomes more complicated because of the coupling on the coordinates. The plate rotates around the x and y -axis that requires two perpendicular control inputs to the plate. The control job is to regulate the position of the ball at a certain location in the plate by changing the angle of the plate. This is indeed not an easy task because the ball does not stay in one point on the plate that moves with an acceleration that is proportional to the length of the plate.

The main objective is to design a controller for the ball and plate system and that the controller should be capable of manipulating the position of the ball and tracking a reference path with high accuracy. In a more precise manner, one can stabilize the ball and specificy a trajectory then let the ball follow it with the least error and a minimum time. In this regard, there has been a good number of efforts on the designing of tracking controllers from both experimental [1,2] and theoretical perspectives for the plate and ball system. The proportional-integral-derivative (PID) controller is employed in [3], the sliding mode control is considered in [4,5]. Position of the ball is regulated by a feedback control in [6]. A nonlinear control via input-output linearization is demonstrated in [7]. In more detail, the work in [8] considers the stabilization problem of the plate and ball system by an approximate solution of the matching conditions to derive a stabilizing control law. A sliding mode controller is used and compared with a linear quadratic control for the plate and ball system in [9]. A cost-effective implementation on the Stewart platform with rotary actuators is demonstrated in experiments. Further, in [10], a neutral network-based PID control structure is proposed for the nonlinear plate and ball system wherein the controller parameters of PID are adjusted by neutral networks during control process.

Unlike the aforementioned rich literature in control techniques applied for the plate and ball system, there are few contributions presented in regards to employing an H_∞ controller for the same problem. For instance, [11] represents a double feedback loop structure where an inner loop is regulated by a DC motor servo controller while outer loop utilizes an H_∞ controller based on the solution of Algebraic Riccati Equations (AREs). Morever, the work [12] represents AREs-based H_∞ optimal control for the plate and ball system. [13] demonstrates a shaping weighting function method for the loop-shaping that is applied to the plate and ball system for validation.

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In most of the cases in control design, sensors have the full state information of the plant, and the state is updated by the model and a state-feedback is performed. In this article, we assume that the states of the system are not directly measurable, which holds a more realistic scenario in real-time applications. The proposed controller directly feeds the plant output to the controller for the next action. The main contributions of this paper are twofold. a) Motivated by the present gap of research in H_∞ control, this study proposes a dynamic output-feedback H_∞ control for reference tracking problem of the plate and ball system first time. b) The output-feedback H_∞ controller gains are computed based on the solution of linear matrix inequalities (LMIs) by MATLAB convex optimization toolbox. The proposed controller is then compared with a baseline PID controller designed via PID tuner in Simulink to reveal the enhanced H_∞ tracking performance.

We start with linearizing the nonlinear plant, hence the linear feedback control methods can be applied and the stability of the open-loop can be determined with the linear model in Section 2. Then the output feedback controller H_∞ synthesis is shown and simulation results are given in Section 3. Lastly, conclusions are drawn in Section 4.

2. Modeling

The mathematical model of the plate and ball system is an important step to describe the main dynamics. There are two ways to derive the equations of motion of the ball and plate system, i.e., by using either the Newtonian method or the Lagrangian method. Each method has some advantages over the other depending on the type of the problem [14]. It is important to fully capture the dynamics of each part by deriving the equation, thus the Lagrangian method is seen more suitable both in terms of simplicity and ability to describe the rotational dynamics within the differential equations.

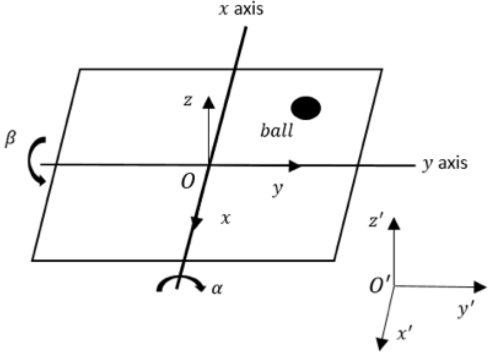

Figure 1. A simplified scheme of ball and plate system.

State variables are chosen as: $x(m)$ is the displacement of the ball along the x-axis, $y(m)$ is the displacement of the ball along the y-axis. The Langrangian equation of a system is defined as

$$\mathcal{L} = T - P. \tag{1}$$

where T is the kinetic energy and P is the potential energy of the system. The generalized coordinates of the ball and plate system is defined

$$\varphi = \begin{bmatrix} \rho \\ \beta \\ \alpha \end{bmatrix}, \tag{2}$$

ρ is the position of the ball, β is the angle between the x-axis of the plate and horizontal direction, α is the angle between the y-axis of the plate and horizontal direction. Note that time notation t is omitted for simplicity in derivations.

The kinetic energy of the plate is:

$$T_{plate} = \frac{1}{2}J_x\dot{\beta}^2 + \frac{1}{2}J_y\dot{\alpha}^2, \tag{3}$$

where J is the moment of inertia of the ball. The kinetic energy of the ball is given by

$$T_{ball} = \frac{1}{2}J_{ball}\dot{q}_{ball}^2 + \frac{1}{2}mv_{ball}^2, \tag{4}$$

where $\dot{q}_{ball}$ is the angular velocity v_{ball} is the linear velocity of the ball. The $\dot{q}_{ball}$ term is defined as the distance of the ball to the center of the plate divided by its radius. i.e.,

$$\dot{q}_{ball} = \frac{\rho}{r}, \tag{5}$$

where $\rho = \sqrt{x^2 + y^2}$. The linear velocity expression of the ball is $v_{ball}^2 = x^2 + y^2$.

$$x = \rho cos\beta + \rho cos\alpha, \tag{6}$$

$$\dot{x} = \dot{\rho}cos\beta - \beta\rho sin\beta + \dot{\rho}cos\alpha - \beta\rho sin\alpha, \tag{7}$$

$$y = \rho sin\beta + \rho sin\alpha, \tag{8}$$

$$\dot{y} = \dot{\rho}sin\beta + \beta\rho cos\beta + \dot{\rho}sin\alpha + \beta\rho cos\alpha, \tag{9}$$

$$T_{ball} = \frac{1}{2}\left[\left(m + \frac{J_{ball}}{r^2}\right)(\dot{x}^2 + \dot{y}^2) + J_{ball}(\dot{\beta}^2 + \dot{\alpha}^2) + m\left(x\dot{\beta} + y\dot{\alpha}\right)^2\right], \tag{10}$$

Eq. (10) is the said to be the kinetic energy of the ball.

The potential energy of the ball is

$$V_{ball} = mgxsin\beta + mgysin\alpha, \tag{11}$$

The Langranian expression of the ball is

$$\mathcal{L} = \frac{1}{2}\left(m + \frac{J_{ball}}{r^2}\right)(\dot{x}^2 + \dot{y}^2) \tag{12}$$

$$+J_{ball}\left(\dot{\beta}^2+\dot{\alpha}^2\right)+m\left(x\dot{\beta}+y\dot{\alpha}\right)^2$$
$$-(mgxsin\beta+mgysin\alpha).$$

The Lagrange equation for both coordinates is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right)-\frac{\partial\mathcal{L}}{\partial x}=0,\qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right)-\frac{\partial\mathcal{L}}{\partial y}=0, \tag{13}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}=\left(m+\frac{J_{ball}}{r^2}\right)\ddot{x}, \tag{14}$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}}=\left(m+\frac{J_{ball}}{r^2}\right)\ddot{y}$$

$$\frac{\partial\mathcal{L}}{\partial x}=mx\dot{\beta}^2+my\dot{\beta}\dot{\alpha}-mgsin\beta\,, \tag{15}$$
$$\frac{\partial\mathcal{L}}{\partial y}=mx\dot{\alpha}^2+my\dot{\beta}\dot{\alpha}-mgsin\alpha.$$

Putting the Euler-Lagrange's equation into the equations above, we obtain two decoupled nonlinear differential equations as

$$\left(m+\frac{J_{ball}}{r^2}\right)\ddot{x}-mx\dot{\beta}^2-my\dot{\beta}\dot{\alpha} \tag{16}$$
$$+mgsin\beta=0.$$
$$\left(m+\frac{J_{ball}}{r^2}\right)\ddot{y}-my\dot{\alpha}^2-mx\dot{\beta}\dot{\alpha} \tag{17}$$
$$+mgsin\alpha=0.$$

Then, the Lagrange expression

$$=\frac{1}{2}\left[\begin{array}{c}\left(J_x\dot{\beta}^2+J_y\dot{\alpha}^2\right)+\left(m+\frac{J_{ball}}{r^2}\right)\left(\dot{x}^2+\dot{y}^2\right)\\ +J_{ball}\left(\dot{\beta}^2+\dot{\alpha}^2\right)+m\left(x\dot{\beta}+y\dot{\alpha}\right)^2\end{array}\right] \tag{18}$$
$$-(mgxsin\beta+mgsin\alpha).$$

The Lagrange equation is for the plate is now given

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right)-\frac{\partial\mathcal{L}}{\partial\beta}=\tau_x, \tag{19}$$
$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right)-\frac{\partial\mathcal{L}}{\partial\alpha}=\tau_y,$$

$$\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right)=(J_x+J_{ball}+mx^2)\dot{\beta}+(mxy\dot{\alpha}), \tag{20}$$
$$\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right)=(J_x+J_{ball}+mx^2)\dot{\alpha}+\left(mxy\dot{\beta}\right),$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\beta}}\right)=(J_x+J_{ball}+mx^2)\ddot{\beta} \tag{21}$$
$$+2mx\dot{x}\dot{\beta}+m\dot{x}y\dot{\alpha}+mx\dot{y}\dot{\alpha}+mxy\ddot{\alpha},$$
$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right)=\ (J_x+J_{ball}+my^2)\ddot{\alpha} \tag{22}$$
$$+2mx\dot{x}\dot{\alpha}+m\dot{x}y\dot{\beta}+mx\dot{y}\dot{\beta}+mxy\ddot{\beta},$$
$$\frac{\partial\mathcal{L}}{\partial\beta}=-mgxcos\beta,\quad \frac{\partial\mathcal{L}}{\partial\alpha}=-mgxcos\alpha. \tag{23}$$

Adding the Euler-Lagrange's equation terms to the plate system

$$(J_x+J_{ball}+mx^2)\ddot{\beta}+2mx\dot{x}\dot{\beta}+m\dot{x}y\dot{\alpha}+ \tag{24}$$
$$mx\dot{y}\dot{\alpha}+mxy\ddot{\alpha}+\ mgxcos\beta=\tau_x,$$

$$(J_x+J_{ball}+my^2)\ddot{\alpha}+2mx\dot{x}\dot{\alpha}+m\dot{x}y\dot{\beta}+ \tag{25}$$
$$mx\dot{y}\dot{\beta}+mxy\ddot{\beta}+\ mgxcos\alpha=\tau_y.$$

Linearization of the ball equations about the plate angle, β and α, assuming $\dot{\beta}$ and $\dot{\alpha}$ are zero, with the small angle condition gives the following linear relations; $sin\beta=\beta$, $sin\alpha=\alpha$,

$$\left(m+\frac{J_{ball}}{r^2}\right)\ddot{x}+mg\beta=0, \tag{26}$$
$$\left(m+\frac{J_{ball}}{r^2}\right)\ddot{y}+mg\alpha=0. \tag{27}$$

The relation between the plate angle and the angle of the gear can be approximately related

$$\beta=\alpha=\frac{d}{L}\theta. \tag{28}$$

Plugging (28) this into (26,27) and taking the Laplace transform of the decoupled equations with zero initial conditions in x and y axis, the following input-output equations are found

$$\left(m+\frac{J_{ball}}{r^2}\right)X(s)s^2=\ -mg\frac{d}{L}\theta(s), \tag{29}$$
$$\left(m+\frac{J_{ball}}{r^2}\right)Y(s)s^2=\ -mg\frac{d}{L}\theta(s). \tag{30}$$

by rearranging the terms, the transfer function from the gear angle to the ball position in x and y axis is obtained

$$Q(s)=\frac{X(s)}{\theta(s)}=\frac{Y(s)}{\theta(s)}=-\frac{mgd}{L\left(\frac{J_{ball}}{r^2}+m\right)}\frac{1}{s^2}. \tag{31}$$

One can write the state-space equations of ball and plate system by considering τ_x and τ_y are torques exerted to the plate in x-axis and y-axis respectively [12]. The states variables defined $[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8]^T=[x,\dot{x},\beta,\dot{\beta},y,\dot{y},\alpha,\dot{\alpha}]^T$ and $U=[u_x,u_y]^T=[\tau_x,\tau_y]^T$.Then the nonlinear state equations

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\\ \dot{x}_8\end{bmatrix}=\begin{bmatrix}x_2\\ K(x_1x_4^2+x_5x_4x_8-gsinx_3\\ x_4\\ 0\\ x_6\\ K(x_5x_8^2+x_1x_4x_8-gsinx_7\\ x_8\\ 0\end{bmatrix}+\begin{bmatrix}0&0\\0&0\\0&0\\1&0\\0&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}\tau_x\\ \tau_y\end{bmatrix}. \tag{32}$$

The values $[\tau_x,\tau_y]^T$ are considered to be 0. $K=\frac{m}{\left(m+\frac{J_{ball}}{r^2}\right)}$, and $J_{ball}=\frac{2}{5}$ for a ball. Then $K=\frac{5}{7}$.

The new state-space equation is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\\ \dot{x}_7\\ \dot{x}_8\end{bmatrix} = \begin{bmatrix}x_2\\ K(x_1x_4^2 - g sin x_3\\ x_4\\ 0\\ x_6\\ K(x_5x_8^2 - g sin x_7\\ x_8\\ 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tau_x\\ \tau_y\end{bmatrix}. \quad (33)$$

The ball and plate system can be regarded as two individual decoupled systems. Writing the x coordinate state equations in state-space format in (34) with the values yields

$\dot{x} = A_p x + B_p u$

$y = C_p x.$

where x is being the state vector, u is the control input, and y is the output vector.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -7 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}\tau_x \quad (34)$$

$$y = [1 \quad 0 \quad 0 \quad 0]x.$$

Note that we omit the time notation. Similar expression can be written for y-axis. As seen, the ball and plate system is an open-loop unstable stable, presenting challenges in control design.

3. Formulations and Results

3.1. PID control

A PID control is designed to compare with H_∞ control for the plate and ball positioning problem in the upcoming section. The gains of the PID is chosen as Kp=2.38, Ki=0.0404, Kd=4.16. The gains are selected based on Matlab's PID control toolbox tuning and successfully stabilize the unstable plate-ball system.

3.2. H_∞ control

In H_∞ control, the design objective is to find a controller that minimizes the worst-case energy amplification over certain frequency ranges. This can be interpreted as minimization of the cost function in the presence of external disturbances. The cost function is given below

$$J = \int_0^\infty [z^T(t)z(t) - \gamma^2 u^T(t)u(t)]dt. \quad (35)$$

where $z(t)$ is the output signal and $u(t)$ is the input signal. The H_∞ norm of a stable transfer function G(s) is the largest input/output root mean square (RMS) gain $i.e.$,

$$\|G\|_\infty = \sup_{\substack{u\in L_2\\ u\neq 0}} \frac{\|z\|_{L_2}}{\|u\|_{L_2}} = \sup_{\omega} \sigma_{max}(G(j\omega)). \quad (36)$$

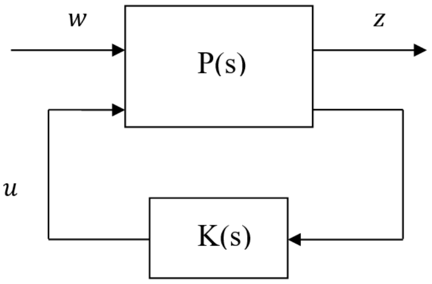

Figure 2. H_∞ controller block diagram.

where $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance with finite energy in the space $L_2[0\ \infty), u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $z(t) = \mathbb{R}^{n_z}$ is the controlled outputs, $y(t) \in \mathbb{R}^{n_y}$ is the measured outputs and $P(s)$ is the augmented plant model with weights and $K(s)$ is being the controller model. The optimal H_∞ controller seeks to minimize $\|F(P,K)\|_\infty$ over all stabilizing LTI controllers $K(s)$.

3.3. L_2 control

The signal $z(t) : [0, \infty) \to \mathbb{R}^n$ is said to be in the space $L^n{}_2\ [0, \infty)$ or simply L_2, if

$$\int_0^\infty z^T(t)z(t)dt < \infty. \quad (37)$$

The 2-norm, denoted by $\|z\|_2$, is defined as

$$\|z\|_2 \triangleq \sqrt{\int_0^\infty z^T(t)z(t)dt}. \quad (38)$$

In this section, we use the MATLAB's Control Toolbox to design the controller for the plate and ball system. The proper weighting functions are used such the regulated outputs remain in specified bound. The weighting error transfer function is chosen such that the output error to be small at lower frequencies, introduces better tracking performance. Similarly, the noise in the system creates high-frequency components that cause the control input to saturate. In order to penalize the input deteriorations at high frequencies, an input weighting function is implemented. We consider Figure 3 for a given closed-loop system where $\tilde{e}, \tilde{u}, \tilde{y}$ represent the weighted outputs of the error, control input and output, respectively. In this design, disturbance (d) and noise (n) are 0.

$$W_e(s) = \frac{1}{s+1}, W_u(s) = \frac{1}{s+10}, \ W_y(s) = \frac{1}{0.5s+1}.$$

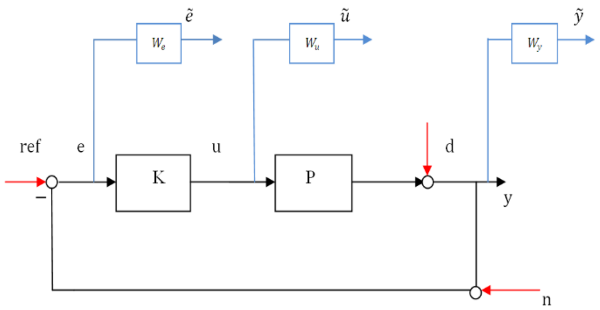

Figure 3. General block diagram for H_∞ synthesis with weighted regulated outputs and disturbances.

In the design of H_∞ control, W_e, W_u, and W_y are used to shape the plant model P in Figure 3. The selection of weighting transfer functions is important that requires experience and skill. We choose them with respect to imposing a reasonable control and enhancing tracking property. The state-space expressions for the weighting functions are:

$$W_e(s) = \begin{bmatrix} A_e & B_e \\ C_e & 0 \end{bmatrix}, W_u(s) = \begin{bmatrix} A_u & B_u \\ C_u & 0 \end{bmatrix},$$
$$W_y(s) = \begin{bmatrix} A_y & B_y \\ C_y & 0 \end{bmatrix},$$

Linear Fractional Transformation (LFT) of the augmented system with omitted time notation t is given as:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_y \\ \dot{x}_e \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_p & 1 & 0 & 0 \\ B_y C_p & A_z & 0 & 0 \\ -B_y C_p & 0 & A_e & 1 \\ 0 & 0 & 0 & A_u \end{bmatrix} \begin{bmatrix} x_p \\ x_y \\ x_e \\ x_u \end{bmatrix} + \begin{bmatrix} 0 & B_p \\ 0 & 0 \\ B_e & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ref \\ u \end{bmatrix}. \tag{39}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ e \end{bmatrix} = \begin{bmatrix} 0 & C_y & 0 & 0 \\ 0 & 0 & C_e & 0 \\ 0 & 0 & 0 & C_u \\ -C_p & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_y \\ x_e \\ x_u \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} ref \\ u \end{bmatrix}. \tag{40}$$

where A_p, B_p, C_p are the state-space model of the linearized model. The open-loop generalized plant model with a new state vector is then written with the following state-space equation

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 \tilde{x}(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 \tilde{x}(t) + D_{21} w(t) + D_{22} u(t) \end{aligned} \tag{41}$$

with $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times n_w}$, $B_2 \in \mathbb{R}^{n\times n_u}$, $C_1 \in \mathbb{R}^{n_z\times n}$, $D_{11} \in \mathbb{R}^{n_z\times n_w}$, $D_{12} \in \mathbb{R}^{n_z\times n_u}$, $C_2 \in \mathbb{R}^{n_y\times n}$, $D_{21} \in \mathbb{R}^{n_y\times n_w}$, $D_{22} \in \mathbb{R}^{n_y\times n_u}$.
Here, $n = 7, n_y = 1, n_u = 1, n_z = 3, n_w = 1$.

We aim to design a dynamic output-feedback controller in the form of

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{42}$$

where $x_c(t) \in \mathbb{R}^n$ is the state of the controller with $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times n_y}$, $C_c \in \mathbb{R}^{n_u\times n}$, and $D_c \in \mathbb{R}^{n_u\times n_y}$. The following theorem is used to construct the output-feedback control gains in terms of a set of LMIs solution.

Theorem [15]: Given the open – loop LFT system governed by (41). Suppose that there exists two symmetric matrices **X, Y** and four data matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$. The following LMIs give the controller matrices.

$$\begin{bmatrix} \mathbf{X}A + \hat{\mathbf{B}}C_2 + (\star) & \star & \star & \star \\ \hat{\mathbf{A}}^T + A + B_2\hat{\mathbf{D}}C_2 & A\mathbf{Y} + B_2\hat{\mathbf{C}} + (\star) & \star & \star \\ (\mathbf{X}B_1 + \hat{\mathbf{B}}D_{21})^T & (B_1 + B_2\hat{\mathbf{D}}D_{21})^T & -\gamma I_{n_w\times n_w} & \star \\ C_1 + D_{12}\hat{\mathbf{D}}C_2 & C_1\mathbf{Y} + D_{12}\hat{\mathbf{C}} & D_{11} + D_{12}\hat{\mathbf{D}}D_{21} & -\gamma I_{n_z\times n_z} \end{bmatrix} < 0, \tag{43}$$

$$\begin{bmatrix} \mathbf{X} & I_{n\times n} \\ I_{n\times n} & \mathbf{Y} \end{bmatrix} > 0. \tag{44}$$

$(\star)$ denotes being symmetric. Then, there exist a controller such that

i) the closed-loop system is stable
ii) the induced L_2-norm of the operator $w \to z$ is bounded by $\gamma > 0$ ($i,e.$, $\|T_{zw}\|_{i,2} < \gamma$)

Once matrices **X, Y,** $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ matrices obtained, the controller matrices are computed in the following steps:

1) Solve for N, M, the factorization problem
$$I - \mathbf{XY} = NM^T. \tag{45}$$

Here, we choose $N := I - XY$, $M := I$.

2) Compute A_c, B_c, C_c and D_c with

$$A_c = \mathrm{N}^{-1}\big(\widehat{\mathbf{A}} - \mathbf{X}(\mathrm{A} - \mathrm{B}_2\widehat{\mathbf{D}}\mathrm{C}_2)\mathbf{Y} - \widehat{\mathbf{B}}\mathrm{C}_2\mathbf{Y} - \mathbf{X}\mathrm{B}_2\widehat{\mathbf{C}}\big)\mathrm{M}^{-\mathrm{T}}, \quad (46)$$

$$B_c = \mathrm{N}^{-1}\big(\widehat{\mathbf{B}} - \mathbf{X}\mathrm{B}_2\widehat{\mathbf{D}}\big), \quad (47)$$

$$C_c = \big(\widehat{\mathbf{C}} - \widehat{\mathbf{D}}\mathrm{C}_2\mathbf{Y}\big)\mathrm{M}^{-\mathrm{T}}, \quad (48)$$

$$D_c = \widehat{\mathbf{D}}. \quad (49)$$

Then the found controller matrices by LMI Control Toolbox of MATLAB are

$$A_c = \begin{bmatrix} -0.176 & 15.8 & 267 & 97 & 13.14 & -9702 & -32501 \\ -2.333 & -2.265 & -21.99 & -7.998 & -1.101 & 799.1 & 2687.4 \\ -0.347 & 1.11 & -12.42 & -1.621 & -0.286 & 219.82 & 738.541 \\ 0.0802 & -0.274 & 32.788 & -6.184 & -0.357 & 254.1 & 853.5 \\ 0.0006 & -0.027 & 0.328 & -0.0515 & -1.052 & 2.662 & 9.047 \\ 0.0045 & -0.002 & -0.031 & -0.024 & -0.0027 & -6.57 & -21.26 \\ -0.0080 & 0.0335 & -382 & -0.0418 & -0.075 & -3.38 & -10.19 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -0.0004 \\ -0.0073 \\ -10.8287 \\ 67.7571 \\ 0.670 \\ 0.3438 \\ -0.967 \end{bmatrix},$$

$$C_c = [0.962 \quad -17.32 \quad -291.1 \quad 105.8 \quad -14.628 \quad 10581 \quad 35549.5],$$

$$D_c = [-0.0003.05].$$

LMI methods have been successfully implemented in the control field in the last decades. The interior-point optimization solver gives the optimal values for controller matrices so that the desired specifications are met. In the above problem, the minimization problem is computed with MATLAB's LMI Toolbox [16] using MATLAB 2018a, and optimal disturbance attenuation value i.e., energy-to-energy norm γ is found as 1.41871. The desired circular reference circle radius is 250 mm and the side length of the square shape reference is set to 500 mm, centered at the origin. Note that the same controller is applied for both x axis and y axis to regulate the ball's position. The obtained results show that the controller design with the LMI approach gives perfect tracking. Notice that the circular trajectory performance in Figure 4 is acceptable whereas the square case possesses significant performance deterioration for square reference in Figure 5.

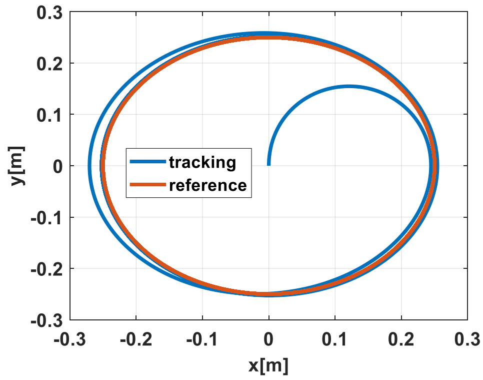

Figure 4. Circular trajectory tracking performance of PID controller.

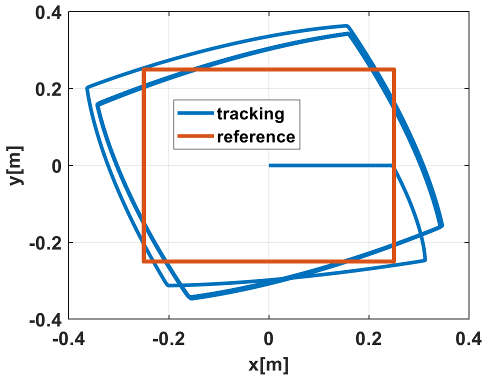

Figure 5. Square trajectory tracking performance of PID controller.

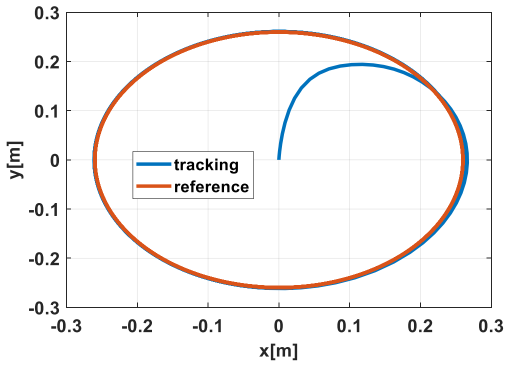

Figure 6. Circular trajectory tracking performance of H_∞ controller.

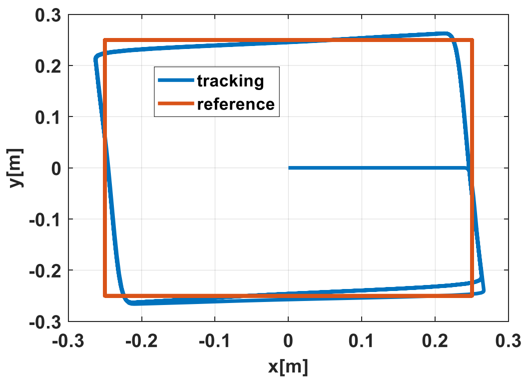

Figure 7. Square trajectory tracking performance of H_∞ controller.

H_∞ controller trajectory tracking of the circular reference from the origin (0,0) of the ball is seen in Figure 6. The tracking problem to make the ball follow the squared path is observed in Figure 7. The ball starts following the reference paths in less than 2 seconds i.e., settling time is less than 2 seconds. To sum up, tracking the references results demonstrate the effectiveness of the proposed solution.

3.4. Performance analysis

The desire of assessing the performance of the controllers is essential due to the need of a reliable and effective application. The closed-loop performance of a control system is measured based on a performance index. Performance index-based analysis is one of the commonly used methods to test whether design requirements are met. To this end, the closed-loop tracking performance of the designed control system is measured with three different performance indices for reference tracking error (of the derived PID and H_∞ controls). The performance indices are:

Integral of the squared value of the error (ISE):

$$ISE = \int_0^\infty e(t)^2 dt. \tag{50}$$

Integral of the absolute value of the error (IAE):

$$IAE = \int_0^\infty |e(t)| dt. \tag{51}$$

Integral of the time-absolute value of the error (ITAE):

$$ITAE = \int_0^\infty t|e(t)| dt. \tag{52}$$

Table 1. Performance comparison of both controllers

Ind.	PID/circ.	PID/sq	H_∞ /circ.	H_∞/sq.
ISE	1.193	2.5	0.8551	1.466
IAE	6.165	10	5.238	7.157
$ITAE$	126.6	200	105.8	143.9

Each of these indices is calculated over an interval of time. The simulation is run for 40 seconds. A performance index is a quantitative measure of the performance of a system, thus it is important to determine system specifications i.e., settling time, rise time, peak overshoot and steady-state performance. The steady-state tracking error is vital in our analysis and it is desired to be zero. To quantitatively compare the both controllers' tracking performance for both references, Table 1 is reported. For instance, the ISE index for PID control is obtained as 1.193 while it is 0.8551 for H_∞ controller in circular trajectory case. It is observed from simulations that since the following square shape reference is a hard task, the tracking performance is deteriorated, proven by numbers. ISE index for PID control is obtained as 2.5 while it is 1.466 for H_∞ controller in square trajectory. Regardless of the tracking shapes and types of performance index, H_∞ controller outperforms.

4. Conclusions

The plate and ball system presents several difficulties in terms of stabilization and control design. This work presents a trajectory tracking control problem for the plate and ball system. We derived the nonlinear model and linearized the plate and ball system that imposes restrictions on the control design step. We use a variety of modern controller design tools to handle the design difficulties. A dynamic output-feedback H_∞ controller is then employed for the presented system based on the solution of linear matrix inequalities. Moreover, the trajectory following performance of the proposed controller is compared with a PID controller to better understand the improvements. It is shown from the simulations that H_∞ controller outperforms on tracking the state variables for both circular and square references.

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Conflict of interest

The author declares no conflict of interest.

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