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ON NEW GRÜSS TYPE INEQUALITIES FOR CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. We use conformable fractional integral, recently introduced by Khalil et. al. and Abdeljavad, to obtain some new integral inequalities of Grüss type. We show two new theorems associated with Grüss inequality, as well as state and show new identities related to this fractional integral operator.

Keywords: Grüss inequality, Riemann-Liouville fractional integrals, conformable fractional integrals.

AMS Subject Classification: 26A33, 26D10, 33B20.

1. Introduction

In 1935, Grüss [5] proved the well known inequality:

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\leq\frac{(M-m)(P-p)}{4} \quad (1)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the conditions

$$m\leq f(x)\leq M,\quad p\leq g(x)\leq P,\quad m,M,p,P\in\mathbb{R},x\in[a,b]. \quad (2)$$

For some recent counterparts, generalizations of Grüss inequality, the reader is refer to [7, 8]. The Beta function:

$$B\left(a,b\right)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt,\qquad a,b>0,$$

where $\Gamma\left(\alpha\right)=\int_0^\infty e^{-u}u^{\alpha-1}du$ is Gamma function.

Definition 1.1. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ are defined by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\qquad x>a$$

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and

$$J^{\alpha}_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b$$

respectively. Here $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$.

In [2], Dahmani *et al.* gave following theorems for the Grüss inequalities.

Theorem 1.1. *Let f and g be two integrable functions on $[0,\infty)$ satisfying the condition (2) on $[0,\infty)$. Then for all $t>0,\ \alpha>0.$, we have:*

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\alpha}g(t)\right|\le\left(\frac{t^{\alpha}}{2\Gamma(\alpha+1)}\right)^2(M-m)(P-p). \tag{3}$$

Theorem 1.2. *Let f and g be two integrable functions on $[0,\infty)$ satisfying the condition (2) on $[0,\infty)$. Then for all $t>0,\ \alpha>0,\ \beta>0$, we have:*

$$\begin{aligned}&\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}fg(t)+\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}fg(t)-J^{\alpha}f(t)J^{\beta}g(t)-J^{\beta}f(t)J^{\alpha}g(t)\right)^2\\ \le&\left[\left(M\frac{t^{\alpha}}{\Gamma(\alpha+1)}-J^{\alpha}f(t)\right)\left(J^{\beta}f(t)-m\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right.\\ &\left.+\left(J^{\alpha}f(t)-m\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(M\frac{t^{\beta}}{\Gamma(\beta+1)}-J^{\beta}f(t)\right)\right]\\ &\times\left[\left(P\frac{t^{\alpha}}{\Gamma(\alpha+1)}-J^{\alpha}f(t)\right)\left(J^{\beta}f(t)-p\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right.\\ &\left.+\left(J^{\alpha}f(t)-p\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(P\frac{t^{\beta}}{\Gamma(\beta+1)}-J^{\beta}f(t)\right)\right]\end{aligned}\tag{4}$$

In [6], Khalil *et al.* define a new well-behaved simple fractional derivative called conformable fractional derivative depending just on the basic limit definition of the derivative. They also defined the fractional integral of order $0<\alpha\le 1$ only.

In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha>0$.

Definition 1.2. *Let $\alpha\in(n,n+1]$, $n=0,1,2,...$ and set $\beta=\alpha-n$. Then the left conformable fractional integral of any order $\alpha>0$ is defined by*

$$(I^a_{\alpha}f)(t)=\frac{1}{n!}\int_a^t (t-x)^n(x-a)^{\beta-1}f(x)dx \tag{5}$$

Analogously, the right conformable fractional integral of any order $\alpha>0$ is defined by

$$({}^bI_{\alpha}f)(t)=\frac{1}{n!}\int_t^b (x-t)^n(b-x)^{\beta-1}f(x)dx. \tag{6}$$

Recently many authors have presented a number of interesting integral inequalities using conformable fractional integrals. For instance, see [4, 9, 10, 11, 12, 13, 14].
Note that, we present our new results associated with the conformable fractional integral using the left-sided conformable fractional integral , only. Moreover, we admit $a=0$ in (5) in order to get

$$(I_{\alpha}f)(t)=\frac{1}{n!}\int_0^t (t-x)^n x^{\beta-1}f(x)dx.$$

The main purpose of this paper is to establish some new Grüss type inequalities in the form of conformable fractional integral.

2. Grüss Type Inequalities

Lemma 2.1. *Let f be an integrable function on $[0,\infty)$ satisfying the condition (2) on $[0,\infty)$. Then for all $t>0$, $\alpha \in (n,n+1]$, n=0,1,2,..., we have:*

$$\begin{aligned} & \frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(f^2)(t)-\left(I_{\alpha}(f)\right)^2(t) \\ = & \left(\frac{1}{n!}Mt^{\alpha}B(n+1,\alpha-n)-(I_{\alpha}f)(t)\right)\left(I_{\alpha}(f)(t)-\frac{1}{n!}mt^{\alpha}B(n+1,\alpha-n)\right) \\ & -\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(M-f(t))(f(t)-m). \end{aligned} \tag{7}$$

Proof. For any $x,y\in[0,\infty)$, we have

$$\begin{aligned} & (M-f(y))(f(x)-m)+(M-f(x))(f(y)-m) \\ & -(M-f(x))(f(x)-m)-(M-f(y))(f(y)-m) \\ = & f^2(x)+f^2(y)-2f(x)f(y). \end{aligned} \tag{8}$$

Multiplying (8) by $\frac{1}{n!}(t-x)^n x^{\alpha-n-1}$ and integrating the resulting identity with respect to x over $[0,t]$ we have

$$\begin{aligned} & (M-f(y))\left(I_{\alpha}(f)(t)-\frac{m}{n!}t^{\alpha}B(n+1,\alpha-n)\right) \\ & +\left(\frac{M}{n!}t^{\alpha}B(n+1,\alpha-n)-I_{\alpha}(f)(t)\right)(f(y)-m) \\ & -I_{\alpha}(M-f(t))(f(t)-m)-\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)(M-f(y))(f(y)-m) \\ = & I_{\alpha}(f^2)(t)+\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)f^2(y)-2I_{\alpha}(f)(t)f(y) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \int_0^t (t-x)^n x^{\alpha-n-1}dx &= \int_0^1 (t-tu)^n (ut)^{\alpha-n-1}du \\ &= t^{\alpha}B(n+1,\alpha-n) \end{aligned}$$

Now, we multiplying (9) by $\frac{1}{n!}(t-y)^n y^{\alpha-n-1}$ and integrating the resulting identity with respect to y over $[0,t]$ we have

$$\begin{aligned} & \left(\frac{M}{n!}t^{\alpha}B(n+1,\alpha-n)-(I_{\alpha}f)(t)\right)\left(I_{\alpha}(f)(t)-\frac{m}{n!}t^{\alpha}B(n+1,\alpha-n)\right) \\ & +\left(\frac{M}{n!}t^{\alpha}B(n+1,\alpha-n)-I_{\alpha}(f)(t)\right)\left((I_{\alpha}f)(t)-\frac{m}{n!}t^{\alpha}B(n+1,\alpha-n)\right) \\ & -\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(M-f(t))(f(t)-m) \\ & -\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(M-f(t))(f(t)-m) \\ = & \frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(f^2)(t)+\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)I_{\alpha}(f^2)(t)-2I_{\alpha}(f)(t)I_{\alpha}(f)(t) \end{aligned}$$

So we have (7) and the proof is completed. □

Theorem 2.1. *Let f and g be two integrable functions on $[0,\infty)$ satisfying the condition (2) on $[0,\infty)$. Then for all $t>0$, $\alpha\in(n,n+1]$, n=0,1,2,..., we have*

$$\begin{aligned}&\left|\frac{t^\alpha}{n!}B(n+1,\alpha-n)I_\alpha(fg)(t)-(I_\alpha f(t))(I_\alpha g(t))\right|\\ \leq&\left(\frac{t^\alpha}{2n!}B(n+1,\alpha-n)\right)^2(M-m)(P-p).\end{aligned}\tag{10}$$

Proof. We define

$$H(x,y):=(f(x)-f(y))(g(x)-g(y));\qquad x,y\in[a,b].\tag{11}$$

We multiplying both sides of obtained identity by $\frac{1}{(n!)^2}(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}$ and integrating the resulting identity with respect to over $[0,t]$ we have

$$\begin{aligned}&\frac{1}{(n!)^2}\int_0^t\int_0^t(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}H(x,y)dxdy\\ =&\frac{2}{n!}t^\alpha B(n+1,\alpha-n)I_\alpha(fg)(t)-2(I_\alpha f(t))(I_\alpha g(t)).\end{aligned}\tag{12}$$

Using Cauchy-Schwarz inequality $\left(\int fgh=\int f^{1/2}gf^{1/2}h\leq\left(\int fg^2\right)^{1/2}\left(\int fh^2\right)^{1/2}\right)$, we have

$$\begin{aligned}&\frac{1}{(n!)^2}\int_0^t\int_0^t(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}H(x,y)dxdy\\
=&\frac{1}{(n!)^2}\int_0^t\int_0^t(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}(f(x)-f(y))(g(x)-g(y))dxdy\\
\leq&\frac{1}{(n!)^2}\left(\int_0^t\int_0^t(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}(f(x)-f(y))^2dxdy\right)^{1/2}\\
&\times\left(\int_0^t\int_0^t(t-x)^n t^{\alpha-n-1}(t-y)^n t^{\alpha-n-1}(g(x)-g(y))^2dxdy\right)^{1/2}\\
=&\frac{1}{n!}\Bigg[\int_0^t\Bigg(\int_0^t(t-x)^n t^{\alpha-n-1}f^2(x)dx-2\int_0^t(t-x)^n t^{\alpha-n-1}f(x)f(y)dx\\
&+\int_0^t(t-x)^n t^{\alpha-n-1}f^2(y)dx\Bigg)(t-y)^n t^{\alpha-n-1}dy\Bigg]^{1/2}\\
&\times\frac{1}{n!}\Bigg[\int_0^t\Bigg(\int_0^t(t-x)^n t^{\alpha-n-1}g^2(x)dx-2\int_0^t(t-x)^n t^{\alpha-n-1}g(x)g(y)dx\\
&+\int_0^t(t-x)^n t^{\alpha-n-1}g^2(y)dx\Bigg)(t-y)^n t^{\alpha-n-1}dy\Bigg]^{1/2}\end{aligned}$$

$$
\begin{aligned}
= &\left(\frac{1}{n!}(I_\alpha f^2)(t)\int_0^t (t-y)^n t^{\alpha-n-1}dy - \frac{2}{n!}(I_\alpha f)(t)\int_0^t (t-y)^n t^{\alpha-n-1}f(y)dy\right.\\
&\left.+\frac{1}{n!}t^\alpha B(n+1,\alpha-n)\int_0^t (t-y)^n t^{\alpha-n-1}f^2(y)\right)^{1/2}\\
&\times\left(\frac{1}{n!}(I_\alpha g^2)(t)\int_0^t (t-y)^n t^{\alpha-n-1}dy - \frac{2}{n!}(I_\alpha g)(t)\int_0^t (t-y)^n t^{\alpha-n-1}g(y)dy\right.\\
&\left.+\frac{1}{n!}t^\alpha B(n+1,\alpha-n)\int_0^t (t-y)^n t^{\alpha-n-1}g^2(y)\right)^{1/2}\\
= &\left(\frac{2t^\alpha}{n!}B(n+1,\alpha-n)(I_\alpha f^2)(t) - 2(I_\alpha f)^2\right)\left(\frac{2t^\alpha}{n!}B(n+1,\alpha-n)(I_\alpha g^2)(t) - 2(I_\alpha g)^2\right).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&\left(\frac{1}{n!}t^\alpha B(n+1,\alpha-n)I_\alpha(fg)(t) - (I_\alpha f(t))(I_\alpha g(t))\right)^2 \qquad (13)\\
\leq &\left(\frac{1}{n!}t^\alpha B(n+1,\alpha-n)(I_\alpha f^2)(t) - (I_\alpha f(t))^2\right)\\
&\times\left(\frac{1}{n!}t^\alpha B(n+1,\alpha-n)(I_\alpha g^2)(t) - (I_\alpha g(t))^2\right)
\end{aligned}
$$

Since $(M-f(x))(f(x)-m)\geq 0$ and $(P-g(x))(g(x)-p)\geq 0$, we have

$$\frac{1}{n!}t^\alpha B(n+1,\alpha-n)I_\alpha(M-f(t))(f(t)-m)\geq 0$$

and

$$\frac{1}{n!}t^\alpha B(n+1,\alpha-n)I_\alpha(P-g(t))(g(t)-P)\geq 0.$$

So, from Lemma 2.1, we have

$$
\begin{aligned}
&\frac{1}{n!}t^\alpha B(n+1,\alpha-n)(I_\alpha f^2)(t) - (I_\alpha f(t))^2 \qquad (14)\\
\leq &\left(\frac{M}{n!}t^\alpha B(n+1,\alpha-n) - (I_\alpha f)(t)\right)\left((I_\alpha f)(t) - \frac{m}{n!}t^\alpha B(n+1,\alpha-n)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{n!}t^\alpha B(n+1,\alpha-n)(I_\alpha g^2)(t) - (I_\alpha g(t))^2 \qquad (15)\\
\leq &\left(\frac{P}{n!}t^\alpha B(n+1,\alpha-n) - (I_\alpha g)(t)\right)\left((I_\alpha g)(t) - \frac{p}{n!}t^\alpha B(n+1,\alpha-n)\right).
\end{aligned}
$$

By using the inequalities (14), (15) and (13), we get

$$
\begin{aligned}
&\left(\frac{1}{n!}t^\alpha B(n+1,\alpha-n)I_\alpha(fg)(t) - (I_\alpha f(t))(I_\alpha g(t))\right)^2\\
\leq &\left(\frac{M}{n!}t^\alpha B(n+1,\alpha-n) - (I_\alpha f)(t)\right)\left((I_\alpha f)(t) - \frac{m}{n!}t^\alpha B(n+1,\alpha-n)\right)\\
&\times\left(\frac{P}{n!}t^\alpha B(n+1,\alpha-n) - (I_\alpha g)(t)\right)\left((I_\alpha g)(t) - \frac{p}{n!}t^\alpha B(n+1,\alpha-n)\right). \qquad (16)
\end{aligned}
$$

Then, using the elementary inequality $4rs \leq (r+s)^2$ $r,s \in \mathbb{R}$, we obtain

$$\begin{aligned} & 4\left(\frac{M}{n!}t^{\alpha}B(n+1,\alpha-n)-(I_{\alpha}f)(t)\right)\left((I_{\alpha}f)(t)-\frac{m}{n!}t^{\alpha}B(n+1,\alpha-n)\right) \\ \leq & \left(\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)(M-m)\right)^{2} \end{aligned} \tag{17}$$

and

$$\begin{aligned} & 4\left(\frac{P}{n!}t^{\alpha}B(n+1,\alpha-n)-(I_{\alpha}f)(t)\right)\left((I_{\alpha}f)(t)-\frac{p}{n!}t^{\alpha}B(n+1,\alpha-n)\right) \\ \leq & \left(\frac{1}{n!}t^{\alpha}B(n+1,\alpha-n)(P-p)\right)^{2}. \end{aligned} \tag{18}$$

From (16), (17) and (18), we get desired result. □

Remark 2.1. *If we take $\alpha = n+1$ in Theorem 2.1, the inequality (10) becomes inequality (3).*

Theorem 2.2. *Let f and g be two integrable functions on $[0,\infty)$. Then for all $t > 0$, $\alpha \in [n, n+1)$ and $\beta \in [k, k+1)$, n,k=0,1,2,..., we have*

$$\begin{aligned} & \Bigg(\frac{t^{\alpha}}{n!}B(n+1,\alpha-n)(I_{\beta}fg)(t)+\frac{t^{\beta}}{k!}B(k+1,\beta-k)(I_{\alpha}fg)(t) \\ & -(I_{\alpha}f)(t)(I_{\beta}g)(t)-(I_{\beta}f)(t)(I^{\alpha}g)(t)\Bigg)^{2} \\ & \leq \left(\frac{t^{\alpha}}{n!}B(n+1,\alpha-n)(I_{\beta}f^{2})(t)+\frac{t^{\beta}}{k!}B(k+1,\beta-k)(I_{\alpha}f^{2})(t)-2(I_{\alpha}f)(t)(I_{\beta}f)(t)\right) \\ & \times\left(\frac{t^{\alpha}}{n!}B(n+1,\alpha-n)(I_{\beta}g^{2})(t)+\frac{t^{\beta}}{k!}B(k+1,\beta-k)(I_{\alpha}g^{2})(t)-2(I_{\alpha}g)(t)(I_{\beta}g)(t)\right). \end{aligned}$$

Proof. Multiplying (11) by $\frac{1}{n!k!}(t-x)^{n}t^{\alpha-n-1}(t-y)^{k}t^{\beta-k-1}$ and integrating the resulting identity with respect to x and y over $(0,t)^2$, we get

$$\begin{aligned} & \frac{1}{n!k!}\int_{0}^{t}\int_{0}^{t}(t-x)^{n}t^{\alpha-n-1}(t-y)^{k}t^{\beta-k-1}H(x,y)dxdy \\ = & \frac{t^{\alpha}}{n!}B(n+1,\alpha-n)(I_{\beta}fg)(t)+\frac{t^{\beta}}{k!}B(k+1,\beta-k)(I_{\alpha}fg)(t) \\ & -(I_{\alpha}f)(t)(I_{\beta}g)(t)-(I_{\beta}f)(t)(I^{\alpha}g)(t). \end{aligned}$$

Then, applying Cauchy-Schwarz inequality for double integrals similarly Theorem 2.1, we obtain desired results. □

Lemma 2.2. *Let f be an integrable function on $[0,\infty)$ satisfying the condition (2) on $[0,\infty)$. Then for all $t>0$, $\alpha\in(n,n+1]$, $\beta\in[k,k+1)$,n,k=0,1,2,..., we have:*

$$\begin{aligned}
&\frac{t^\alpha}{n!}(I_\beta f^2)(t)+\frac{t^\beta}{k!}(I_\alpha f^2)(t)-2(I^\alpha f)(t)(I_\beta f)(t)\\
=&\left(\frac{Mt^\alpha}{n!}B(n+1,\alpha-n)-(I_\alpha f)(t)\right)\left((I_\beta f)(t)-\frac{mt^\beta}{k!}B(k+1,\beta-k)\right)\\
&+\left(\frac{Mt^\beta}{n!}B(k+1,\beta-n)-(I_\beta f)(t)\right)\left((I_\alpha f)(t)-\frac{mt^\alpha}{n!}B(n+1,\alpha-n)\right)\\
&-\frac{t^\alpha}{n!}B(n+1,\alpha-n)(I_\beta)(M-f(t))(f(t)-m)\\
&-\frac{t^\beta}{k!}B(k+1,\alpha-k)(I_\alpha)(M-f(t))(f(t)-m).
\end{aligned}$$

Proof. Multiplying (9) by $\frac{1}{k!}(t-y)^k t^{\beta-k-1}$ and integrating the resulting identity with respect to y from 0 to t, we have

$$\begin{aligned}
&\left(I_\alpha f(t)-\frac{mt^\alpha}{n!}B(n+1,\alpha-n)\right)\frac{1}{k!}\int_0^t(t-y)^k t^{\beta-n-1}(M-f(y))dy\\
&+\left(\frac{Mt^\alpha}{!n}B(n+1,\alpha-n)-I_\alpha f(t)\right)\frac{1}{k!}\int_0^t(t-y)^k t^{\beta-n-1}(f(y)-m)dy\\
&-I_\alpha\left((M-f(t))(f(t)-m)\right)\frac{1}{k!}\int_0^t(t-y)kt^{\beta-n-1}dy\\
&-\frac{t^\alpha}{k!}B(n+1,\alpha-n)\int_0^t(t-y)^k t^{\beta-n-1}(M-f(y))(f(y)-m)dy\\
=&\ \frac{t^\alpha}{n!}B(n+1,\alpha-n)(I_\beta f^2)(t)+\frac{t^\beta}{k!}B(k+1,\beta-k)(I_\alpha f^2)(t)-2(I_\alpha f)(t)(I_\beta f)(t).
\end{aligned}$$

So, we obtain desired results. □

Theorem 2.3. *Let f and g be two integrable functions on $[0,\infty)$ satisfying the conditions (2) on $[0,\infty)$. Then for all $t>0$, $\alpha\in[n,n+1)$, $\beta\in[k,k+1)]$, n,k=0,1,2,..., we have*

$$\begin{aligned}
&\left(\frac{t^\alpha}{n!}B(n+1,\alpha-n)(I_\beta fg)(t)+\frac{t^\beta}{k!}B(k+1,\alpha-k)(I_\alpha fg)(t)\right.\\
&\left.-(I_\alpha f)(t)(I_\beta g)(t)-(I_\beta f)(t)(I_\alpha g)(t)\right)^2\\
&\leq\left[\left(\frac{Mt^\alpha}{n!}B(n+1,\alpha-n)-(I_\alpha f)(t)\right)\left((I_\beta f)(t)-\frac{mt^\beta}{k!}B(k+1,\beta-k)\right)\right.\\
&\left.+\left((I_\alpha f)(t)-\frac{mt^\alpha}{n!}B(n+1,\alpha-n)\right)\left(\frac{Mt^\beta}{k!}B(k+1,\beta-k)-(I_\beta f)(t)\right)\right]\\
&\left[\left(\frac{Pt^\alpha}{n!}B(n+1,\alpha-n)-(I_\alpha g)(t)\right)\left((I_\beta g)(t)-\frac{pt^\beta}{k!}B(k+1,\beta-k)\right)\right.\\
&\left.+\left((I_\alpha g)(t)-\frac{pt^\alpha}{n!}B(n+1,\alpha-n)\right)\left(\frac{Pt^\beta}{k!}B(k+1,\beta-k)-(I_\beta g)(t)\right)\right].
\end{aligned}\tag{19}$$

Proof. Since $(M - f(x))(f(x) - m) \geq 0$ and $(P - g(x))(g(x) - p)$, we can write

$$-\frac{t^{\alpha}}{n!}B(n+1, \alpha-n)(I_{\beta}(M-f(t))(f(t)-m)) - \frac{t^{\beta}}{k!}B(k+1, \beta-k)(I_{\alpha}(M-f(t))(f(t)-m)) \leq 0 \quad (20)$$

and

$$-\frac{t^{\alpha}}{n!}B(n+1, \alpha-n)(I_{\beta}(P-g(t))(g(t)-p)) - \frac{t^{\beta}}{k!}B(k+1, \beta-k)(I_{\alpha}(P-g(t))(g(t)-p)) \leq 0. \quad (21)$$

Applying Lemma 2.2 to f and g, then using Teorem 2.2, (20) and (21), we obtained desired result. □

Remark 2.2. *If we take $\alpha = \beta$ we obtain Theorem 2.1.*

Remark 2.3. *If we take $\alpha = n + 1$ in Theorem 2.3 , we get inequality (4).*

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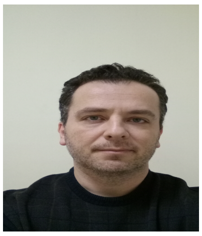

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