Fixed Point Iteration and Newton's Method

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Abstract

In this study, we examined the Newton-Rapson method from fixed point iterations. With a few examples, we proved the validity of the method again.

Key words

Newton-Rapson method, fixed point iterations and fixed point.

1.Introduction

Takahashi [6] introduced the concept of convex metric space which is a more general space and each linear normed space is a special case of a convex metric space. In 2005, Tian[6] gave some sufficient and necessary conditions such that the Ishikawa iteration sequence for an asymptotically quasi-nonexpansive mapping to converge to a fixed point in convex metric spaces. In 2009, Wang and Liu [6] gave some sufficient and necessary conditions for an Ishikawa iteration sequence with errors to approximate a common fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces.

Fixed Point Theory is a beautiful mixture of analysis , topology and geometry. Topological ideas are present in almost all the areas of today's mathematics. The subject of topology itself consists of several different branches; such as point set topology, algebraic topology and differential topology, which have relatively little in common.

Fixed point theorems give the conditions under which mappings have solutions. Over the last fifty years or so, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

In particular, fixed point techniques have been applied in such diverse fields as Biology, Chemistry, Economics, Engineering, Game theory and Physics.[1]

Fixed Point Theorems with Applications to Economics and Game Theory by Kim Border (1985) is a complement, not a substitute, explaining various forms of the fixed point principle such as the KKMS theorem and some of the many theorems of Ky Fan, along with the concrete details of how they are actually applied in economic theory. Fixed Point Theory by Dugundji and Granas (2003) is, even more than this book, a comprehensive treatment of the topic. Its fundamental point of view audience and technical base are quite different, but it is still a work with much to offer to economics. [2]

In the previous two lectures we have seen some applications of the mean value theorem. We now see another application. In this lecture we discuss the problem of finding approximate solutions of the equation

$$g(x) = 0 \qquad\qquad (1)$$

In some cases it is possible to find the exact roots of the equation (1), for example, when $g(x)$ is a quadratic or cubic polynomial. Otherwise, in general, one is interested in finding approximate solutions using some methods. Here, we will discuss a method called fixed point iteration method and a particular case of this method called Newton's method. [3]

2. Fixed Point Iteration Method

In this method, we first rewrite the equation (1) in the form

$$x = f(x) \tag{2}$$

in such a way that any solution of the equation (2), which is a fixed point of f , is a solution of equation (1). Then consider the following algorithm.

2.1. Algorithm:

Start from any point x_0 and consider the recursive process

$$x_{n+1} = f(x_n); \quad n = 0; 1; 2; \ldots \tag{3}$$

If f is continuous and (x_n) converges to some p_0 then it is clear that p_0 is a fixed point of f and hence it is a solution of the equation (1). Moreover, x_n can be considered as an approximate solution of the equation (1).

First let us illustrate whatever we said above with an example.

Example 2.1: We know that there is a solution for the equation $x^3 - 3x + 1 = 0$ in [0; 1]. We rewrite the equation in the form

$$x = \frac{1}{3}(x^3 + 1)$$

and define the process

$$x_{n+1} = \frac{1}{3}(x^3 + 1)$$

We have already seen in a tutorial class that if $0 \leq x_0 \leq 1$ then (x_0) satisfies the Cauchy criterion and hence it converges to a root of the above equation. We also note that if we start with $x_0 = 2$ then the recursive process does not converge.

It is clear from the above example that the convergence of the process (3) depends on f and the starting point x_0. Moreover, in general, showing the convergence of the sequence (x_n) obtained from the iterative process is not easy. For this reason, we can ask the following problems.

Problems:

Under what assumptions on f and x_0 , does Algorithm one converge ? When does the sequence (x_n) obtained from the iterative process (3) converge ? The following result is a consequence of the mean value theorem.

Theorem 2.1   Let f : [a; b] $\rightarrow$ [a; b] be a differentiable function such that

$$|f'(x)| \leq \alpha < 1 \quad \text{for all } x \in [a; b] \tag{4}$$

Then f has exactly one fixed poin t $p_0 \in$ [a; b] and the sequence (x_n) defined by the process (3), with a starting point $x_0 \in$ [a; b], converges to p_0.

Proof : By the intermediate value property f has a fixed point, say p_0 .

The convergence of (x_n) to p_0 follows from the following inequalitie

$$|x_n - p_0| = |f(x_{n-1}) - f(p_0)| \leq \alpha |x_{n-1} - p_0|$$

$$\leq \alpha^2 |x_{n-2} - p_0|$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\leq \alpha^n |x_0 - p_0| \rightarrow 0$$

If p_1 is a fixed point then

$$|p_1 - p_0| = |f(p_1) - f(p_0)| \leq \alpha |p_1 - p_0| < |p_1 - p_0| \text{ . This implies that } p_1 = p_0 \text{ .}$$

Example 2.2

2.2.1: Let us take the problem given in Example 2.1 where $f(x) = \frac{1}{3}(x^3 + 1)$. Then

$$f : [0; 1] \to [0; 1] \quad \text{and} \quad |f(x)'| < 1/3 \quad \text{for all } x \in [0; 1]$$

2.2. Consider $h(x) : [0; 2] \to R$ defined by $h(x) = (1 + x)^{1/3}$. Observe that h maps $[0, 2]$ onto itself.

Moreover $|h'(x)| \leq 1/3 < 1$ for $x \in [0; 2]$. By the previous theorem the sequence (x_n) defined by

$x_{n+1} = (1 + x_n)^{1/3}$ converges to a root of $x^3 - x - 1 = 0$ in the interval $[0; 2]$.

In practice, it is often dificult to check the condition $h([a; b]) \subseteq [a; b]$ given in the previous theorem. We now present a variant of Theorem 2.1.

Theorem 2.2 Let p_0 be a fixed point of $f(x)$. Suppose $f(x)$ is differentiable on $[p_0 - \varepsilon, p_0 + \varepsilon]$ for some $\varepsilon > 0$ and f satisfies the condition $|f'(x)| \leq \alpha < 1$ for all $x \in [p_0 - \varepsilon, p_0 + \varepsilon]$. Then the sequence (x_n) defined , with a starting point $x \in [p_0 - \varepsilon, p_0 + \varepsilon]$, converges to p_0.

Proof : By the mean value theorem $f([p_0 - \varepsilon, p_0 + \varepsilon]) \subseteq [p_0 - \varepsilon, p_0 + \varepsilon]$.

Therefore, the proof follows from the previous theorem. The previous theorem essentially says that if the starting point is sufficiently close to the fixed point then the chance of convergence of the iterative process is high.

Remark : If f is invertible then p_0 is a fixed point of f if and only if p_0 is a fixed point of $f^{-1}(x)$. In view of this fact, sometimes we can apply the fixed point iteration method for $f^{-1}(x)$ instead of f. For understanding, consider $f(x) = 2x+1$ then $|f'(x)| = 2$ for all x. So the fixed point iteration

3.Newton's Method or Newton-Raphson Method :

The following iterative method used for solving the equation $f(x) = 0$ is called Newton's method.

3.1 Algorithm : $x_{n+1} = \frac{f(x_n)}{f'(x_n)}$ $n = 0; 1; 2; \ldots$

It is understood that here we assume all the necessary conditions so that xn is well defined. If we
Take

$$g(x) = x - \frac{f(x)}{f'(x)}$$

then Algorithm 3.1 is a particular case of Algorithm 2.1. So we will not get in to the convergence analysis of Algorithm 3.1. Instead, we will illustrate Algorithm 3.1 with an example.

Example 3.1: Suppose $f(x) = x^3 + 3$ and we look for the positive root of $f(x) = 0$. Since $f'(x) = 3x^2$, the iterative process of Newton's method is

$$x_{n+1} = \frac{1}{3}\left(x_n + \frac{3}{x_n}\right) \quad n = 0; 1; 2; \ldots$$

We have already discussed this sequence in a tutorial class. Geometric interpretation of the iterative process of Newton's method : Suppose we have found $(x_n; f(x_n))$. To find x_{n+1}, we approximate the graph $y = f(x)$ near the point $(x_n; f(x_n))$.

the tangent :

$$f(x) - f(x_n) = f'(x_n)\,(x - x_n)$$

Note that x_{n+1} is the point of intersection of the x-axis and the tangent at x_n .

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