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REDUCED DIFFERENTIAL TRANSFORMS APPROACH FOR HIGHLY NONLINEAR SYSTEM OF TWO DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

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Abstract. Analytical solution of highly nonlinear system of two dimensional Volterra integral equations is studied by the reduced differential transform method [RDTM]. We present a new property of RDTM to acquire the recursive relation which is used to get analytical solution of the above mentioned two dimensional system. Results of the numerical examples obtained by RDTM are compared with the existing results obtained by TDDTM. Though solutions obtained by RDTM and TDDTM are same, RDTM has significant advantage over TDDTM that is RDTM generates the solution of the nonlinear problem by operating the multivariable function with respect to a desired variable only not on all of their independent variables unlike in TDDTM so that RDTM reduces the time consumption than TDDTM.
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1. Introduction and Preliminaries

Numerous science and engineering fields like electric circuit analysis, elasticity, electromagnetism and electrodynamics, heat and mass transfer, visco elasticity, chemical and electrochemical process are modeled by the two dimensional highly nonlinear system of integral equations [1] - [6]. Motivated by these applications, we consider the following system of two dimensional nonlinear Volterra integral equations [7],

$$f_i(x,t)+\sum_{j=0}^{n}\int_0^t\int_0^x k_{i,j}(x,y,r,s,u_j(r,s))drds = u_i(x,t), i=1,2,3\cdots\cdots,n. \tag{1.1}$$

where $u(x,t)$ is unknown function, $k_{i,j}(x,y,r,s,u_j(r,s))$ and $f_i(x,t)$, i, $j=1,2,\cdots,n$, are given continuous functions on $D=[0\times X]*[0\times T]$ and $E=(x,y,r,s,u): 0\leq r\leq x\leq X$, $0\leq s\leq t\leq T, -\infty\leq u\leq\infty$ and it has the following degenerate form,

$$K_{i,j}(x,y,r,s)=\sum_{m=1}^{p}a_m(x,t)b_m(r,s,u_j(r,s)), i=1,2,\cdots n, j=1,2,\cdots n. \tag{1.2}$$

In [8], Pachpattei has discussed the existence, uniqueness and other properties of solutions of Volterra integral and integro-differential equations in two variables. Han and Zhang [9] studied the asymptotic error expansion of two-dimensional Volterra integral equation by iterated collocation. Tari [7] presented the existence and uniqueness solution of nonlinear volterra partial integral and integro-differential equations. In [10], Kwapisz discussed the weighted norms and existence and uniqueness of L_p solutions for integral equations in several variables. Also, some numerical approaches can be found out in the literature for different two dimensional nonlinear volterra partial integral and integro-differential equations (one can refer [11–15]).

In [16], authors have applied two dimensional differential transform method[TDDTM] for solving (1.1). Though the TDDTM is a fine tool to acquire analytical solution of higher dimensional problems, it also comes across some deficiencies like it operates the multivariable function with respect to all of their independent variables which leads to complicated computation and consuming more time. To avoid these kind of difficulties, Keskin [17] introduced the reduced differential transform method[RDTM] and it operates the multivariable function with respect to desired variable so that it reduces the computational size. Several works have been carried out by this method in the recent years for various nonlinear problems (see [11, 18–22] and references therein).

In this work, we use the RDTM to solve (1.1) by giving extension to the theorem which was given by [7] and [11]. To show the proficiency of the proposed method, we compare the results obtained by the RDTM and the existing results by TDDTM [16].

2. Reduced Differential Transform Method

As in reference [11] and [17], the basic definition of the RDTM is introduced as follows:

Consider a function of two variables $w(x,t)$, and suppose that it can be represented as a product of two single-variable function, that is, $w(x,t) = f(x)\, g(t)$. Based on the properties of one dimensional differential transform, the function $w(x,t)$ can be represented as,

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)x^j = \sum_{k=0}^{\infty} U_k(x)t^k,$$

where $U_k(x)$ is called t-dimensional spectrum function of u(x,t).

The reduced differential transform of an analytic function $u(x,t)$ at $t = 0$ is defined as,

$$U_k(x) = \frac{1}{k!}\Big[\frac{\partial^k u(x,t)}{\partial t^k}\Big]_{t=0}, \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(x)$ is the transformed function.

The reduced differential inverse transform of $U_k(x)$ is defined as,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{2.2}$$

and from (2.1) and (2.2), we have,

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big[\frac{\partial^k u(x,t)}{\partial t^k}\Big]_{t=0} t^k. \tag{2.3}$$

The fundamental theorems that can be deduced from (2.1) and (2.2) are given in the 1, [11, 17]:

Table 1. Fundamental theorems of RDTM

S. No	Original function	Transformed function
1	$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
2	$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$
3	$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
4	$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
5	$w(x,t) = \frac{\partial^r u(x,t)}{\partial t^r}$	$W_k(x) = (k+1)(k+2)....(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
6	$w(x,t) = \frac{\partial u(x,t)}{\partial x}$	$W_k(x) = \frac{\partial}{\partial x}(U_k(x))$
7	$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x)$
8	$w(x,t) = \int_0^x \int_0^t u(r,s) dr ds$	$W_k(x) = \begin{cases} 0, & \text{k} = 0 \ , \\ \int_0^x \frac{U_{k-1}(r)}{k} dr, & \text{k} \geq 1. \end{cases}$

3. New Property of RDTM

In this section, we give the extension of the theorem which was given by [7] and [11] for solving (1.1).

Theorem 3.1. *If* $w(x,t) = \int_0^x \int_0^t u_1(r,s)u_2(r,s)u_3(r,s) \ldots\ldots u_{n-1}(r,s)u_n(r,s) dr ds$ *then*

$$W_k(x) = \frac{1}{k} \int_0^x \Big[\sum_{k_1=0}^{k-1} \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_{n-2}=0}^{k_{n-3}} \cdots \sum_{k_3=0}^{k_2} \sum_{k_2=0}^{k_1} U_{1_{[k_{n-1}]}}(r) U_{2_{[k_{n-2}-k_{n-1}]}}(r) U_{3_{[k_{n-3}-k_{n-2}]}}(r) \ldots$$
$$U_{n-2_{[k_2-k_3]}}(r) U_{n-1_{[k_1-k_2]}}(r) U_{n_{[k-k_1-1]}}(r) \Big] dr$$

Proof. This theorem will be proved by Mathematical induction.
For $n = 2$,

$$w(x,t) = \int_0^x \int_0^t u_1(r,s)u_2(r,s) dr ds.$$

We write the above equation by Leibnitz rule as,

$$\begin{aligned}
\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] &= \frac{\partial^k}{\partial t^k}\Big[\int_0^x \int_0^t u_1(r,s)u_2(r,s)ds\Big],\\
&= \int_0^x \frac{\partial^{k-1}}{\partial t^{k-1}}\Big[u_1(r,t)u_2(r,t)\Big]dr,\\
&= \int_0^x \Big[\sum_{k_1=0}^{k-1}\frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{\partial^{k_1}}{\partial t^{k_1}}[u_1(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_2(r,t)]\Big]dr,\\
k!W_k(x) &= \int_0^x \Big[\sum_{k_1=0}^{k-1}\frac{(k-1)!}{k_1!(k-k_1-1)!}U_{1_{[k_1]}}(r)(k_1)!(k-k_1-1)!U_{2_{[k-k_1-1]}}(r)\Big]dr,
\end{aligned}$$

$$W_k(x) = \frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1} U_{1_{[k_1]}}(r)U_{2_{[k-k_1-1]}}(r)\Big]dr$$

Therefore, this theorem holds for $n=2$.
For $n=3$,

$$w(x,t) = \int_0^x \int_0^t u_1(r,s)u_2(r,s)u_3(r,s)drds,$$

By Leibnitz rule, we write the above equation as,

$$\begin{aligned}
\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] &= \frac{\partial^k}{\partial t^k}\Big[\int_0^x \int_0^t u_1(r,s)u_2(r,s)u_3(r,s)drds\Big]\\
&= \int_0^x \big[\frac{\partial^{k-1}}{\partial t^{k-1}}u_1(r,t)u_2(r,t)u_3(r,t)\big]dr\\
&= \int_0^x \big[\sum_{k_1=0}^{k-1}\frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{\partial^{k_1}}{\partial t^{k_1}}[g(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_3(r,t)]\big]dr
\end{aligned}$$

where $g(x,t) = u_1(r,t)u_2(r,t)$.

$$\begin{aligned}
\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \big[&\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1}\frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{(k_1)!}{k_2!(k_1-k_2)!}\times\\
&\frac{\partial^{k_2}}{\partial t^{k_2}}[u_1(r,t)]\frac{\partial^{k_1-k_2}}{\partial t^{k_1-k_2}}[u_2(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_3(r,t)]\big]dr.
\end{aligned}$$

Using (2.1) in the above equation, we get,

$$\begin{aligned}
k!W_k(x) = \int_0^x \big[&\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1}\frac{(k-1)!(k_1)!(k_2)!(k_1-k_2)!(k-k_1-1)!}{(k_1)!(k-k_1-1)!(k_2)!(k_1-k_2)!}\times\\
&U_{1_{[k_2]}}(r)U_{2_{[k_1-k_2]}}(r)U_{3_{[k-k_1-1]}(r)}\Big]dr
\end{aligned}$$

$$W_k(x) = \frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1} U_{1_{[k_2]}}(r)U_{2_{[k_1-k_2]}}(r)U_{3_{[k-k_1-1]}}(r)\Big]dr$$

Therefore, theorem holds for $n = 3$.
For $n = 4$,

$$w(x,t) = \int_0^x \int_0^t u_1(r,s)u_2(r,s)u_3(r,s)u_4(r,s)drds,$$

By Leibnitz rule, we write the above equation as,

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \frac{\partial^k}{\partial t^k}\Big[\int_0^x \int_0^t u_1(r,s)u_2(r,s)u_3(r,s)u_4(r,s)drds\Big]$$

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \frac{\partial^{k-1}}{\partial t^{k-1}}\Big[u_1(r,t)u_2(r,t)u_3(r,t)u_4(r,t)\Big]dr$$

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \Big[\sum_{k_1=0}^{k-1} \frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{\partial^{k_1}}{\partial t^{k_1}}[g_1(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_4(r,t)]\Big]dr$$

where $g_1(r,t) = u_1(r,t)u_2(r,t)u_3(r,t)$.

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1} \frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{(k_1)!}{(k_2)!(k_1-k_2)!}\times$$
$$\frac{\partial^{k_2}}{\partial t^{k_2}}[g_2(r,t)]\frac{\partial^{k_1-k_2}}{\partial t^{k_1-k_2}}[u_3(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_4(r,t)]\Big]dr$$

where $g_2(r,t) = u_1(r,t)u_2(r,t)$.

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2} \frac{(k-1)!}{k_1!(k-k_1-1)!}\frac{(k_1)!}{k_2!(k_1-k_2)!}\frac{(k_2)!}{(k_3)!(k_2-k_3)!}\times$$
$$\frac{\partial^{k_3}}{\partial t^{k_3}}[u_1(r,t)]\frac{\partial^{k_2-k_3}}{\partial t^{k_2-k_3}}[u_2(r,t)]\frac{\partial^{k_1-k_2}}{\partial t^{k_1-k_2}}[u_3(r,t)]\frac{\partial^{k-k_1-1}}{\partial t^{k-k_1-1}}[u_4(r,t)]\Big]dr$$

$$\frac{\partial^k}{\partial t^k}\big[w(x,t)\big] = \int_0^x \Big[\frac{1}{k!}\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2} \frac{(k-1)!(k_1)!(k_2)!(k_3)!(k_2-k_3)!(k_1-k_2)!(k-k_1-1)!}{k_1!(k-k_1-1)!k_2!(k_1-k_2)!k_3!(k_2-k_3)!}\times$$
$$U_{1_{(k_3)}}(r)U_{2_{(k_2-k_3)}}(r)U_{3_{(k_1-k_2)}}(r)U_{4_{(k-k_1-1)}}(r)\Big]dr$$

$$W_k(x) = \frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_2} U_{1_{(k_3)}}(r)U_{2_{(k_2-k_3)}}(r)U_{3_{(k_1-k_2)}}(r)U_{4_{(k-k_1-1)}}(r)\Big]dr$$

$$\vdots$$

In general,

$$W_k(x) = \frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_{n-1}=0}^{k_{n-2}}\sum_{k_{n-2}=0}^{k_{n-3}} \cdots\cdots \sum_{k_3=0}^{k_2}\sum_{k_2=0}^{k_1} U_{1_{[k_{n-1}]}}(r)U_{2_{[k_{n-2}-k_{n-1}]}}(r)U_{3_{[k_{n-3}-k_{n-2}]}}(r)$$
$$\cdots\cdots U_{n-2_{[k_2-k_3]}}(r)U_{n-1_{[k_1-k_2]}}(r)U_{n_{[k-k_1-1]}}(r)\Big]dr$$

which completes the proof. □

4. Description of the method

In this section, we present the RDTM for solving (1.1).
By taking reduced differential transform on both sides of (1.1) , we have,

$$RDT\Big[f_i(x,t)+\sum_{j=0}^{n}\int_0^t\int_0^x k_{i,j}(x,y,r,s,u_j(r,s))drds=u_i(x,t)\Big], i=1,2,3,\cdots,n. \quad (4.1)$$

Applying the fundamental theorems into (4.1), we obtain the following recurrence relation,

$$F_{n_{(k)}}(x)+\frac{1}{k}G_{n_{(k)}}(x)=U_{n_{(k)}}(x), \quad (4.2)$$

where $F_{n_{(k)}}(x), G_{n_{(k)}}(x)$ and $U_{n_{(k)}}(x)$ are the transformed functions of

$$f_n(x,t), \int_0^t\int_0^x k_{i,j}(x,y,r,s,u_j(r,s))drds$$

and $u_i(x,t)$ respectively.
By iterative calculations on (4.2), we obtain the values of $U_{1_{(k)}}(x)\dots U_{n_{(k)}}(x)$ as,

$$\begin{gathered}U_{1_{(0)}}(x)=\eta_0(x), U_{1_{(1)}}(x)=\eta_1(x), U_{1_{(2)}}(x)=\eta_2(x), U_{1_{(3)}}(x)=\eta_3(x),\dots\dots\\ U_{2_{(0)}}(x)=\alpha_0(x), U_{2_{(1)}}(x)=\alpha_1(x), U_{2_{(2)}}(x)=\alpha_2(x),\dots\dots\\ \vdots\\ U_{n_{(0)}}(x)=\beta_0(x), U_{n_{(1)}}(x)=\beta_1(x), U_{n_{(2)}}(x)=\beta_2(x),\dots\dots\dots \quad (4.3)\end{gathered}$$

One can get the solution of (1.1) by substituting (4.3) into (2.2).

5. Illustrative Examples

Two different test examples are considered in this section to illustrate the effectiveness of the RDTM.

Example 5.1. Consider the nonlinear system of Volterra integral equations,

$$\begin{aligned}u(x,t)&=x^2t-\frac{x^5t^3}{15}-\frac{x^{10}t^7}{70}+\int_0^x\int_0^t\big[u(r,s)\big]^2+\big[v(r,s)\big]^3drds,\\ v(x,t)&=x^3t^2+\frac{x^7t^5}{35}-\frac{x^7t^4}{28}+\int_0^x\int_0^t\big[u(r,s)\big]^3-\big[v(r,s)\big]^2drds, \quad (5.1)\end{aligned}$$

whose exact solution was found to be [16]:

$$u(x,t)=x^2t; v(x,t)=x^3t^2. \quad (5.2)$$

By taking the reduced differential transform on both sides of (5.1), the following recurrence relation is obtained.

$$U_k(x)=x^2\delta(k-1)-\frac{x^{15}}{15}\delta(k-3)-\frac{x^{10}}{70}\delta(k-7)+\frac{1}{k}\int_0^x\big[\sum_{k_1=0}^{k-1}U_{k_1}(r)U_{k-k_1-1}(r)\big]dr+$$

$$\frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1} V_{k_2}(r)V_{k_1-k_2}(r)V_{k-k_1-1}(r)\Big]dr, \tag{5.3}$$

$$V_k(x) = x^3\delta(k-2)+\frac{x^7}{35}\delta(k-5)-\frac{x^7}{28}\delta(k-4)+\frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1}\sum_{k_2=0}^{k_1} U_{k_2}(r)U_{k_1-k_2}(r)U_{k-k_1-1}(r)\Big]dr-$$

$$\frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1} V_{k_1}(r)V_{k-k_1-1}(r)\Big]dr. \tag{5.4}$$

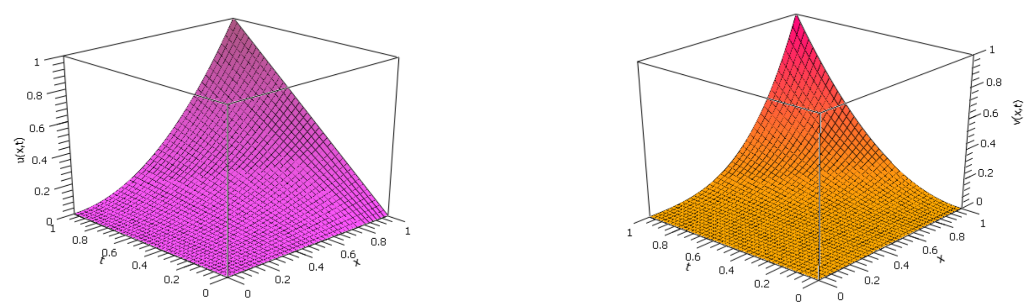
By iterative calculations on (5.3) and (5.4), we have,

$$U_k(x) = \begin{cases} x^2, & k=1\ , \\ 0, & \text{otherwise.} \end{cases} \quad V_k(x) = \begin{cases} x^3, & k=2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Substituting (5.5) into (2.2), we have,

$$u(x,t) = x^2t; v(x,t) = x^3t^2 \tag{5.6}$$

which is the exact solution of the problem.



(a) Plot of the surface $u(x,t) = x^2t$ (b) Plot of the surface $v(x,t) = x^3t^2$

FIGURE 1. Graphical representation of the solutions $u(x,t) = x^2t$ and $v(x,t) = x^3t^2$

Example 5.2. Consider the nonlinear system of Volterra integral equations,

$$u(x,t) = e^xt + \frac{x^2}{2} - \frac{t^3}{6} + \frac{e^{2x}t^3}{6} - \frac{x^2e^t}{2} + \int_0^t\int_0^x \Big[v(r,s) - [u(r,s)]^2\Big]drds,$$

$$v(x,t) = xe^t - \frac{x^2}{2} + \frac{t^2}{2} - \frac{e^xt^2}{2} + \frac{x^2e^t}{2} + \int_0^t\int_0^x \Big[u(r,s) - v(r,s)\Big]drds, \tag{5.7}$$

whose exact solution was found to be [16]:

$$u(x,t) = e^x t; v(x,t) = xe^t. \tag{5.8}$$

By taking the reduced differential transform on both sides of (5.7), the following recurrence relation is obtained.

$$\begin{aligned} U_k(x) = {} & e^x \delta(k-1) + \frac{x^2}{2}\delta(k-0) - \frac{\delta(k-3)}{6} + \frac{e^{2x}\delta(k-3)}{6} - \frac{x^2}{2k!} \\ & - \frac{1}{k}\int_0^x \Big[\sum_{k_1=0}^{k-1} U_{k_1}(r)U_{k-k_1-1}(r)\Big]dr + \int_0^x \frac{V_{k-1}(r)}{k}dr. \end{aligned} \tag{5.9}$$

$$\begin{aligned} V_k(x) = {} & -\frac{x^2\delta(k-0)}{2} + \frac{\delta(k-2)}{2} - \frac{e^x\delta(k-2)}{2} + \frac{x^2}{2k!} + \frac{1}{k}\int_0^x U_{k-1}(r)dr \\ & + \frac{x}{k!} - \int_0^x \frac{V_{k-1}(r)}{k}dr. \end{aligned} \tag{5.10}$$

By iterative calculations on (5.9) and (5.10), we have,

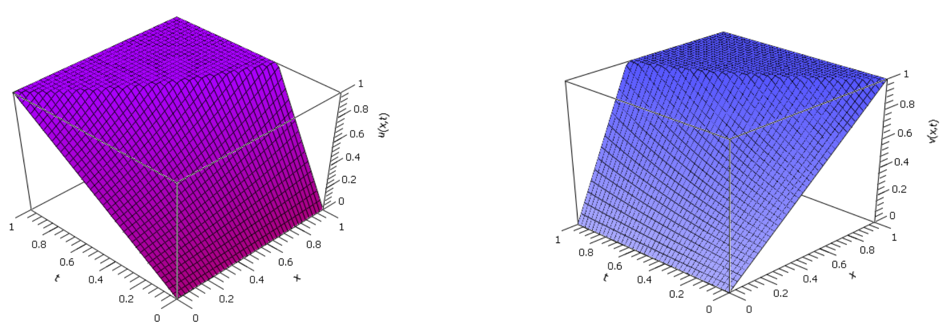
$$U_k(x) = \begin{cases} e^x, & k = 1, \\ 0, & \text{otherwise}. \end{cases}$$

$$V_0(x) = x, V_1(x) = x, V_2(x) = \frac{x}{2!}, V_3(x) = \frac{x}{3!} \ldots\ldots\ldots \tag{5.11}$$

Substituting (5.11) into (2.2), we have,

$$u(x,t) = e^x t, v(x,t) = xe^t, \tag{5.12}$$

which is the exact solution of the problem.



(a) Plot of the surface $u(x,t) = e^x t$ (b) Plot of the surface $v(x,t) = xe^t$

FIGURE 2. Graphical representation of the solutions $u(x,t) = e^x t$ and $v(x,t) = xe^t$

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6. Conclusion

We presented a new property of RDTM and it was successfully applied to solve the system of two dimensional highly nonlinear Volterra integral equations. From the solution of examples, we can see that RDTM and TDDTM have given the same exact solution. Although solutions obtained by RDTM and TDDTM are same, RDTM has significant advantage over TDDTM that is RDTM generates the solution of the nonlinear problem by operating the multivariable function with respect to a desired variable only not on all of their independent variables unlike in TDDTM. Due to this reason RDTM reduces the time consumption than TDDTM.

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