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f-Asymptotically $\mathcal{I}_2^{\sigma\theta}$-Equivalence for Double Set Sequences

Küme Dizilerinin f-Asimptotik $\mathcal{I}_2^{\sigma\theta}$-Denkliği

Erdinç Dündar[1,*], Nimet Pancaroğlu Akın[2]

[1]Afyon Kocatepe University, Science and Arts Faculty, Department of Mathematics, Afyonkarahisar, Turkey
[2]Afyon Kocatepe University, Education Faculty, Department of Mathematics and Science Education, Afyonkarahisar, Turkey

Abstract

Recently, Pancaroğlu Akın et al. (2018) defined and studied f-asymptotically $\mathcal{I}_{\sigma\theta}$-statistical equivalence for sequences of sets. In this paper, firstly, we denote the notions of strongly asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence, f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence, strongly f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence for double set sequences. Secondly, we investigate some relationships and important properties among these new notions. Then, we denoted asymptotically $\mathcal{I}_2^{\sigma\theta}$-statistical equivalence for double set sequences. Also, we examine inclusion and necessity relations between them.

Keywords: Asymptotic equivalence, Lacunary invariant convergence, Modulus function, $\mathcal{I}_2$-convergence, Wijsman convergence

Öz

Son zamanlarda, Pancaroğlu Akın vd. (2018) küme dizileri için f-asimptotik $\mathcal{I}_{\sigma\theta}$-istatistiksel denkliğini tanımladılar ve çalıştılar. Bu makalede öncelikli olarak, çift küme dizileri için kuvvetli asimptotik $\mathcal{I}_2^{\sigma\theta}$-denkliği, f-asimptotik $\mathcal{I}_2^{\sigma\theta}$-denkliği, kuvvetli f-asimptotik $\mathcal{I}_2^{\sigma\theta}$-denkliği tanımları verildi. İkinci olarak, bu kavramların bazı önemli özellikleri ve arasındaki ilişkiler araştırıldı. Daha sonra, çift küme dizilerinde asimptotik $\mathcal{I}_2^{\sigma\theta}$-istatistiksel denklik kavramı tanımlandı. Ayrıca, bu kavramlar arasındaki kapsama ve gerektirme incelendi.

Anahtar Kelimeler: Asimptotik denklik, Lacunary invariant yakınsaklık, Modülüs fonksiyonu, $\mathcal{I}_2$-yakınsaklık, Wijsman yakınsaklık

1. Introduction

Many mathematicians studied statistical convergence which is a generalization of usual convergence and ideal convergence which is a generalization of statistical convergence of real numbers. Statistical convergence was firstly introduced by Fast (1951) and Schoenberg (1959), separately and some authors researched these concepts in metric spaces and normed spaces. Recent times, statistical convergence was extended to the double sequences by Mursaleen and Edely (2003). Kostyrko et al. (2000) introduced and examined $\mathcal{I}$-convergence which is a generalization of statistical convergence of real numbers. Das et al. (2008) denoted the notion of ideal convergence of double sequence and examined some properties of this concept. The notion of statistical convergence of sequences of set was defined by Nuray and Rhoades (2012). Also, they examined some important properties of this concept. After that, some authors expanded the convergence of the real number sequences to the convergence of set sequences and examined the summability feature.

The notion of invariant convergence have analyzed some authors (Nuray and Savaş 1994, Pancaroğlu and Nuray 2013, 2014, 2015, Raimi 1963, Savaş and Nuray 1993, Schafer 1972, Ulusu et al. 2018). Nuray et al. (2011) denoted the notions of invariant uniform density of subsets E of $\mathbb{N}$, ideal invariant convergence and examined inclusion and necessity relations among ideal invariant convergence, invariant convergence and p-strongly invariant convergence. Recently, ideal invariant convergence for double set sequences introduced by Tortop and Dündar (2018) and Wijsman $\mathcal{I}_2^{\sigma\theta}$-convergence for double set sequences denoted by Dündar and Pancaroğlu Akın (2020).

The notion of asymptotically equivalence and some important properties of this notion investigated by some mathematicians (Dündar et al. 2020, Kişi et al. 2015, Savaş 2013, Ulusu and

*Corresponding Author: edundar@aku.edu.tr

Erdinç Dündar orcid.org/0000-0002-0545-7486
Nimet Pancaroğlu Akın orcid.org/0000-0003-2886-3679

Gülle 2019). Recently, the notion of asymptotically ideal invariant equivalence of set sequences introduced by Ulusu and Gülle (2019). Also, the notion of asymptotically $\mathcal{J}_2^{\sigma}$ -equivalence of double sequences studied by Dündar et al. (2020).

Some authors applying a modulus function f, denote some new notions and give implication theorems (Kişi et al. 2015, Maddox 1986, Nakano 1953, Pancaroğlu and Nuray 2015, Pehlivan and Fisher 1995). The notion of lacunary f-ideal equivalent sequences was denoted by Kumar and Sharma (2012). The concept of f-asymptotically lacunary ideal equivalence of set sequences was introduced by Kişi et al. (2015). The notions of f-asymptotically ideal invariant and lacunary ideal invariant statistical equivalence of set sequences were given Pancaroğlu Akın and Dündar (2018) and Pancaroğlu Akın et al. (2018). Dündar and Pancaroğlu Akın (2019) studied f-asymptotically $\mathcal{J}_2^{\sigma}$-equivalence for double sequences of sets.

We now note some of the basic definitions and concepts we use throughout the article (see, Baronti and Papini 1986, Beer 1985, 1994, Das et al. 2008, Kişi et al. 2015, Kostyrko et al. 2000, Kumar and Sharma 2012, Maddox 1993, Marouf 1993, Nuray et al. 2011, Pancaroğlu Akın et al. 2018, Raimi 1963, Schaefer 1972, Tortop and Dündar 2018, Ulusu and Nuray 2016).

Let $u=(u_k)$ and $v=(v_k)$ be non-negative sequences. If $\lim_{k\to\infty}\frac{u_k}{v_k}=1$, then $u=(u_k)$ and $v=(v_k)$ are told to be asymptotically equivalent (showed by $u\sim v$).

Let (Y,ρ) be a metric space, $y\in Y$ and any non-empty subset C of Y, then we define the way from y to C by $d(y,C)=\inf_{c\in C}\rho(y,c)$.

Later, we let (Y,ρ) be a metric space and C,D,C_k and $D_k\,(k=1,2,\ldots)$ be non-empty closed subsets of Y.

A sequence $\{C_k\}$ is Wijsman convergent to C if $\lim_{k\to\infty}d(y,C_k)=d(y,C)$ for every $y\in Y$. In this instance, it is showed by $W-\lim_{k\to\infty}C_k=C$.

If $\sup_k d(y,C_k)<\infty$ for each $y\in Y$, then $\{C_k\}$ is said to be bounded. We show the space of all bounded sequences of sets by L_∞.

Let $C_k,D_k\subseteq Y$ such that $d(y,C_k)>0$ and $d(y,D_k)>0$ for all $y\in Y$. The sequences $\{C_k\}$ and $\{D_k\}$ are asymptotically equivalent if for all $y\in Y, \lim_{k\to\infty}\frac{d(y,C_k)}{d(y,D_k)}=1$ (denoted by $C_k\sim D_k$).

Let $C_k,D_k\subseteq Y$ such that $d(y,C_k)>0$ and $d(y,D_k)>0$ for each $y\in Y$. If for all $\varepsilon>0$ and all $y\in Y, \lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n: \left|\frac{d(y,C_k)}{d(y,D_k)}-L\right|\ge\varepsilon\right\}\right|=0$, then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent of multiple L (showed by $C_k\overset{WS_L}{\sim}D_k$) and if $L=1$, then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent.

Let $\sigma:\mathbb{N}\to\mathbb{N}$ be a mapping and ϕ be a continuous linear functional on the space of real bounded sequences ℓ_∞. ϕ is an invariant mean or a σ-mean, if the belowing terms hold:

1. $\phi(u)\ge 0$, when the sequence $u=(u_n)$ has $u_n\ge 0$, for all n,
2. $\phi(e)=1$, where $e=(1,1,1\ldots)$,
3. $\phi(u_{\sigma(n)})=\phi(u)$, for all $u\in\ell_\infty$.

Suppose that the mappings σ are injective and such that $\sigma^m(j)\neq j$, for all $j,m\in\mathbb{N}$, where $\sigma^m(j)$ is the *m*th iterate of σ at j. Therefore, for all $u\in c\,\phi(u)$ equals to $\lim u$ which is the extension of the limit functional on c, where $c=\left\{x=(x_k):\lim_k x_k \textit{ exists}\right\}$.

If the equality $\sigma(j)=j+1$ exists, then σ-mean is named a Banach limit, generally.

Now, we give the definition of ideal. $\mathcal{J}\subseteq 2^{\mathbb{N}}$ is named an ideal, if the bellowings provide:

(i) $\phi\in\mathcal{J}$, (ii) For all $E,F\in\mathcal{J}$ we have $E\cup F\in\mathcal{J}$, (iii) For every $E\in\mathcal{J}$ and all $F\subseteq E$ we have $F\in\mathcal{J}$.

Let $\mathcal{J}\subseteq 2^{\mathbb{N}}$ be an ideal. $\mathcal{J}\subseteq 2^{\mathbb{N}}$ is named non-trivial provided that $\mathbb{N}\notin\mathcal{J}$. For a non-trivial ideal $\mathcal{J}$ and for each $n\in\mathbb{N}$ provided that $\{n\}\in\mathcal{J}$, then $\mathcal{J}$ is admissible ideal. Later, we think about that $\mathcal{J}$ is an admissible ideal.

For a nontrivial ideal $\mathcal{J}_2$ of $\mathbb{N}\times\mathbb{N}$ if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ pertain to $\mathcal{J}_2$ for each $i\in\mathbb{N}$, then $\mathcal{J}_2$ is strongly admissible ideal. Later, we take $\mathcal{J}_2$ as a strongly admissible ideal.

If we take an ideal as a strongly admissible ideal, then it is obvious that the ideal we receive is a admissible ideal.

Let $C\subseteq\mathbb{N}\times\mathbb{N}$ and

$$s_{mk}=\min_{i,j}\left|C\cap\{(\sigma(i),\sigma(j)),(\sigma^2(i),\sigma^2(j)),\ldots,(\sigma^m(i),\sigma^k(j))\}\right|$$

and

$$S_{mk}=\max_{i,j}\left|C\cap\{(\sigma(i),\sigma(j)),(\sigma^2(i),\sigma^2(j)),\ldots,(\sigma^m(i),\sigma^k(j))\}\right|$$

If the limits $\underline{V_2}(C):=\lim_{m,k\to\infty}\frac{s_{mk}}{mk}$ and $\overline{V_2}(C):=\lim_{m,k\to\infty}\frac{S_{mk}}{mk}$ exists then $\underline{V_2}(C)$ is named a lower and $\overline{V_2}(C)$ is named an upper σ-uniform density of the set C, in order of. If $\underline{V_2}(C) = \overline{V_2}(C)$, then $V_2(C) = \underline{V_2}(C) = \overline{V_2}(C)$ is named the σ-uniform density of C. Show by $\mathcal{J}_2^{\sigma}$ the taxon of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(C) = 0$.

Later, we let C_{ij}, D_{ij}, C, D be any nonempty closed subsets of Y.

If for every $y \in Y$,

$$\lim_{m,k\to\infty} \frac{1}{mk} \sum_{i,j=1,1}^{m,k} d(y, C_{\sigma^i(s),\sigma^j(t)}) = d(y,C),$$

uniformly in then, is told to be invariant convergent to in.

If for each $\varphi > 0$,

$$A(\varphi, y) = \{(i,j) : | d(y, C_{ij}) - d(y,C) | \geq \varphi\}$$

belongs to $\mathcal{J}_2^{\sigma}$ that is, $V_2(A(\varphi,y)) = 0$ then, $\{C_{ij}\}$ is told to be Wijsman $\mathcal{J}_2^{\sigma}$-convergent or $\mathcal{J}_{W_2}^{\sigma}$-convergent to C. In the present case, we write $C_{ij} \to C(\mathcal{J}_{W_2}^{\sigma})$ and by $\mathcal{J}_{W_2}^{\sigma}$ we show the set of all Wijsman $\mathcal{J}_2^{\sigma}$-convergence of double set sequences.

For C_{ij}, D_{ij} are be non-empty closed subsets of Y define as $d(y; C_{ij} \cup D_{ij})$ follows:

$$d(y; C_{ij}, D_{ij}) = \begin{cases} \dfrac{d(y, C_{ij})}{d(y, D_{ij})}, & y \notin C_{ij} \cup D_{ij} \\ L, & y \in C_{ij} \cup D_{ij}. \end{cases}$$

If for each $\varphi > 0$ and every $y \in Y$

$$\left\{(m,k) :\in \mathbb{N} \times \mathbb{N} : \frac{1}{mk} \sum_{i,j=1,1}^{m,k} | d(y; C_{ij}, D_{ij}) - L | \geq \varphi \right\}$$

belongs to $\mathcal{J}_2^{\sigma}$ then, double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be strongly asymptotically $\mathcal{J}_2^{\sigma}$-equivalent of multiple L (showed by $C_{ij} \overset{[W_{\mathcal{J}_2^{\sigma}}^L]}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be strongly asymptotically $\mathcal{J}_2^{\sigma}$-equivalent.

Let $f : [0,\infty) \to [0,\infty)$ be a function. If bellowing terms hold for f then, it is named a modulus function:

1. $f(t) = 0 \Leftrightarrow t = 0$,
2. $f(t+s) \leq f(t) + f(s)$,
3. f is increasing,
4. f is continuous from the right at 0.

Later, we take f as a modulus function.

f may be unbounded (for instance $f(t) = t^q, 0 < q < 1$) or bounded (for instance $f(t) = \dfrac{t}{t+1}$).

If for each $\varphi > 0$ and for each

$$y \in Y, \{(i,j) \in \mathbb{N} \times \mathbb{N} : f(| d(y; C_{ij}, D_{ij}) - L |) \geq \varphi\} \in \mathcal{J}_2^{\sigma},$$

then the double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be f-asymptotically $\mathcal{J}_2^{\sigma}$-equivalent of multiple L (showed by $C_{ij} \overset{W_{\mathcal{J}_2^{\sigma}}^L(f)}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be f-asymptotically $\mathcal{J}_2^{\sigma}$-equivalent.

If for every $\varphi > 0$ and for every $y \in Y$,

$$\left\{(m,k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mk} \sum_{i,j=1,1}^{m,k} f(| d(y; C_{ij}, D_{ij}) - L |) \geq \varphi \right\} \in \mathcal{J}_2^{\sigma}$$

then, $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be strongly f-asymptotically $\mathcal{J}_2^{\sigma}$-equivalent of multiple L (showed by $C_{ij} \overset{[W_{\mathcal{J}_2^{\sigma}}^L(f)]}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be strongly f-asymptotically $\mathcal{J}_2^{\sigma}$-equivalent.

If for every $\varphi > 0, \delta > 0$ and all $x \in X$,

$$\left\{(m,k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mk} | \{i \leq m, j \leq k : | d(y; C_{ij}, D_{ij}) - L | \geq \delta \right\} \in \mathcal{J}_2^{\sigma},$$

then $\{C_{ij}\}$ and $\{D_{ij}\}$ are told to be asymptotically $\mathcal{J}_2^{\sigma}$-statistical equivalent of multiple L (showed by $C_{ij} \overset{W_{\mathcal{J}_2^{\sigma}}^L(S)}{\sim} D_{ij}$) and if $L = 1$, then asymptotically $\mathcal{J}_2^{\sigma}$-statistical equivalent.

A double sequence $\theta_2 = \{(k_r, j_u)\}$ is named double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \quad \text{and} \quad j_0 = 0, \bar{h}_u = j_u - j_{u-1} \to \infty,$$
as $r, u \to \infty$.

We use the following notations afterwards:

$$k_{ru} = k_r j_u, h_{ru} = h_r \bar{h}_u, I_{ru} = \begin{Bmatrix} (k,j) : k_{r-1} < k < k_r \\ and \ j_{u-1} < j < j_u \end{Bmatrix}.$$

After this, we take $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence, $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} = \min_{m,n} | C \cap \{(\sigma^k(m), \sigma^j(n)) : (k,j) \in I_{ru}\} |$$

and

$$S_{ru} = \max_{m,n} | C \cap \{(\sigma^k(m), \sigma^j(n)) : (k,j) \in I_{ru}\} |.$$

If the limits $\underline{V_2^{\theta}}(C) := \lim_{r,u\to\infty} \dfrac{s_{ru}}{h_{ru}}$ and $\overline{V_2^{\theta}}(C) := \lim_{r,u\to\infty} \dfrac{S_{ru}}{h_{ru}}$ consist, then those are nameded a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set C, in order of. Provided that $\underline{V_2^{\theta}}(C) = \overline{V_2^{\theta}}(C)$, then $V_2^{\theta}(C) = \underline{V_2^{\theta}}(C) = \overline{V_2^{\theta}}(C)$ is named the lacunary σ-uniform density of C. Show by $\mathcal{J}_2^{\sigma\theta}$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^{\theta}(C) = 0$.

Later, we take $\mathcal{J}_2^{\sigma\theta}$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

Lemma 1 (Pehlivan and Fisher 1995). Let f be a modulus and $0<\delta<1$. Then, for each $u\geq\varphi$ we have $f(t)\leq 2f(1)\varphi^{-1}t$.

2. f-Asymptotically $\mathcal{J}_2^{\sigma\theta}$-Equivalence of Double Sequences of Sets

Definition 2.1. If for each $\varphi>0$ and every $y\in Y$,

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}|d(y;C_{kj},D_{kj})-L|\geq\varphi\right\}\in\mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be strongly asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L (showed by $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}$) and if $L=1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Definition 2.2. If for each $\varphi>0$ and every $y\in Y$,

$$\{(k,j)\in I_{ru}:f(|d(y;C_{kj},D_{kj})-L|)\geq\varphi\}\in\mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L (showed by $C_{kj}\overset{W^L_{\mathcal{J}_2^{\sigma\theta}}(f)}{\sim}D_{kj}$) and if $L=1$, then $\{C_{kj}\}$ and are told to be f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Definition 2.3. If for each $\varphi>0$ and every $y\in Y$,

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)\geq\varphi\right\}\in\mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be strongly f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L showed by $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}$) and if $L=1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be strongly f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Theorem 2.1. For each $y\in Y$, we have

$$C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}\Rightarrow C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}.$$

Proof. Let $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}$ and $\varphi>0$. Select $0<\delta<1$ such that $f(z)<\varphi$ for $0\leq z\leq\delta$. Then, for each $y\in Y$, we can write

$$\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)$$
$$=\frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |d(y;C_{kj},D_{kj})-L|\leq\delta}}f(|d(y;C_{kj},D_{kj})-L|)$$
$$+\frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |d(y;C_{kj},D_{kj})-L|>\delta}}f(|d(y;C_{kj},D_{kj})-L|)$$

and so by Lemma 1

$$\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)$$
$$<\gamma+\left(\frac{2f(1)}{\delta}\right)\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}|d(y;C_{kj},D_{kj})-L|.$$

Thus, for every any $\varepsilon>0$ and every $y\in Y$

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)\geq\varepsilon\right\}$$
$$\subseteq\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|\geq\frac{(\varepsilon-\gamma)\delta}{2f(1)}\right\}.$$

Since $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}$ then, it is clear that the latter set pertains to $\mathcal{J}_2^{\sigma\theta}$ and thus, the first set pertains to $\mathcal{J}_2^{\sigma\theta}$. This proves that $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}$.

Theorem 2.2. If $\lim\limits_{z\to\infty}\frac{f(z)}{z}=\alpha>0$, then

$$C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}\Leftrightarrow C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}.$$

Proof. If $\lim\limits_{z\to\infty}\frac{f(z)}{z}=\alpha>0$, then we get $f(z)\geq\alpha z$ for every $z\geq 0$. Assume that $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}$. Since for each $y\in Y$

$$\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)$$
$$\geq\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}\alpha(|d(y;C_{kj},D_{kj})-L|)$$
$$=\alpha\left(\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}|d(y;C_{kj},D_{kj})-L|\right),$$

then, for every $\varepsilon>0$, we get

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}|d(y;C_{kj},D_{kj})-L|\geq\varepsilon\right\}$$
$$\subseteq\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}}f(|d(y;C_{kj},D_{kj})-L|)\geq\alpha\varepsilon\right\},$$

for each $y\in Y$. Since $C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}$, it is clear that the latter set pertains to $\mathcal{J}_2^{\sigma\theta}$ and thus, the first set pertains to $\mathcal{J}_2^{\sigma\theta}$. This proves that

$$C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim}D_{kj}\Leftrightarrow C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}.$$

Definition 2.4. If for each $\varepsilon>0$, each $\varphi>0$ and every $y\in Y$,

$$\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{ru}}|\{(k,j)\in I_{ru}:|d(y;C_{kj},D_{kj})$$
$$-L|\geq\varepsilon\}|\geq\varphi\}\in\mathcal{J}_2^{\sigma\theta},$$

then the sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be asymptotically $\mathcal{J}_2^{\sigma\theta}$-statistical equivalent of multiple L (showed by $C_{kj}\overset{W^L_{\mathcal{J}_2^{\sigma\theta}}(S)}{\sim}D_{kj}$) and if $L=1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are told to be asymptotically $\mathcal{J}_2^{\sigma\theta}$-statistical equivalent.

Theorem 2.3. For each $y\in Y$, we have

$$C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim}D_{kj}\Rightarrow C_{kj}\overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(S)]}{\sim}D_{kj}$$

Proof. Assume that $C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kj}$) and $\varepsilon > 0$ be given. As for every $x \in X$

$$\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f(|d(y;C_{kj},D_{kj})-L|)$$
$$\geq \frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |d(y;C_{kj},D_{kj})-L|\geq\varepsilon}} f(|d(y;C_{kj},D_{kj})-L|)$$
$$\geq f(\varepsilon).\frac{1}{h_{ru}}|\{(k,j)\in I_{ru}: |d(y;C_{kj},D_{kj})-L|\geq\varepsilon\}|,$$

it bellows that for any $\varphi > 0$ and every $y \in Y$,

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{ru}}|\{(k,j)\in I_{ru}: |d(y;C_{kj},D_{kj})-L| \geq \varepsilon\}| \geq \frac{\gamma}{f(\varepsilon)}\right\}$$
$$\subseteq \left\{(r,u)\in\mathbb{N}\times\mathbb{N}: \frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f(|d(y;C_{kj},D_{kj})-L|) \geq \gamma\right\}.$$

Since $C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma}}(f)]}{\sim} D_{kj}$, then the latter set pertains to $\mathcal{J}_2^{\sigma\theta}$. Then, by the definition of an ideal, the first set pertains to $\mathcal{J}_2^{\sigma\theta}$ and so, $C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(S)]}{\sim} D_{kj}$.

Theorem 2.4. If f is bounded, then for every $y \in Y, C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kj} \Leftrightarrow C_{kj} \overset{W^L_{\mathcal{J}_2^{\sigma\theta}}(S)}{\sim} D_{kj}$.

Proof. Assume that f is bounded and let $C_{kj} \overset{W^L_{\mathcal{J}_2^{\sigma\theta}}(S)}{\sim} D_{kj}$. Since f is bounded there exists a $K > 0$ such that $|f(y)| \leq K$ for every $y \geq 0$. Further using fact, we get

$$\frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f\big(|d(y;C_{kj},D_{kj})-L|\big)$$
$$= \frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |d(y;C_{kj},D_{kj})-L|\geq\varepsilon}} f\big(|d(y;C_{kj},D_{kj})-L|\big)$$
$$+\frac{1}{h_{ru}}\sum_{\substack{(k,j)\in I_{ru}\\ |d(y;C_{kj},D_{kj})-L|<\varepsilon}} f\big(|d(y;C_{kj},D_{kj})-L|\big)$$
$$\leq \frac{K}{h_{ru}}\Big|\{(k,j)\in I_{ru}: |d(y;C_{kj},D_{kj})-L| \geq \varepsilon\}\Big| + f(\varepsilon),$$

for each $y \in Y$. This proves that $C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kf}$.

3. References

Baronti, M., Papini, P. 1986. Convergence of sequences of sets, In: Methods of Functional Analysis in Approximation Theory. Birkhäuser, Basel, pp. 133-155.

Beer, G. 1985. On convergence of closed sets in a metric space and distance functions. *Bull. Aust. Math. Soc.,* 31: 421-432.

Beer, G. 1994. Wijsman convergence: A survey. *Set-Valued Anal.*, 2(1-2): 77-94.

Das, P., Kostyrko, P., Wilczyński, W., Malik, P. 2008. $\mathcal{J}$ and $\mathcal{J}^*$-convergence of double sequences. *Mathematica Slovaca,* 58(5): 605-620.

Dündar, E., Pancaroğlu Akın N. 2019. f-Asymptotically $\mathcal{J}_2^{\sigma}$-equivalence of double sequences of sets. *AKU J. Sci. Eng.,* 19: 79-86.

Dündar, E., Pancaroğlu Akın, N. 2020. Wijsman lacunary $\mathcal{J}_2$-invariant convergence of double sequences of sets. *Honam Mathematical J.* (in press).

Dündar, E., Ulusu, U. and Nuray, F. 2020. On asymptotically ideal invariant equivalence of double sequences. *Thai J. Math.* (in press).

Fast, H. 1951. Sur la convergence statistique. *Colloq. Math.*, 2: 241–244.

Kişi, Ö., Gümüş, H., Nuray, F. 2015. $\mathcal{J}$-Asymptotically lacunary equivalent set sequences defined by modulus function. *Acta Universitatis Apulensis*, 41: 141-151.

Kostyrko, P., Šalát, T., Wilczyński, W. 2000. $\mathcal{J}$-convergence. *Real Anal. Exchange*, 26(2): 669-686.

Kumar, V., Sharma, A. 2012. Asymptotically lacunary equivalent sequences defined by ideals and modulus function. *Mathematical Sciences*, 6(23): 5.

Maddox, I. J. 1986. Sequence spaces defined by a modulus. *Math. Proc. Camb. Phil. Soc.*, 100: 161-166.

Marouf, M. 1993. Asymptotic equivalence and summability. *Int. J. Math. Math. Sci.*,16(4): 755-762.

Mursaleen, M., Edely O. H. H. 2003. Statistical convergence of double sequences. *J. Math. Anal. App*l., 288: 223-231.

Nakano, H. 1953. Concave modulars. *J. Math. Soc. Japan*, 5: 29-49.

Nuray, F., Rhoades, B. E. 2012. Statistical convergence of sequences of sets. *Fasc. Math.*, 49: 87-99.

Nuray, F., Savaş, E. 1994. Invariant statistical convergence and -invariant statistical convergence. *Indian J. Pure Appl. Math.*, 10: 267-274.

Nuray, F., Gök, H., Ulusu, U. 2011. $\mathcal{J}_\sigma$-convergence, *Math. Commun*. 16: 531-538.

Pancaroğlu Akın, N., Dündar, E. 2018. Asymptotically $\mathcal{J}$-invariant statistical equivalence of sequences of set defined by a modulus function. *AKU Journal of Science Engineering*, 18(2): 477-485.

Pancaroğlu Akın, N., Dündar, E., Ulusu, U. 2018. Asymptotically $\mathcal{J}_{\sigma\theta}$-statistical equivalence of sequences of set defined by a modulus function. *Sakarya University Journal of Science*, 22(6): 1857-1862.

Pancaroğlu, N., Nuray, F. 2013. On invariant statistically convergence and lacunary invariant statistically convergence of sequences of sets. *Progress in Applied Mathematics,* 5(2): 23-29.

Pancaroğlu, N., Nuray, F. 2014. Invariant statistical convergence of sequences of sets with respect to a modulus function. *Abstract and Applied Analysis*, Article ID 818020, 5 pages.

Pancaroğlu, N., Nuray, F. 2015. Lacunary invariant statistical convergence of sequences of sets with respect to a Modulus Function. *Journal of Mathematics and System Science*, 5: 122-126.

Pehlivan, S., Fisher, B. 1995. Some sequences spaces defined by a modulus. *Mathematica Slovaca*, 45: 275-280.

Raimi, RA. 1963. Invariant means and invariant matrix methods of summability. *Duke Math. J.*, 30: 81-94.

Savaş, E., Nuray, F. 1993. On σ-statistically convergence and lacunary σ-statistically convergence. *Math. Slovaca,* 43(3): 309-315.

Savaş, E. 2013. On $\mathcal{J}$-asymptotically lacunary statistical equivalent sequences. *Adv. Differ. Equ.*, (111) (2013): doi:10.1186/1687-1847-2013-111

Schaefer, P. 1972. Infinite matrices and invariant means. *Proc. Amer. Math. Soc.*, 36: 104-110.

Schoenberg, I. J. 1959. The integrability of certain functions and related summability methods. *Amer. Math. Monthly,* 66: 361-375.

Tortop, Ş., Dündar, E. 2018. Wijsman $\mathcal{J}_2$-invariant convergence of double sequences of sets. *Journal of Inequalities and Special Functions*, 9(4): 90-100.

Ulusu, U., Nuray, F. 2016. Lacunary $\mathcal{J}_\sigma$-convergence. *2nd International Conference on Analysis and Its Applications*, pp: 321, Kırşehir.

Ulusu, U., Dündar, E., Nuray, F. 2018. Lacunary $\mathcal{J}_2$-invariant convergence and some properties. *International Journal of Analysis and Applications*, 16(3): 317-327.

Ulusu, U., Gülle, E. 2019. Asymptotically $\mathcal{J}_\sigma$-equivalence of sequences of sets. *Sakarya University Journal of Science*, 23(5): 718-723.