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Discontinuous Density Function Identification

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Abstract. The work is devoted to the identification step density function of a string. The inverse problem consists of recovering constant densities ρ_i of eigenvalue problem. It is shown that if we use only the natural frequencies of the boundary value problem itself to restore the step density, then this inverse problem has an infinite number of solutions $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ in $\mathbb{R}^n$ and unique solution in a sufficiently small area $\Omega \subset \mathbb{R}^n$. For the uniqueness of the recovery of the step density of a string, the natural frequencies of one boundary value problem are not enough. We need to use the natural frequencies of the two boundary problems. To uniquely reconstruct a step density function, we need to use natural frequencies of the boundary value problem itself and natural frequencies of another boundary problem, which differs from the first one only by one boundary condition. In M. Krein uniqueness theorems, to restore the continuous density function, we used all the eigenvalues of the two problems. In contrast to the M. Krein uniqueness theorems, for the uniqueness of the recovery of the n-step density function, we need a finite number of eigenvalues.

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1. Introduction

A large number of papers are devoted to direct and inverse problems [6,12–21]. This article deals with the problem of identifying the densities of two materials of which the string consists by two natural frequencies of its oscillations. The closest to our article are works of M.G. Gasymov, R. Carlson, H.M. Hüseynov, and E.N. Akhmedova where problems of restoring discontinuous coefficients by transform operator and Weyl function were considered [1–5]. In contrast to these works, we use neither the transformation operator nor Weyl function, but eigenvalues to restore discontinuous coefficients. In Krein's papers [7–11] considered the problem of identifying a continuous variable string density from the spectra of two problems. We also restore the string density, but the difference between our task is that we restore not a continuous, but a discontinuous step function. In this case, to restore the density, we use not two infinite sets of eigenvalues, but a finite number of them.

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2. Identification of A Step Density Function by Eigenvalues of One Boundary Problem

We consider Sturm-Liouville boundary problem for string on the inteval $[0, \pi]$ with a piecewise step density function

$$\rho(x) = \begin{cases} \rho_1, & 0 \le x < a, \\ \rho_2, & a < x \le \pi \end{cases}$$

:

$$-y'' + q(x)y = \lambda\rho(x)y, \quad x \in [0, a) \cup (a, \pi], \tag{2.1}$$

with boundary conditions

$$y(0) = 0, \quad y(\pi) = 0, \tag{2.2}$$

and the transmission conditions

$$y(-a) - y(+a) = 0, \quad y'(-a) - y'(+a) = 0, \tag{2.3}$$

where λ is a complex parameter, $q(x) \in C(0, \pi)$, $\rho_1, \rho_2 = const \neq 0$.

Let $y_{1i}(x, \lambda)$ and $y_{2i}(x, \lambda)$ be linearly independent solutions of Equation

$$-y_i'' + q(x)y_i = \lambda\rho_i(x)y_i \tag{2.4}$$

satisfying the conditions

$$y_{1i}(0, \lambda) = 1, \quad y'_{1i}(0, \lambda) = 0, \quad y_{2i}(0, \lambda) = 0, \quad y'_{2i}(0, \lambda) = 1. \tag{2.5}$$

The characteristic determinant of the problem (2.1)–(2.3) has the following form

$$\Delta_1(\lambda) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & y_{12}(\pi) & y_{22}(\pi) \\ y_{11}(-a) & y_{21}(-a) & -y_{12}(+a) & -y_{22}(+a) \\ y'_{11}(-a) & y'_{21}(-a) & -y'_{12}(+a) & -y'_{22}(+a) \end{vmatrix}. \tag{2.6}$$

The function $\Delta_1(\lambda)$ is a transcendental function with respect to variables ρ_1 and ρ_2. Therefore, if λ_1 and λ_2 are the first two eigenvalues, then the system of equations $\Delta_1(\lambda_1) = 0$, $\Delta_1(\lambda_2) = 0$ has an infinite number of solution pairs (ρ_1, ρ_2). However, in some small neighborhood of the point (ρ_1, ρ_2) this solution is unique.

Let's demonstrate this statement with a concrete example.

Example 2.1. Let the eigenvalues of the problem (2.1)–(2.3) are the following numbers $\lambda_1 = 0.40361$, $\lambda_2 = 1.3761$, $a = 1$, and $q(x) = 1 + 2x$.

Linearly independent solutions of Equation (2.4) satisfying the conditions (2.5) have the following forms

$$\begin{aligned} y_{11}(x, \lambda) = 1 + \left(\frac{1}{2} - \frac{1}{2}\rho_1\,\lambda^2\right)x^2 + \frac{1}{24}(-1 + \rho_1 \cdot \lambda^2)^2\,x^4 + \cdots - \\ - \frac{\rho_1^{13}\,\lambda^{26}\,x^{29}}{2255551529015230090444800000} + O(x^{30}). \end{aligned} \tag{2.7}$$

$$\begin{aligned} y_{21}(x, \lambda) = x + \frac{1}{6}x^4 + \left(\frac{1}{6} - \frac{1}{6}\rho_1\,\lambda^2\right)x^3 + \cdots + \\ \frac{\rho_1^{14}\,\lambda^{28} \cdot x^{29}}{884176199373970195454361600000} + O(x^{30}), \end{aligned} \tag{2.8}$$

$$\begin{aligned} y_{12}(x, \lambda) = 1 + \frac{1}{3}x^3 + \left(\frac{1}{2} - \frac{1}{2}\rho_2\,\lambda^2\right)x^2 + \cdots - \\ - \frac{1}{2255551529015230090444800000}\rho_2^{13}\,\lambda^{26}\,x^{29} + O(x^{30}), \end{aligned} \tag{2.9}$$

$$\begin{aligned} y_{22}(x, \lambda) = x + \frac{1}{6}x^4 + \left(\frac{141961}{1307674368000} - \frac{21601}{186810624000}\rho_2\,\lambda^2\right)x^{15} + \cdots - \\ + \frac{1}{884176199373970195454361600000}\rho_2^{14}\,\lambda^{28}\,x^{29} + O(x^{30}) \end{aligned} \tag{2.10}$$

Here, linearly independent solutions of Equation (2.4) are not represented as Airy functions, but are expanded in a series in x and λ, since this approach is more convenient for solving the problem of identification of ρ_1 and ρ_2.

Substituting (2.7)–(2.10) in (2.6), we get the system of equations $\Delta_1(\lambda_1) = 0$, $\Delta_1(\lambda_2) = 0$. Solving this system of equations using a package of analytical calculations, we obtain an infinite number of solutions

$$\begin{gathered} \ldots, \quad \{\rho_1 = -0.82197, \quad \rho_2 = 0.35146\}, \quad \{\rho_1 = 1.0000, \quad \rho_2 = 2.0000\}, \\ \{\rho_1 = 2.3412, \quad \rho_2 = 3.1076\}, \quad \{\rho_1 = 5.4088, \quad \rho_2 = 5.2162\}, \\ \{\rho_1 = 8.7059, \quad \rho_2 = 6.8780\}, \quad \ldots \end{gathered} \tag{2.11}$$

Now consider another Sturm-Liouville boundary problem for string on the inteval $[0,\pi]$ with a piecewise step density function

$$\rho(x) = \begin{cases} \rho_1, & 0 \le x < a, \\ \rho_2, & a < x \le \pi \end{cases}$$

:

$$-y'' + q(x)y = \lambda\rho(x)y, \quad x \in [0,a) \cup (a,\pi], \tag{2.12}$$

with another boundary conditions

$$y'(0) = 0, \quad y(\pi) = 0, \tag{2.13}$$

and the transmission conditions

$$y(-a) - y(+a) = 0, \quad y'(-a) - y'(+a) = 0, \tag{2.14}$$

where λ is a complex parameter, $q(x) \in C(0,\pi)$, $\rho_1, \rho_2 = const \neq 0$.

The characteristic determinant of the problem (2.12)–(2.14) has the following form

$$\Delta_2(\lambda) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & y_{12}(\pi) & y_{22}(\pi) \\ y_{11}(-a) & y_{21}(-a) & -y_{12}(+a) & -y_{22}(+a) \\ y'_{11}(-a) & y'_{21}(-a) & -y'_{12}(+a) & -y'_{22}(+a) \end{vmatrix}. \tag{2.15}$$

The function $\Delta_2(\lambda)$ is a transcendental function with respect to variables ρ_1 and ρ_2. Therefore, if λ_{12} and λ_{22} are the first two eigenvalues, then the system of equations $\Delta_2(\lambda_{12}) = 0$, $\Delta_1(\lambda_{22}) = 0$ has an infinite number of solution pairs (ρ_1, ρ_2). However, in some small neighborhood of the point (ρ_1, ρ_2) this solution is unique.

Example 2.2. Let the eigenvalues of the problem (2.12)–(2.14) are the following numbers $\lambda_{12} = 0.42722$, $\lambda_{22} = 1.6204$, $a = 1$, and $q(x) = 1 + 2x$.

Substituting (2.7)–(2.10) in (2.15), we get the system of equations $\Delta_1(\lambda_{12}) = 0$, $\Delta_1(\lambda_{22}) = 0$. Solving this system of equations using a package of analytical calculations, we obtain an infinite number of solutions

$$\begin{gathered} \ldots, \quad \{\rho_1 = 1.0000, \quad \rho_2 = 2.0000\}, \quad \{\rho_1 = 3.4126, \quad \rho_2 = 3.7606\}, \\ \{\rho_1 = 5.3436, \quad \rho_2 = 4.9578\}, \quad \{\rho_1 = 24.680, \quad \rho_2 = 31.881\}, \\ \{\rho_1 = 45.776, \quad \rho_2 = 54.028\}, \quad \ldots \end{gathered} \tag{2.16}$$

3. Identification of A Step Density Function by Eigenvalues of Two Boundary Problem

M.G. Krein [7–11] used for the uniqueness of the restoration of continuous density not one spectrum, but two. This suggests that, for the uniqueness of the restoration of the step density, we use the eigenvalues of two boundary value problems. Indeed, the eigenvalues of the two problems allow us to show the uniqueness of the step density recovery.

Example 3.1. Let the eigenvalues of the problem (2.1)–(2.3) are the following numbers $\lambda_1 = 0.40361$, $\lambda_2 = 1.3761$, the eigenvalues of the problem (2.12)–(2.14) are the following numbers $\lambda_{12} = 0.42722$, $\lambda_{22} = 1.6204$, $a = 1$, and $q(x) = 1 + 2x$.

Solving this system of equations $\Delta_1(\lambda_1) = 0$, $\Delta_1(\lambda_2) = 0$ using a package of analytical calculations, we obtain an infinite number of solutions (2.11). Solving this system of equations $\Delta_2(\lambda_{12}) = 0$, $\Delta_2(\lambda_{22}) = 0$ using a package of analytical calculations, we obtain an infinite number of solutions (2.16). The intersection of these solutions is the only solution $\{\rho_1 = 1.0000, \quad \rho_2 = 2.0000\}$.

From the foregoing, the following follows:

Theorem 3.2. *To uniquely reconstruct a two-step function, you need to use two eigenvalues (two natural frequencies) of one boundary problem and two eigenvalues (two natural frequencies) of another boundary problem, which differs from the first one only by one boundary condition.*

Theorem 3.3. *To uniquely reconstruct a two-step function, you need to use two eigenvalues (two natural frequencies) of one boundary problem and two eigenvalues (two natural frequencies) of another boundary problem, which differs from the first one only by one boundary condition.*

4. Conclusion

Thus, for the uniqueness of the recovery of the step density of a string, the eigenvalues of one boundary value problem are not enough. We need to use the eigenvalues of the two boundary problems. To uniquely reconstruct a two-step function, you need to use two eigenvalues (two natural frequencies) of one boundary problem and two eigenvalues (two natural frequencies) of another boundary problem, which differs from the first one only by one boundary condition.

A similar situation in the general case. For a unique restoration of the n step density, we need to use n eigenvalues of a certain boundary value problem and n eigenvalues of another boundary problem, which differs from the first only in the boundary condition.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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