On Δ-Uniform and Δ-Pointwise Convergence on Time Scale

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Abstract

In this article, we define the concept of Δ-Cauchy, Δ-uniform convergence and Δ-pointwise convergence of a family of functions \( \{f_j\}_{j \in \mathbb{J}} \), where \( \mathbb{J} \) is a time scale. We study the relationships between these notions. Moreover, we introduced sufficient conditions for interchangeability Δ-limitation with Riemann Δ-integration or Δ-differentiation. Also, we obtain the analogue of the well-known Dini's Theorem.

Keywords: Δ-Convergence; Δ-Cauchy; Statistical convergence.

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Zaman Skalası Üzerinde Δ-Düzgün ve Δ-Noktasal Yakınsaklık

Öz

Bu makalede \( \mathbb{J} \) bir zaman skalası olmak üzere, \( \{f_j\}_{j \in \mathbb{J}} \) fonksiyon ailesi için Δ-Cauchy, Δ-düzgün yakınsaklık ve Δ-noktasal yakınsaklık kavramları verilerek bu kavramlar arasındaki ilişkiler incelenmiştir. Δ-limit ile Riemann Δ-integralli ve Δ-türevin yer değişme problemi araştırılarak Dini Teoreminin farklı bir versiyonu elde edilmiştir.

Anahtar Kelimeler: Δ-Yakınsaklık; Δ-Cauchy; İstatistiksel yakınsaklık.
1. Introduction and Preliminaries

The time scale calculus was introduced in 1989 by German mathematician Stefan Hilger [1]. It is a unification of the theory of differential equations with that of difference equations. This theory was developed to a certain extent in [2] by Hilger.

The notion of statistical convergence for complex number sequences was introduced by Fast in [3]. Schoenberg gave some properties of this concept [4]. Fridy progressed with the statistically Cauchy and showed the equivalence of these concepts in [5].

In recent years, there are many studies based on the density function, which is defined on some subsets of time scale. For instance, first author and Tan [6] gave the notions of \( \Delta \)-Cauchy and \( \Delta \)-convergence of a function defined on time scale by using \( \Delta \)-density. The notion of \( m \)- and \( (\lambda, m) \)- uniform density of a set and the concept of \( m \)- and \( (\lambda, m) \)- uniform convergence on a time scale were presented by Altin et al. [7]. Also, Altin et al. gave \( \lambda \)-statistical convergence on time scale and examined some of its features [8]. Some fundamental properties of Lacunary statistical convergence and statistical convergence on time scale investigated by Turan and Duman in [9].

Let \( S \) be the collection of all subsets of time scale \( J \) in the form of \([a, b)\), where \([a, a) = \emptyset\). Then \( S \) is a semiring on \( J \). The set function \( m \) defined by \( m([a, b)) = b - a \) is a measure on \( S \). The outer measure \( m^* : S \to [0, \infty) \) generated by \( m \) is defined by

\[
m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) : A \subset \bigcup_{n=1}^{\infty} [a_n, b_n) \right\}.
\]

The family of all \( m^* \)-measurable (it is also called \( \Delta \)-measurable) sets \( \mathcal{M} = \mathcal{M}(m^*) \) is a \( \sigma \)-algebra and it is well known that from the measure theory the restriction of \( m^* \) to \( \mathcal{M} \), which we denote by \( \mu_\Delta \), is a measure. This measure is called Lebesgue \( \Delta \)-measure on \( J \).

**Definition 1.** [5] Let \( A \subset \mathbb{N} \), and

\[
A_n = \sum_{m \leq n, m \in A} 1.
\]

The asymptotic density of \( A \) is defined by \( \delta(A) = \lim_n n^{-1} A_n \), which is also called natural density. The real number sequence \( x = (x_n) \) is statistically convergent to \( l \) if for each \( \epsilon > 0 \), \( \delta\{n \in \mathbb{N} : |x_n - l| \geq \epsilon \} = 0 \); in this case we write \( \text{st-lim } x = l \).

From now on we assume that \( \sup J = \infty \) and \( J \) has a minimum for the time scale \( J \).
Definition 2. (Δ-Density) [6] Let $B$ be a subset of $\mathbb{J}$ such that $B \in \mathcal{M}$ and $a = \min \mathbb{J}$. Δ-density of $B$ in $\mathbb{J}$ is defined by

$$\delta_\Delta(B) := \lim_{j \to \infty} \frac{\mu_\Delta(B \cap [a, j])}{\sigma(j) - a}$$

provided that this limit exists.

A property of points of $\mathbb{J}$ is said to hold Δ-almost everywhere (or Δ-almost all $j \in \mathbb{J}$) if the set of points in $\mathbb{J}$ at which it fails to hold has zero Δ-density. The expression Δ-almost everywhere abbreviated to Δ-a.e.

Definition 3. (Δ-Convergence) [6] If for every $\epsilon > 0$, the inequality $|g(j) - l| < \epsilon$ holds Δ-a.e. on $\mathbb{J}$, then $g: \mathbb{J} \to \mathbb{R}$ is called Δ-convergent to $l \in \mathbb{R}$ (or has Δ-limit). In this case we write $\Delta \lim_{j \to \infty} f(j) = l$.

Definition 4. (Δ-Cauchy) [6] The function $g: \mathbb{J} \to \mathbb{R}$ is Δ-Cauchy provided that for each $\epsilon > 0$, there exist $K = K(\epsilon) \subset \mathbb{J}$ and $j_0 \in \mathbb{J}$ such that $\delta_\Delta(K) = 1$ and $|g(j) - g(j_0)| < \epsilon$ holds for all $j \in K$.

Note that the Δ-density, Δ-Cauchy and Δ-Convergence coincide with the natural density, statistical Cauchy and statistical convergence respectively whenever $\mathbb{J}$ is the natural numbers.

2. Δ-Pointwise and Δ-Uniform Convergence

In this section, we will deal with the family of functions $\{f_j\}_{j \in \mathbb{J}}$ whose elements defined on any subset of real numbers.

Definition 5. (Δ-Pointwise Convergence) Let $B \subset \mathbb{R}$ and for each $j \in \mathbb{J}$, $f_j$ and $f$ be real valued functions on $B$. The family $\{f_j\}_{j \in \mathbb{J}}$ converges Δ-pointwise to $f$ on $B$, if for each given $\epsilon > 0$ and $t \in B$, the inequality $|f_j(t) - f(t)| < \epsilon$ holds Δ-a.e. on $\mathbb{J}$. This notion is abbreviated as $\{f_j\}_{j \in \mathbb{J}} \to f$ on $B$.

Definition 6. (Δ-Uniform Convergence) Let $B \subset \mathbb{R}$ and for each $j \in \mathbb{J}$, $f_j$ and $f$ be real valued functions on $B$. The family $\{f_j\}_{j \in \mathbb{J}}$ converges Δ-uniformly to $f$ on $B$, if for each given $\epsilon > 0$, the inequality $|f_j(t) - f(t)| < \epsilon$ holds Δ-a.e. on $\mathbb{J}$ and for all $t \in B$. In this case we write $\{f_j\}_{j \in \mathbb{J}} \Rightarrow f$ on $B$. 

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Definition 7. (Δ-Uniform Cauchy) Let $B \subseteq \mathbb{R}$ and $\{f_j\}$ be a family of real valued functions defined on $B$. The family $\{f_j\}_{j \in J}$, Δ-uniform Cauchy on $B$, if for all $\epsilon > 0$ there exists a subset $K = K(\epsilon)$ of $J$ and $j_0 \in J$ such that $\delta_\Delta(K) = 1$ and $|f_j(t) - f_{j_0}(t)| < \epsilon$ for all $j \in K$ and for all $t \in B$.

Example 8. Let $J = [0, \infty)$ and $B \subseteq \mathbb{R}$. We denote the irrational and rational numbers in $[0, \infty)$ by $\mathbb{I}_{[0, \infty)}$ and $\mathbb{Q}_{[0, \infty)}$, respectively. We consider the functions $f_j : B \to \mathbb{R}$ $(j \in J)$ defined as:

$$f_j(t) = \begin{cases} \sin j t, & \text{if } j \in \mathbb{Q}_{[0, \infty)} \\ 0, & \text{if } j \in \mathbb{I}_{[0, \infty)} \end{cases}.$$ 

Since the set $\mathbb{Q}_{[0, \infty)}$ has zero density in $J$, the density of $\mathbb{I}_{[0, \infty)}$ is one. Hence, $\{f_j\}_{j \in J} \Rightarrow f = 0$ on $B$.

It is easily seen that Δ-uniform convergence implies Δ-pointwise convergence, but the converse is not always true as we can see from the following counter-example.

Example 9. Let $J = [1, \infty)$ and $j \in J$. Consider the functions $f_j : [0, \infty) \to \mathbb{R}$ defined as:

$$f_j(t) = \begin{cases} t, & \text{if } j \in \mathbb{Q}_{[1, \infty)} \\ 0, & \text{if } j \in \mathbb{I}_{[0, \infty)} \end{cases}.$$ 

Although $\{f_j\}_{j \in J}$ is Δ-pointwise convergent to $f = 0$, it is not Δ-uniform convergent.

The proof of the following theorem is clear.

Theorem 10. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real valued functions defined on $B \subseteq \mathbb{R}$. If $(f_n)_{n \in \mathbb{N}}$ converges uniformly (pointwise) to $f$, then $(f_n)_{n \in \mathbb{N}}$ converges Δ-uniformly (Δ-pointwise) to $f$.

Theorem 11. Let $\{f_j\}_{j \in J}$ be a family of real valued functions defined on $B \subseteq \mathbb{R}$. If $\{f_j\}_{j \in J} \Rightarrow f$ on $B$, then $\{f_j\}_{j \in J} \Rightarrow f$ on $B$ if and only if

$$\Delta - \limsup_{f \to \infty} \sup_{t \in B} |f_j(t) - f(t)| = 0.$$

Theorem 12. Let $\{f_j\}_{j \in J}$ be a family of real valued functions defined on $B \subseteq \mathbb{R}$. $\{f_j\}_{j \in J} \Rightarrow f$ on $B$ if and only if it is Δ-uniform Cauchy on $B$.

Proof. Necessity is obvious. Let $\{f_j\}_{j \in J}$ be Δ-uniform Cauchy on $B$. For a given $\epsilon > 0$ there exists $j_0 \in J$ and $K \subseteq J$ such that $\delta_\Delta(K) = 1$, the inequality
$$|f_j(t) - f_{j_0}(t)| < \frac{\epsilon}{2},$$ \hspace{1cm} (1)\]

holds for all $j \in K$ and $t \in B$. Let $g_t: \mathbb{J} \rightarrow \mathbb{R}$ defined by $g_t(f) = f_j(t)$ for each $t \in B$. For each fixed $t$

$$|g_t(j) - g_t(j_0)| = |f_j(t) - f_{j_0}(t)| < \epsilon,$$

holds $\Delta$-a.e. on $\mathbb{J}$. Therefore, the functions $g_t$, $(t \in B)$ are $\Delta$-Cauchy. These functions have $\Delta$-limit. Let $f(t) = \Delta \lim_{j \to \infty} g_t(j)$. As $j \to \infty$, the $\Delta$-limit of (1) yields

$$|f(t) - f_{j_0}(t)| \leq \frac{\epsilon}{2},$$ \hspace{1cm} (2)\]

In view of inequalities (1) and (2), one can get

$$|f_j(t) - f(t)| \leq |f_j(t) - f_{j_0}(t)| + |f_{j_0}(t) - f(t)| < \epsilon,$$

for all $j \in K$ and for all $t \in B.$

**Theorem 13.** Let $\mathbb{T}$ and $\mathbb{J}$ be two time scales and $[\alpha, \beta] \subset B \subset \mathbb{T}$. If $f_j \in C_{rd}(B, \mathbb{R}) := \{f | f: B \rightarrow \mathbb{R} \text{ is rd - continuous} \}$ for all $j \in \mathbb{J}$, and $\{f_j\}_{j \in \mathbb{J}} \rightrightarrows f$, then $f \in C_{rd}(B, \mathbb{R})$ and

$$\Delta - \lim_{j \to \infty} \int_{\alpha}^{\beta} f_j(t) \Delta t = \int_{\alpha}^{\beta} f(t) \Delta t.$$

**Proof.** Let any positive $\epsilon$ be given. In accordance with $\Delta$-uniform convergence, the time scale $\mathbb{J}$ has a subset $K$ such that $\delta_\Delta(K) = 1$ and the inequality

$$|f_j(t) - f(t)| < \frac{\epsilon}{3}$$

holds for all $j \in K$ and for all $t \in B$.

Let $j_0 \in K$ and $t_0 \in B$ are arbitrary. We consider two cases. In the first case we assume that $t_0$ is left-dense. From rd-continuity of $f_{j_0}$, we can find $\delta > 0$ such that

$$|f_{j_0}(\xi) - f_{j_0}(\eta)| < \frac{\epsilon}{3},$$

for any $\xi, \eta \in (t_0 - \delta, t_0)$. If $t_n \to t_0^-$ as $n \to \infty$, then there exists natural number $n_0$ such that $n, m > n_0$ imply $t_m, t_n \in (t_0 - \delta, t_0)$ and

$$|f_{j_0}(t_n) - f_{j_0}(t_m)| < \frac{\epsilon}{3}.$$ \hspace{1cm} (3)\]
Hence, for \( m, n > n_0 \), we have
\[
|f(t_n) - f(t_m)| = |f(t_n) - f_{j_0}(t_n) + f_{j_0}(t_n) - f_{j_0}(t_m) + f_{j_0}(t_m) - f(t_m)|
\]
\[
\leq |f(t_n) - f_{j_0}(t_n)| + |f_{j_0}(t_n) - f_{j_0}(t_m)|
\]
\[
+ |f_{j_0}(t_m) - f(t_m)|
\]
\[
< \epsilon. \tag{4}
\]

Therefore, the function \( f \) has finite left-sided limit at \( t_0 \).

In the second case we assume that \( t_0 \) is right-dense. Then all functions \( f_j \) are continuous at \( t_0 \). If \( t_n \to t_0 \) as \( n \to \infty \), then there exists natural number \( n_0 \) such that \( n, m > n_0 \) imply \( t_m, t_n \in (t_0 - \delta, t_0 + \delta) \) and (3-4) holds. This is implies continuity of \( f \) at \( t_0 \). Therefore, \( f \) is Riemann \( \Delta \)-integrable on every subinterval \( [\alpha, \beta] \subset \mathbb{B} \). So, we obtain the inequality
\[
\left| \int_\alpha^\beta f_j(t) \Delta t - \int_\alpha^\beta f(t) \Delta t \right| \leq \int_\alpha^\beta |f_j(t) - f(t)| \Delta t < \frac{\epsilon}{3}(\beta - \alpha),
\]
for every \( j \in K \) that completes our proof.

**Theorem 14.** Let \( \mathbb{T} \) and \( \mathbb{J} \) be two time scales and \( [\alpha, \beta] \subset \mathbb{T} \). Suppose that the functions
\[
f_j: [\alpha, \beta] \to \mathbb{R} \quad (j \in \mathbb{J})
\]
satisfies the following conditions on \( [\alpha, \beta] \):

1. \( f_j \) has Hilger derivative and its Hilger derivative \( f_j^\Delta \) is rd-continuous,

2. \( \{f_j\}_{j \in \mathbb{J}} \to f \),

3. \( \{f_j^\Delta\}_{j \in \mathbb{J}} \Rightarrow g \).

Then \( f \) has Hilger derivative on \( [\alpha, \beta] \) and \( f^\Delta(t) = g(t) \) for all \( t \in [\alpha, \beta] \).

**Proof.** \( g \) is rd-continuous on \( [\alpha, \beta] \) by Theorem 13 and so \( g \) is Riemann \( \Delta \)-integrable on this interval. By the help of Theorem 13, we have
\[
\int_\alpha^\ell g(s) \Delta s = \Delta - \lim_{j \to \infty} \int_\alpha^\ell f_j^\Delta f(s) \Delta s = \Delta - \lim_{j \to \infty} (f_j(t) - f_j(\alpha)) = f(t) - f(\alpha),
\]
for all $t \in [\alpha, \beta]$. Since the left hand-side of the last equality has Hilger derivative, the right hand-side also has, and it follows that $f^\Delta(t) = g(t)$ for all $t \in [\alpha, \beta]$.

**Theorem 15.** (Dini’s Theorem) Let $X$ be a compact metric space. Let $f : X \to \mathbb{R}$ be a continuous function and the functions $f_j : X \to \mathbb{R}$, $(j \in \mathbb{J})$ are continuous for $\Delta$-almost all $\mathbb{J}$. If the following two conditions are satisfied:

1. $\{f_j\}_{j \in \mathbb{J}} \to f$ on $X$,

2. $f_j(x) \leq f_i(x)$ for all $x \in X$ and $\Delta$-almost all $i, j \in \mathbb{J}$ such that $i < j$,

then $\{f_j\}_{j \in \mathbb{J}} \rightrightarrows f$ on $X$.

**Proof.** There exists a subset $K_1 \subset \mathbb{J}$ with $\Delta$-density 1. Moreover, for each $j \in K_1$ the functions $f_j$ are continuous, and

$$f_j(x) \leq f_i(x) \quad \text{for all} \quad x \in X,$$

holds for all $i, j \in K_1$ such that $i < j$. For each $j \in K_1$, define $g_j = f_j - f$. Then $\{g_j\}_{j \in K_1}$ is a family of continuous functions on the compact metric space $X$ that converges $\Delta$-pointwise to 0. Furthermore,

$$0 \leq g_j(x) \leq g_i(x),$$

for all $x \in X$ and $i, j \in K_1$ such that $i < j$.

Let $\varepsilon > 0$ and define

$$G_j = \{x \in X : g_j(x) < \varepsilon\}, \quad (j \in K_1).$$

Since $g_j$ is continuous, then $G_j$ is an open set and $G_i \subseteq G_j$ for each $i, j \in K_1$ such that $i < j$.

Let $x_0 \in X$ be arbitrary. Since $\Delta\lim_{j \to \infty} g_j(x_0) = 0$, then there exists a subset $K_2 \subset \mathbb{J}$ such that $\delta_\Delta(K_2) = 1$ and the inequality $|g_j(x_0)| < \varepsilon$ holds for all $j \in K_2$. If we set $K = K_1 \cap K_2$ then $\delta_\Delta(K) = 1$ and $g_j(x_0) = |g_j(x_0)| < \varepsilon$ for all $j \in K$. Thus $x_0 \in G_j$ for all $j \in K$, and thus, we have

$$X = \bigcup_{j \in K} G_j.$$
Since $K$ is compact and $G_i \subset G_j$ when $i < j$, then there is a $j_0 \in K$ with $G_{j_0} = X$. Then we have $G_j = X$ for all $j \in K$ such that $j > j_0$. This implies that $f_j(x) - f(x) = g_j(x) < \varepsilon$ for all $x \in X$ and $j \in K$ such that $j > j_0$. Consequently, $\{f_j\}_{j \in J} \nrightarrow f$ on $X$.

References


