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The analysis of some spherical mechanism movements and joint designs by the new SLERP ınterpolations

Bazı küresel mekanizma hareketlerinin ve mafsal tasarımlarının yeni SLERP interpolasyonları ile incelenmesi

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The Analysis of Some Spherical Mechanism Movements and Joint Design by The New SLERP Interpolations

Highlights

- *The sequential and the sequential fast SLERP are firstly introduced.*
- *The links between the joints in the spherical mechanisms are designd by geometric SLERP.*
- *The Serret-Frenet frame and curvatures of the geometric SLERP interpolation are computed.*

Grafik Özet (Graphical Abstract)

In this paper, the design and the trajectory motion equations of the joints in the spherical mechanisms were presented by the methods of the sequential SLERP.

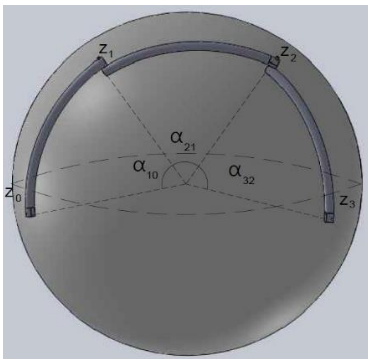

Figure. The design of the links between the joints in the 3DoF spherical mechanism by the geometric SLERP

Aim

The paper aims to design the spherical mechanisms and theri motion via some kinds of SLERP and analyze their geometric properties.

Design & Methodology

Appliying an alternative method to spherical mechanism motions and spherical mechanism designs.

Originality

Some new SLERP interpolations are introduced and used for the first time in spherical mechanism design and motion.

Findings

The sequential and the sequential fast SLERP in the spherical mechanisms firstly by utilizing the definition of the SLERP interpolation are firstly introduced. Furthermore, with the help of the definition of the geometric SLERP, the design of the links between the joints in the spherical mechanisms has been drawn.

Conclusion

The design of the links between the joints in the spherical mechanisms has been drawn using the geometric SLERP. Then the Serret-Frenet frame and curvatures of the geometric SLERP interpolation have been computed in order to find the location and the curvatures of these joints in the spherical mechanism.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

The Analysis of Some Spherical Mechanism Movements and Joint Designs by The New SLERP Interpolations

Araştırma Makalesi / Research Article

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ABSTRACT

The movements of the spherical mechanisms take an important place in the robotic studies. In this paper, the design and the trajectory motion equations of the joints in the spherical mechanisms were presented by the methods of the sequential SLERP, sequential fast SLERP, sequential SLERP with De-Moivre, and modular SLERP interpolation that we firstly defined. Additionally, the Serret Frenet frame and curvatures of the geometrical SLERP interpolation curve for the locations of the joints in the spherical mechanisms were computed, and then, several numerical examples on the interpolation equations that produce the motion equations of the spherical mechanisms with 2 and 3-degree of freedom were given at the end of the paper.

Keywords: Quaternion, sequential SLERP, sequential fast SLERP, modular SLERP, interpolation.

Bazı Küresel Mekanizma Hareketlerinin ve Mafsal Tasarımlarının Yeni SLERP İnterpolasyonları ile İncelenmesi

ÖZ

Küresel mekanizmaların hareketleri robotik çalışmalarda önemli bir yer almaktadır. Çalışmamızda küresel mekanizmalardaki mafsalların tasarımı ve yörünge hareket denklemleri ilk defa tanımladığımız dizisel SLERP, dizisel fast SLERP, De-Moivre ile dizisel SLERP ve modüler SLERP interpolasyon yöntemleri ile verilmiştir. Bununla birlikte, küresel mekanizmalardaki mafsalların konumları için geometrik SLERP interpolasyon eğrisinin Serret-Frenet çatısı ve eğrilikleri hesaplanarak sayısal bir örnek verilmiştir. Ayrıca makalenin sonunda, iki ve üç serbestlik dereceli küresel mekanizmaların hareket denklemlerini üreten interpolasyon denklemleri ile ilgili birkaç sayısal örneklendirme de yapılmıştır.

Anahtar Kelimeler: Kuaterniyon, dizisel SLERP, hızlı dizisel SLERP, modüler SLERP, interpolasyon

1. INTRODUCTION

A famous Irish mathematician Sir William Rowan Hamilton (1843) brought quaternions in the literature by writing the quaternion equation $q \equiv s + ix + jy + kz$ in which $ij = k$ and $ji = -k$, $jk = i$, $kj = -i$, , $ik = -j$, $ki = j$ and $s, x, y, z \in \mathbb{R}$, on the Brougham Bridge, with an inspiration which came while he was going to meeting in the royal family to be a chairman. Since the quaternions with non-commutative reel algebra yield most of the properties of the complex numbers, they also called hypercomplex numbers. Furthermore, since they easily enable the rotation motion of a point in the space, the quaternions have a quite wide implementation area. Especially, in such areas as mechanics, kinematics, robotics, and physics in which geometric movements occur, quaternions have a big advantageous usage in terms of representation and operation. Therefore, a lot of studies on the quaternions have been conducted in various disciplines [1-6]. Sheomake (1985) defined the SLERP interpolation between two unit quaternions for the first time [7]. Dam et al. (1998) carried on various animation studies by utilizing the methods of interpolation [8]. Hast et al. (2003-2004) analyzed the shadings with the spherical linear interpolation by using the De Moivre formula and also made the definition of the fast incremental SLERP by using the orthogonalization method of Gram-Schmidt [9,10]. Kremer (2008) yielded the definition of the normalized linear interpolation and then did new researches on the SLERP interpolation by using the quaternion algorithms [11]. Jafari et al. 2014) analyzed the Bezier curves by utilizing the spherical linear interpolation [12]. In addition, Kuşak Samancı et al.(2015), Karadağ et al.(2019), Incesu (2021) obtained some properties of the Bezier curves [13-15]. On the other hand, the mechanisms and machines theory was firstly launched into analyzing in Russia in 1800 and then spread around the world. Also, in 1969, the second convention for

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machines and mechanisms had been assembled in Poland, and there, the importance of the kinematic in the mechanisms was emphasized. One of the pioneers of the spherical mechanisms, Leonhard Euler (1776) defined the rotation of a solid object about a point, firstly. In recent years, studies have shown that the quaternions play an essential role in the rotation motions of the spherical mechanisms, as well. Alizade(2005), Gezgin(2006), and Kilit(2007) analyzed the motions of the spherical mechanisms by using the quaternions. Moreover, they also defined sequential and modular rotation new methods for the first time [16-18].

The robotic studies have an important place in mechanical engineering. Furthermore, the spherical mechanisms comprise a substantial part of the field. However, we've realized that even though the quaternion structures had been used in the spherical mechanisms in some studies conducted before, the interpolations hadn't been used. Thus, in this study, we have analyzed the equations of the motions of the SLERP interpolations and the spherical mechanisms as an alternative to the quaternion studies which are used in the design of the spherical mechanism before. The sequential SLERP, sequential fast SLERP, sequential SLERP with De-Moivre, and modular SLERP have been defined firstly to analyze the motion of the spherical mechanisms. Also, the geometric SLERP interpolation curve has been benefited for the designs of the spherical mechanism joints. Then, a numeric example has been given by computing the Serret-Frenet frame and the curvatures of the SLERP interpolations. We expect that our study will prepare a theoretical background for engineering studies.

2. MATERIAL and METHOD

A quaternion is defined by the equation $q \equiv s + ix + jy + kz$ where $s, x, y, z \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$, $ij = k$ ve $ji = -k$, $jk = i$, $kj = -i$, and $ik = -j$, $ki = j$. Let H and H_1 be the set of quaternions and the unit quaternions, respectively**.** The two quaternion multiplication $q = s + ix + jy + kz$ and $q' = s' + ix' + jy' + kz'$ is defined by

$$[s,\mathbf{v}] \times [s',\mathbf{v}'] = \begin{bmatrix} ss' - xx' - yy' - zz', \\ s\left(ix' + jy' + kz'\right) + s'\left(ix + jy + kz\right) \\ +\left(yz' - y'z\right)i + \left(zx' - z'x\right)j + \left(xy' - x'y\right)k \end{bmatrix}$$

where $q, q' \in H$ [1-6]. The spherical linear quaternion interpolations are the biggest arcs drawn by two-unit quaternions on the sphere surface. Let $t \in [0,1]$ be a parameter and $q_0, q_1 \in H_1$, the spherical linear interpolation between q_0 and q_1 is defined by

$$SLERP(q_0, q_1, t) = q_0\left(q_0^{-1}q_1\right)^t = q_0\frac{\sin\left((1-t)\theta\right)}{\sin\theta} + q_1\frac{\sin(t\theta)}{\sin\theta}$$

where the angle value is computed by $q_0.q_1 = \cos\theta$ in [7-8]. When the Gram-Schmidt orthogonalization method is applied for the unit quaternion $q_0 \in H$, the fast incremental SLERP interpolation $q(n) = q_1\cos(nK_\theta) + q_0\sin(nK_\theta)$ occurs, where the quaternion q_0 is orthogonal of the quaternion q_1. Also, if there is a digit k**,** the angle between the quaternions is computed by equation $K_\theta = \dfrac{\cos^{-1}(q_1.q_2)}{k}$ [9]. Then, the geometric SLERP interpolation between the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ which are taken in the three-dimensional space is given by the equation

$$Slerp(\mathbf{v_1}, \mathbf{v_2}, t) = \mathbf{v}(\mathbf{t}) = \frac{\sin(1-t)\alpha}{\sin\alpha}\mathbf{v_1} + \frac{\sin t\alpha}{\sin\alpha}\mathbf{v_2}$$

where $t \in [0,1]$ and the angle between the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ is $\alpha = arc\cos(\mathbf{v_1}.\mathbf{v_2})$ [12].

Let the vectors $\mathbf{r_i}$ to be rotated about the axes $\mathbf{m}_i$ as much as the angle θ_i $(i = 1, ..., n)$. Here, in order to get the vector $\mathbf{r_{i+1}}$, the vector $\mathbf{r_i}$ is rotated by the quaternion q_i**.** Thus**,** the vector $\mathbf{r_{i+1}}$ is yielded by the equation $\mathbf{r_{i+1}} = q_i(r_i)q_i^{-1}$. The rotations of the solid objects by using the sequential method and the quaternion operators are defined by $\mathbf{r}_{n+1} = q_n q_{n-1} ... q_2 q_1 (r_i) q_1^{-1} q_2^{-1} ... q_{n-1}^{-1} q_n^{-1}$ in Figure 1 [16-18].

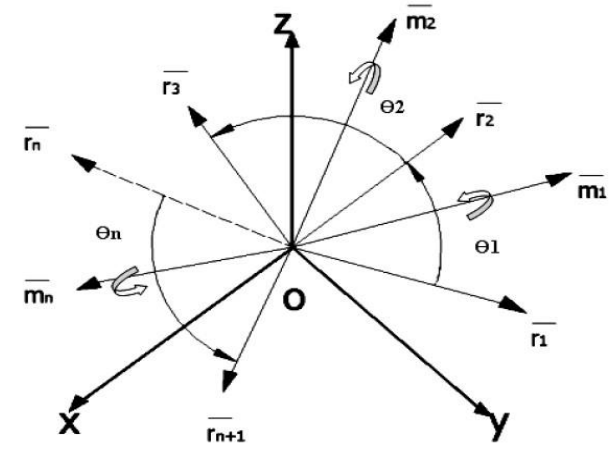

Figure 1. The sequential rotation of the solid objects by quaternions [16-18]

The new vectors $\overline{r_1^{'}}$ and $\overline{m_1^{'}}$ occur as a result of the rotation of the all $\overline{m_i}$ except $\overline{m_n}$**,** and $\overline{r_1}$ about the $\overline{m_n}$ up to the θ_n**,** and this process continues until the $\overline{r_1^{n}}$ occurs. At the end of this process, with the help of the quaternion of the solid objects, the modular rotation

$r_1^n = q_n^* q_{n-1}^* ... q_2^* q_1^* \left(r_1\right) q_1^{*-1} q_2^{*-1} ... q_{n-1}^{*-1} q_n^{*-1}$ has been defined by using $r_1^i = q_i^* \left(r_1\right) q_i^{*-1}$ and $m_j^i = q_i^* \left(m_j^{i-1}\right) q_i^{*-1}$ in Figure 2 [16-18].

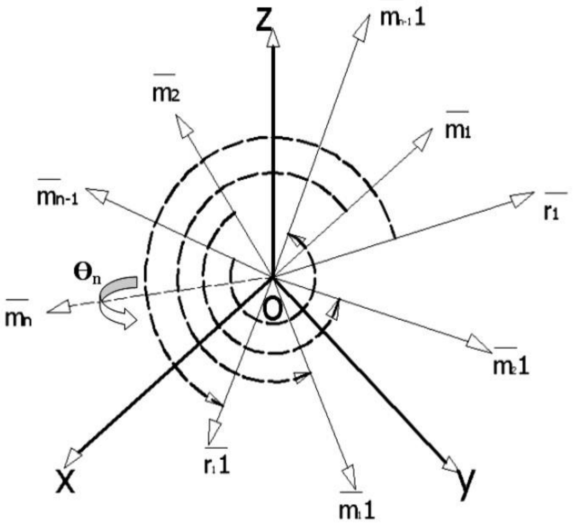

Figure 2. The modular rotation method of the solid objects [16-18]

3. MAIN RESULTS

In this section, the interpolation definitions of the sequential SLERP, sequential fast SLERP, sequential SLERP with De-Moivre, and the modular SLERP with De-Moivre are firstly given. The link design between joints has been made by using geometric SLERP. Finally, the Frenet frame and the curvature calculations of the SLERP interpolation have been defined.

3.1. The Sequential and The Modular SLERP Interpolation In The Spherical Mechanisms

Definition 3.1.1. Let $t \in [0,1]$ be a parameter and $q_{(i-1)}, q_i \in H_1$, the SLERP interpolation equation of the great arc which is drawn from $q_{(i-1)}$ to q_i rotations by the robotic arm in the spherical mechanisms is obtained

$$q_{(i-1)i} = SLERP\left(q_{(i-1)}, q_i, t\right)$$
$$= q_{(i-1)} \frac{\sin\left(\left(1-t\right)\alpha\right)}{\sin\alpha} + q_i \frac{\sin\left(t\alpha\right)}{\sin\alpha}$$

where the angle between two unit quaternions $q_{(i-1)}$ and q_i are computed by $\cos\alpha = q_{(i-1)}.q_i$ in Figure 3 [3].

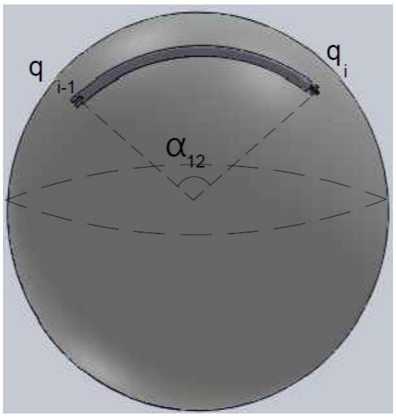

Figure 3. The formation of a path according to the two quaternion points of the robotic arm by the SLERP method

Definition 3.1.2. (The Sequential SLERP Method) Let $t \in \left[0,1\right]$ be a parameter and $q_{(i-1)}, q_i, q_{(i+1)} \in H_1$. Then, the equation yielding the three successive rotation through the spherical mechanisms is $q_{(i-1)}, q_i \in H_1$ where $t \in \left[0,1\right]$. The equation of the position points of the solid objects on the four-dimensional quaternionic hypersphere $r_{n+1} = q_{(n-1)n}^{-1} ... q_{23} q_{12} r_1 q_{12}^{-1} q_{23}^{-1} ... q_{(n-1)n}^{-1}$ is defined as the SLERP method, and this equation is obtained by using the $q_{(i-1)i}\left(t\right) = SLERP\left(q_{(i-1)}, q_i, t\right)$ and the angle $\cos\alpha = q_{(i-1)}.q_i$ between the two quaternions.

Definition 3.1.3. (The Sequential Fast SLERP Method) Let $q_0, q_1, q_i, q_{i+1} \in H_1$ be the unit quaternions where $i = 1, 2, ..., n$. When the quaternions q_i instead of q_0 and q_{i+1} instead of q_1 are taken in the Fast SLERP equation

$$q\left(n\right) = q_0 \cos\left(nk_\theta\right) + q_1 \sin\left(nk_\theta\right),$$

the quaternion $q_{bi} = q_{i+1} - \left(q_i.q_{i+1}\right) q_i$ is obtained. The angle between these quaternions is computed by $\cos\theta = q_i.q_{i+1}$ for $i = 1, 2, ..., n$. The Fast SLERP interpolation between the unit quaternions q_i and q_{i+1} is denoted by $q_i\left(k\right) = q_i \cos\left(k\theta_N\right) + q_{b_i} \sin\left(k\theta_N\right)$ where $0 \le k \le N$ and $\theta_N = \theta / N$. In this equation, the variable k can take an arbitrary value as $k = k_0$ or, for convenience in the computations, it can also be a mean value like $k = 0.5$. When $k = 0.5$ in the Fast SLERP interpolation, the Fast SLERP interpolation value $q_i = \cos\frac{\theta_i}{2} + m_i \sin\frac{\theta_i}{2}$ is obtained. When the position vectors of the joints in the spherical mechanisms are r_i, the rotation process by the unit quaternions q_i of the each of the vector r_i is obtained by $r_{i+1} = q_i\left(r_i\right)q_i^{-1}$ where $i = 1, 2, ..., n$. Then, when these rotations are sequentially written in the sequential SLERP equation which is given in definition 3.1, this method formed by $r_{n+1} = q_n q_{n-1} ... q_2 q_1\left(r_1\right)q_1^{-1}q_2^{-1}...q_{n-1}^{-1}q_n^{-1}$ will be called **the sequential Fast SLERP method.**

Definition 3.1.4. (The Sequential Fast SLERP Method with De-Moivre) Let $q_i, q_{i+1} \in H_1$ be a unit quaternion. When the rule of De-Moivre is used for the complex number $Z = \cos\theta_N + i\sin\theta_N$, the number $Z^k = \cos\left(k\theta_N\right) + i\sin\left(k\theta_N\right)$ is obtained. Then, when this statement is written in the equation $q_i\left(k\right) = \left(q_i, q_{i+1}\right)Z^k$ where $i = 1, 2, ..., n$, the quaternions $q_i \in H_1$ are obtained. Thus, when these quaternions are sequentially applied to the end joint position, the equation $r_{n+1} = q_n q_{n-1} ... q_2 q_1\left(r_1\right)q_1^{-1}q_2^{-1}...q_{n-1}^{-1}q_n^{-1}$ is obtained. This method will be called The Fast SLERP with De-Moivre.

Definition 3.1.5. (The Modular SLERP Interpolation) Let r_i and m_i be the position vectors of the joints in the spherical mechanisms. In this method, all m_i, except r_1 and m_n, rotate about the vector m_n as much as θ_n, and the new vectors become r_1' and m_1'. Then, this process continues until obtaining the vector r_1^n. When the angle between the quaternions $q_i = \cos\dfrac{\theta_i}{2} + m_i\sin\dfrac{\theta_i}{2}$ is computed by $\cos\phi = q_{(i-1)i}.q_{i(i+1)}$ and then, is written in the SLERP equation $q_{i(i+1)}^{*} = q_i\dfrac{\sin\left(\left(1-t\right)\theta_i\right)}{\sin\theta_i} + q_{i+1}\dfrac{\sin\left(t\theta_i\right)}{\sin\theta_i}$, the equation of the SLERP interpolation is $q_{i(i+1)}^{*}\left(t\right) = q_{i(i+1)}^{*}\left(0.5\right)$ where $t = 0.5$. While the rotation of the position vector r_1 by obtained quaternions is yielded by the equation $r_1^i = q_{i(i+1)}^{*}\left(0.5\right)\left(r_1\right)q_{i(i+1)}^{*-1}\left(0.5\right)$, the rotation of the axes m_j^i is yielded by the equation $m_j^i = q_i^{*}\left(m_j^{i-1}\right)q_i^{*-1}$. Thus, the modularly occurred method $r_1^n = q_n^{*}q_{n-1}^{*}...q_{23}^{*}q_{12}^{*}\left(r_1\right)q_{12}^{*-1}q_{23}^{*-1}...q_{n-1}^{*-1}q_n^{*-1}$ will be called the **Modular SLERP interpolation method**.

3.2. The Analysis of the Rotation Motions By The SLERP Interpolation In the Spherical Mechanisms

In this section, the rotation motions of the joints in the spherical mechanisms are analyzed by the SLERP interpolations.

3.2.1. The Analysis of The Rotation Motion In The Spherical Mechanisms of Two-Degree of Freedom by The SLERP Interpolation

Let the unit quaternions yielding the rotations of the joint vectors Z_2, Z_1, Z_0 of a spherical mechanism of two-degree of freedom with $q_{Z_2}, q_{Z_1}, q_{Z_0} \in H_1$. The SLERP interpolation of the unit quaternions q_{Z_2} and q_{Z_1} is yielded by the equation

$$q_{21} = SLERP\left(q_{Z_2}, q_{Z_1}, t\right) = q_{Z_2}\frac{\sin\left(\left(1-t\right)\alpha_{21}\right)}{\sin\alpha_{21}} + q_{Z_1}\frac{\sin\left(t\alpha_{21}\right)}{\sin\alpha_{21}}$$

where the angle $\alpha_{Z_{21}}$ demonstrates the angle between the quaternions q_{Z_2} and q_{Z_1}, and it is computed by the equation $\cos\alpha_{Z_{21}} = q_{Z_2}.q_{Z_1}$. The rotation equation of the vector Z_2, which is the end joint of the spherical mechanism of two-degree of freedom, by the unit quaternion q_{21} is shown by $Z_2^1 = q_{21}\left(Z_2\right)q_{21}^{-1}$. The SLERP interpolation of the unit quaternions q_{Z_1} and q_{Z_0} is given by the equation

$$q_{10} = SLERP\left(q_{Z_1}, q_{Z_0}, t\right) = q_{Z_1}\frac{\sin\left(\left(1-t\right)\alpha_{10}\right)}{\sin\alpha_{10}} + q_{Z_0}\frac{\sin\left(t\alpha_{10}\right)}{\sin\alpha_{10}}$$

where the angle $\alpha_{Z_{10}}$ represents the angle between the quaternions q_{Z_1} and q_{Z_0}, and it is computed by the equation $\cos\alpha_{Z_{10}} = q_{Z_1}.q_{Z_0}$. The equation of rotation of the vector Z_2, which is the end joint of the spherical mechanism of two-degree of freedom, by the unit quaternion q_{10} is shown by the equation $Z_2^2 = q_{10}\left(Z_2^1\right)q_{10}^{-1}$. Also, when the rotation trajectories of the end joint are written by the SLERP method, the rotation motion by the equation $Z_2^2 = q_{10}q_{21}\left(Z_2\right)q_{21}^{-1}q_{10}^{-1}$ has been modeled. Similar operations can be done for the modular motion or the other joint vectors.

3.2.2. The Analysis of The Rotation Motion In The Spherical Mechanisms of Three Degree of Freedom by The SLERP Interpolation

Let the rotation motion for the joints Z_3, Z_2, Z_1, Z_0 in a spherical mechanism with three-degree of freedom be described by the unit quaternions

$q_{Z_3}, q_{Z_2}, q_{Z_1}, q_{Z_0} \in H_1$. The interpolation of the quaternions by SLERP rotation motion is computed by

$$q_{32} = SLERP\left(q_{Z_3}, q_{Z_2}, t\right) = q_{Z_3}\frac{\sin\left(\left(1-t\right)\alpha_{32}\right)}{\sin\alpha_{32}} + q_{Z_2}\frac{\sin\left(t\alpha_{32}\right)}{\sin\alpha_{32}}$$

$$q_{21} = SLERP\left(q_{Z_2}, q_{Z_1}, t\right) = q_{Z_2}\frac{\sin\left(\left(1-t\right)\alpha_{21}\right)}{\sin\alpha_{21}} + q_{Z_1}\frac{\sin\left(t\alpha_{21}\right)}{\sin\alpha_{21}}$$

$$q_{10} = SLERP\left(q_{Z_1}, q_{Z_0}, t\right) = q_{Z_1}\frac{\sin\left(\left(1-t\right)\alpha_{10}\right)}{\sin\alpha_{10}} + q_{Z_0}\frac{\sin\left(t\alpha_{10}\right)}{\sin\alpha_{10}}$$

where the angles are computed by the equations $\cos\alpha_{Z_{32}} = q_{Z_3}.q_{Z_2}$, $\cos\alpha_{Z_{21}} = q_{Z_2}.q_{Z_1}$ and $\cos\alpha_{Z_{10}} = q_{Z_1}.q_{Z_0}$. Then, the rotation motions of the intended joint are respectively done. For example, the position vectors of the joint Z_3 by the rotation by the sequential method are respectively transformed to the vectors $Z_3^1 = q_{32}\left(Z_3\right)q_{32}^{-1}$, $Z_3^2 = q_{21}\left(Z_3^1\right)q_{21}^{-1}$ and $Z_3^3 = q_{10}\left(Z_3^2\right)q_{10}^{-1}$. If the position vector of the motion chain result is wanted to get, it can be computed by writing the equation $Z_3^3 = q_{10}q_{21}q_{32}\left(Z_3\right)q_{32}^{-1}q_{21}^{-1}q_{10}^{-1}$.

3.3. The Design of The Links Between The Joints of The Spherical Mechanisms by The Geometric SLERP

The links between the joints in the spherical mechanisms of 2DoF and 3DoF in the study of Gezgin (2006) and Kilit (2007) are firstly formed by using the geometric SLERPs in this section of our study.

3.3.1. The Design of The Links Between The Joints of The Spherical Mechanisms of Two-Degree of Freedom by The Geometric SLERP

Let $\mathbf{z_0}, \mathbf{z_1} \in R^3$ be the position vectors of the joints of the spherical mechanisms with two-degree of freedom. The equation of the link between the first and the second joints vectors $\mathbf{z_0}$ and $\mathbf{z_1}$ which are taken in this spherical mechanism is obtained by the geometric SLERP method as

$$\mathbf{z_{01}} = Slerp\left(\mathbf{z_0}, \mathbf{z_1}, t\right) = \mathbf{z_0}\frac{\sin\left(\left(1-t\right)\alpha_{10}\right)}{\sin\alpha_{10}} + \mathbf{z_1}\frac{\sin\left(t\alpha_{10}\right)}{\sin\alpha_{10}}.$$

Now, the equation of the link between the second and the third joint vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ is computed by

$$\mathbf{z}_{12} = Slerp\left(\mathbf{z}_1, \mathbf{z}_2, t\right) = \mathbf{z}_1\frac{\sin\left(\left(1-t\right)\alpha_{21}\right)}{\sin\alpha_{21}} + \mathbf{z}_2\frac{\sin\left(t\alpha_{21}\right)}{\sin\alpha_{21}}.$$

Thus, the equations of the links between the joints of the spherical mechanisms of two-degree of freedom have been computed by the help of the geometric SLERP seen as in Figure 4.

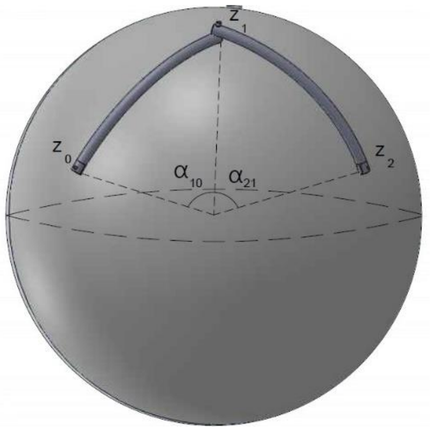

Figure 4. The design of the links between the joints in the 2DoF spherical mechanism by the geometric SLERP

3.3.2. The Design of The Links Between The Joints of The Spherical Mechanisms of Three-Degree of Freedom by The Geometric SLERP

Let the position vectors of the spherical mechanisms of three-degree of freedom to be $\mathbf{z_0}, \mathbf{z_1}, \mathbf{z_2} \in R^3$ The equation of the first link between the joint vectors $\mathbf{z_0}$ and $\mathbf{z_1}$ in the sphere mechanism is denoted by

$$\mathbf{z_{01}} = Slerp\left(\mathbf{z_0}, \mathbf{z_1}, t\right) = \mathbf{z_0}\frac{\sin\left(\left(1-t\right)\alpha_{10}\right)}{\sin\alpha_{10}} + \mathbf{z_1}\frac{\sin\left(t\alpha_{10}\right)}{\sin\alpha_{10}}.$$

Now, the equation of the second link between the second and the third joint vectors $\mathbf{z}_1$ and $\mathbf{z}_2$

$$\mathbf{z}_{12} = Slerp\left(\mathbf{z}_1, \mathbf{z}_2, t\right) = \mathbf{z}_1\frac{\sin\left(\left(1-t\right)\alpha_{21}\right)}{\sin\alpha_{21}} + \mathbf{z}_2\frac{\sin\left(t\alpha_{21}\right)}{\sin\alpha_{21}}$$

is obtained by using the angle α_{21}. The equation of the third link between the vectors $\mathbf{z_2}$ and $\mathbf{z}_3$ is given by

$$\mathbf{z}_{23} = Slerp\left(\mathbf{z}_2, \mathbf{z}_3, t\right) = \mathbf{z}_2\frac{\sin\left(\left(1-t\right)\alpha_{32}\right)}{\sin\alpha_{32}} + \mathbf{z}_3\frac{\sin\left(t\alpha_{32}\right)}{\sin\alpha_{32}}$$

using the angle α_{32}. Thus, the equations of the links between the joints of the spherical mechanisms of the three-degree of freedom have been computed with the help of the geometric SLERP seen as in Figure 5.

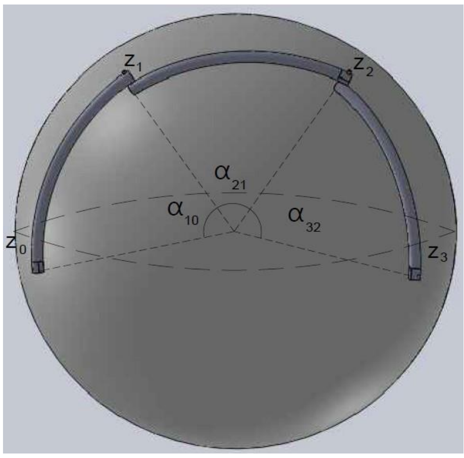

Figure 5. The design of the links between the joints in the 3DoF spherical mechanism by the geometric SLERP

Theorem 3.3.1 (The Serret-Frenet Frame and Curvatures of The Geometric SLERP)
Let $\mathbf{z_0}=(z_{0x},z_{0y},z_{0z})$ and $\mathbf{z_1}=(z_{1x},z_{1y},z_{1z})$ be the unit vectors in the Euclid-3 space and $\beta=Slerp(\mathbf{z_0},\mathbf{z_1},t)$ be the geometric SLERP interpolation formed by the unit vectors $\mathbf{z_0}$ and $\mathbf{z_1}$. The Serret-Frenet frame of the curve β **is**

$$\mathbf{T}=\pm\frac{\mathbf{z_0}\cos(1-t)\alpha+\mathbf{z_1}\cos(t\alpha)}{A},$$

$$\mathbf{N}=\pm\frac{1}{A\sin\alpha}\left\{\begin{array}{l}\left[\cos((1-t)\alpha)+\cos(t\alpha)\cos\alpha\right]\mathbf{z_1}\\-\left[\cos\alpha\cos((1-t)\alpha)+\cos(t\alpha)\right]\mathbf{z_0}\end{array}\right\}$$

$$\mathbf{B}=\frac{\mathbf{z_0}\wedge\mathbf{z_1}}{\sin\alpha},$$

and the curvature and the torsion are obtained by $\kappa=\frac{\sin^3\alpha}{[A]^3}$ and $\tau=0$ where

$$A=\sqrt{\cos^2((1-t)\alpha)+\cos^2(t\alpha)-2\cos((1-t)\alpha)\cos(t\alpha)\cos\alpha}$$

for the abbreviation.

Proof: First, the angle values are computed by the $\mathbf{z_0}.\mathbf{z_1}=\cos\alpha$ between the vectors which are used in the geometric SLERP interpolation. When the first derivative of the SLERP interpolation curve $\beta=Slerp(\mathbf{z_0},\mathbf{z_1},t)=\mathbf{z_0}\frac{\sin((1-t)\alpha)}{\sin\alpha}+\mathbf{z_1}\frac{\sin(t\alpha)}{\sin\alpha}$ of the two vectors is taken, it's computed as

$$\beta'=\mathbf{z_0}\left(\frac{\sin((1-t)\alpha)}{\sin\alpha}\right)'+\mathbf{z_1}\left(\frac{\sin(t\alpha)}{\sin\alpha}\right)'$$

$$=\left(\begin{array}{l}\mathbf{z_{0x}}\frac{-\alpha\cos(1-t)\alpha}{\sin\alpha}+\mathbf{z_{1x}}\frac{\alpha\cos(t\alpha)}{\sin\alpha},\\ \mathbf{z_{0y}}\frac{-\alpha\cos(1-t)\alpha}{\sin\alpha}+\mathbf{z_{1y}}\frac{\alpha\cos(t\alpha)}{\sin\alpha},\\ \mathbf{z_{0z}}\frac{-\alpha\cos(1-t)\alpha}{\sin\alpha}+\mathbf{z_{1z}}\frac{\alpha\cos(t\alpha)}{\sin\alpha}\end{array}\right)$$

The norm of the first derivative of the curve β is denoted as $\|\beta'\|=A.\left|\frac{\alpha}{\sin\alpha}\right|$. The tangent vector field of the non-unit speed geometric SLERP curve β is computed by the equation

$$\mathbf{T}=\frac{\beta'}{\|\beta'\|}=\pm\frac{\mathbf{z_0}\cos(1-t)\alpha+\mathbf{z_1}\cos(t\alpha)}{A}.$$

The binormal vector $\mathbf{B}=\frac{\mathbf{z_0}\wedge\mathbf{z_1}}{\sin\alpha}$ of the geometric SLERP curve β is obtained by using its second derivative

$$\beta''=-\alpha^2\left[\mathbf{z_0}\frac{\sin((1-t)\alpha)}{\sin\alpha}+\mathbf{z_1}\frac{\sin(t\alpha)}{\sin\alpha}\right]$$

in the cross product $\beta'\wedge\beta''=\frac{\alpha^3}{\sin\alpha}.\mathbf{z_0}\wedge\mathbf{z_1}$. Then, the principal normal is found by

$$\mathbf{N}=\mathbf{B}\wedge\mathbf{T}=\pm\frac{1}{A\sin\alpha}\left\{\begin{array}{l}\left[\cos((1-t)\alpha)+\cos(t\alpha)\cos\alpha\right]\mathbf{z_1}\\-\left[\cos\alpha\cos((1-t)\alpha)+\cos(t\alpha)\right]\mathbf{z_0}\end{array}\right\}$$

In this case, the Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ of the geometric SLERP interpolation β has been computed. The curvature of this curve is obtained by

$$\kappa=\frac{\|\beta'\wedge\beta''\|}{\|\beta'\|^3}=\frac{\alpha^3}{\left\|\left|\frac{\alpha}{\sin\alpha}\right|A\right\|^3}=\frac{\sin^3\alpha}{[A]^3},$$

and the torsion is found by

$$\tau=\frac{(\beta'\ \beta''\ \beta''')}{\|\beta'\wedge\beta''\|^2}$$

$$=-\frac{1}{\sin\alpha}\left\langle\mathbf{z_0}\wedge\mathbf{z_1},\left[\mathbf{z_0}\frac{-\alpha\cos(1-t)\alpha}{\sin\alpha}+\mathbf{z_1}\frac{\alpha\sin(t\alpha)}{\sin\alpha}\right]\right\rangle=0.$$

The derivative formulas of the frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ of the geometric SLERP interpolation β are computed by

$$\mathbf{T'}=\frac{|\alpha|\sin^2\alpha}{A^2}\mathbf{N},\quad \mathbf{N'}=-\frac{|\alpha|\sin^2\alpha}{A^2}\mathbf{T},\quad \mathbf{B'}=0$$

and the arch-length of this geometric interpolation curve is found by the equation $\int \|\beta'\| dt = \left|\frac{\alpha}{\sin\alpha}\right| \int A dt.$

4. NUMERIC EXAMPLES

Example 4.1. Let the rotation motions of the joint $\mathbf{Z}_3 = (1,0,0)$ in a spherical mechanism of 3DoF be described by the quaternions

$q_{Z3} = (0,1,0,0)$, $q_{Z2} = (0,0,1,0)$,

$q_{Z1} = (0,0,0,1)$ and $q_{Z0} = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$.

The computation of the angle between q_{Z3} and q_{Z2} is obtained by $q_{Z3}.q_{Z2} = \cos\alpha_{32} = 0$ and $\alpha_{32} = 90^\circ$, and the rotation motion for the value $t = 0.5$ is computed as

$$q_{32} = SLERP\left(q_{Z_3}, q_{Z_2}, 0.5\right) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right).$$

Then, with the help of this quaternion, the joint position is computed by $\mathbf{Z}_3^1 = (0 + 0i + j + 0k)$. Let rotate the joint $\mathbf{Z}_3^1$ by the quaternions $q_{Z2} = (0,0,1,0)$ and $q_{Z1} = (0,0,0,1)$ again. The angle between these quaternions is found from the equation $q_{Z2}.q_{Z1} = \cos\alpha_{21} = 0$ as $\alpha_{21} = 90^\circ$, and the quaternion yielding the rotation motion for the value $t = 0.5$ is computed as

$$q_{21} = SLERP\left(q_{Z_2}, q_{Z_1}, 0.5\right) = \left(0, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

Thus, the position vector, as a result of the rotation of the joint Z_3^1, is computed as $Z_3^2 = (0 + 0i + 0j + k)$. Now, let move the position vector Z_3^2 by the SLERP interpolation of the unit quaternions $q_{Z1} = (0,0,0,1)$ and $q_{Z0} = \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$. The angle between these two quaternions is found from equality $q_{Z1}.q_{Z0} = \cos\alpha_{10} = 0$ as $\alpha_{10} = 90^\circ$ and the interpolation value for the value $t = 0.5$ is computed as

$$q_{10} = SLERP\left(q_{Z_1}, q_{Z_0}, 0.5\right) = \left(0, \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{2}\right).$$

Therefore, the result position vector $Z_3^3 = \left(0 + \frac{\sqrt{3}}{2} i + \frac{1}{2} j + 0k\right)$ has been computed. Thus, the rotation motions of the joint $Z_3 = (1,0,0)$ by the SLERP quaternion interpolation have been designed.

Example 4.2. Let the position vectors of the joints of a spherical mechanism of 2DoF be $\mathbf{z_0} = (1,0,1)$, $\mathbf{z_1} = (0,1,1)$ and $\mathbf{z_2} = (1,1,0)$. The link equation between the joint vectors $\mathbf{z}_0$ and $\mathbf{z}_1$ is computed by the interpolation

$$\begin{aligned}\mathbf{z_{01}} &= Slerp(\mathbf{z_0}, \mathbf{z_1}, t)\\ &= (1,0,1)\frac{\sin(\pi(1-t)/3)}{\sin\pi/3} + (0,1,1)\frac{\sin(\pi t/3)}{\sin\pi/3}\end{aligned}$$

where the angle between the joint vectors is $\alpha_{10} = \pi/3.$ The value of the interpolation at $t = 0.5$ is computed as $\mathbf{z_{01}}(0.5) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. Let compute the equation of the link between the joints $\mathbf{z_1}$ and $\mathbf{z_2}$ after the formation of the first link. Since the angle between the joint vectors $\mathbf{z_1}$ and $\mathbf{z_2}$ is $\alpha_{21} = \pi/3$, the equation of the link between these joints is computed by the geometric SLERP method as

$$\begin{aligned}\mathbf{z_{01}} &= Slerp(\mathbf{z_0}, \mathbf{z_1}, t)\\ &= (0,1,1)\frac{\sin(\pi(1-t)/3)}{\sin\pi/3} + (1,1,0)\frac{\sin(\pi t/3)}{\sin\pi/3}\end{aligned}$$

.

The interpolation value of the computed link at $t = 0.5$ is obtained by $\mathbf{z}_{12}(0.5) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. The values of the initial, mid, and endpoints of the link interpolation equation between the joints are given in Table 1.

Table 1. The values of the initial, mid, and endpoints of the links of the 2DoF mechanism

$t \in [0,1]$	$Slerp(\mathbf{z_0}, \mathbf{z_1}, t)$ interpolation (the link between $\mathbf{z_0}$ and $\mathbf{z_1}$)	$Slerp(\mathbf{z}_1, \mathbf{z}_2, t)$ interpolation (the link between $\mathbf{z}_1$ and $\mathbf{z}_2$)
$t=0$	$\mathbf{z}_0 = (1,0,1)$	$\mathbf{z_1} = (0,1,1)$
$t=0.5$	$\mathbf{z_{01}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	$\mathbf{z}_{12} = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
$t=1$	$\mathbf{z_1} = (0,1,1)$	$\mathbf{z}_2 = (1,1,0)$

Example 4.3. Let the joint vectors of a spherical mechanism of 3DoF be $\mathbf{z_0} = (1,0,1)$, $\mathbf{z_1} = (0,1,1)$, $\mathbf{z_2} = (1,1,0)$ and $\mathbf{z}_3 = (1,1,1)$. The equation of the first link is

$$\mathbf{z_{01}} = Slerp(\mathbf{z_0}, \mathbf{z_1}, t)$$
$$= (1,0,1)\frac{\sin(\pi(1-t)/3)}{\sin \pi/3} + (0,1,1)\frac{\sin(t\pi/3)}{\sin \pi/3},$$

and the value of this interpolation curve at the value $t = 0.5$ is computed by $\mathbf{z_{01}}(0.5) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

Then, the angle between the joint vectors $\mathbf{z_0}$ and $\mathbf{z_1}$ is computed $\alpha_{10} = \pi/3$. Thus, the equation of the second link between the joints $\mathbf{z_1}$ and $\mathbf{z_2}$

$$\mathbf{z}_{12} = Slerp(\mathbf{z}_1, \mathbf{z}_2, t)$$
$$= (0,1,1)\frac{\sin((1-t)\pi/3)}{\sin \pi/3} + (1,1,0)\frac{\sin(t\pi/3)}{\sin \pi/3}$$

is computed by the geometric SLERP equation. The value of this interpolation curve at the value $t = 0.5$ is obtained $\mathbf{z}_{12}(t) = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Now, let compute the equation of the third link of the spherical mechanism. The interpolation curve between the joint vectors $\mathbf{z_2} = (1,1,0)$ and $\mathbf{z}_3 = (1,1,1)$ is given by

$$\mathbf{z}_{23} = Slerp(\mathbf{z}_2, \mathbf{z}_3, t) = \mathbf{z}_2 \frac{\sin((1-t)\alpha_{32})}{\sin \alpha_{32}} + \mathbf{z}_3 \frac{\sin(t\alpha_{32})}{\sin \alpha_{32}}.$$

The vector $\mathbf{z}_{23}(0.5) = (2,2,1)\sqrt{\frac{\sqrt{6}-2}{2\sqrt{6}}}\sqrt{3}$ is computed for the value $t = 0.5$.

Table 2. The values of the initial, mid, and endpoints of the links of the 3DoF mechanism

$t \in [0,1]$	$Slerp(\mathbf{z_0}, \mathbf{z_1}, t)$ interpolation (the link between $\mathbf{z_0}$ and $\mathbf{z_1}$)	$Slerp(\mathbf{z}_1, \mathbf{z}_2, t)$ interpolation (the link between $\mathbf{z}_1$ and $\mathbf{z}_2$)	$Slerp(\mathbf{z}_2, \mathbf{z}_3, t)$ interpolation (the link between $\mathbf{z}_2$ and $\mathbf{z}_3$)
$t=0$	$\mathbf{z}_0 = (1,0,1)$	$\mathbf{z_1} = (0,1,1)$	$\mathbf{z_2} = (1,1,0)$
$t=0.5$	$\mathbf{z_{01}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	$\mathbf{z}_{12} = \left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\mathbf{z}_{23} = (2,2,1)\sqrt{\frac{\sqrt{6}-2}{2\sqrt{6}}}\sqrt{3}$
$t=1$	$\mathbf{z_1} = (0,1,1)$	$\mathbf{z}_2 = (1,1,0)$	$\mathbf{z}_3 = (1,1,1)$

Example 4.4. The geometric SLERP interpolation curve between the initial vector $\mathbf{z_0} = (1,0,0)$ and the end vector $\mathbf{z_1} = (0,1,0)$ is denoted by the equation $\beta = Slerp(z_0, z_1, t) = \left(\sin\left((1-t)\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}t\right), 0\right)$. The tangent vector of this obtained geometric SLERP β is computed as

$$\mathbf{T} = \left(\frac{-\cos\frac{\pi}{2}(1-t)}{\sqrt{\cos^2\frac{\pi}{2}(1-t) + \cos^2\frac{\pi}{2}t}}, \frac{\cos\frac{\pi}{2}t}{\sqrt{\cos^2\frac{\pi}{2}(1-t) + \cos^2\frac{\pi}{2}t}}, 0\right)$$

The binormal vector $\mathbf{B} = (0,0,1)$ and the principal normal vector of the geometric SLERP is obtained by the equations

$$\mathbf{N}=\left(\frac{-\cos\frac{\pi}{2}t}{\sqrt{\cos^2\frac{\pi}{2}(1-t)+\cos^2\frac{\pi}{2}t}},\frac{-\cos\frac{\pi}{2}(1-t)}{\sqrt{\cos^2\frac{\pi}{2}(1-t)+\cos^2\frac{\pi}{2}t}},0\right).$$

The curvature of the interpolation curve is $\kappa=\dfrac{1}{\sqrt{\cos^2(1-t)\frac{\pi}{2}+\cos^2\frac{\pi}{2}t}}$ and the torsion is $\tau=0$. Also, the derivative formulas of the Frenet frame of the geometric SLERP are computed as

$$T'=\left(\frac{-\frac{\pi}{2}\cos\frac{\pi}{2}t}{\sqrt{\left(\cos^2\frac{\pi}{2}(1-t)+\cos^2\frac{\pi}{2}t\right)^3}},\frac{-\frac{\pi}{2}\cos\frac{\pi}{2}(1-t)}{\sqrt{\left(\cos^2\frac{\pi}{2}(1-t)+\cos^2\frac{\pi}{2}t\right)^3}},0\right)$$

$$N'=\left(\frac{\frac{\pi}{2}\cos\frac{\pi}{2}(1-t)}{\sqrt{\left(\cos^2\frac{\pi}{2}(1-t)+\cos^2\frac{\pi}{2}t\right)^3}},\frac{-\frac{\pi}{2}\cos\frac{\pi}{2}t}{\sqrt{\left(\frac{\pi}{2}\cos^2(1-t)+\cos^2\frac{\pi}{2}t\right)^3}},0\right).$$

5. CONCLUSION

In our study, we have defined the sequential and the sequential fast SLERP in the spherical mechanisms firstly by utilizing the definition of the SLERP interpolation. These interpolation methods make us transact faster in robotic arm motions. Also, with the help of the definition of the geometric SLERP, the design of the links between the joints in the spherical mechanisms has been drawn. Additionally, the Serret-Frenet frame and curvatures of the geometric SLERP interpolation have been computed in order to find the location and the curvatures of these joints.

DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Hatice KUŞAK SAMANCI: Put forward the first idea on the stated, developed and wrote the manuscript, contributed at all stages of the article.

Çetin KUŞÇU: Performed the numeric examples and tables. Also, analyzed some calculations and results.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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