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EXISTENCE AND MULTIPLICITY OF WEAK SOLUTIONS FOR PERTURBED KIRCHHOFF TYPE ELLIPTIC PROBLEMS WITH HARDY POTENTIAL

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ABSTRACT. In this paper, we prove the existence of at least three weak solutions for a doubly eigenvalue elliptic systems involving the p-biharmonic equation with Hardy potential of Kirchhoff type with Navier boundary condition. More precisely, by using variational methods and three critical points theorem due to B. Ricceri, we establish multiplicity results on the existence of weak solutions for such problems where the reaction term is a nonlinearity function f which satisfies in the some convenient growth conditions. Indeed, using a consequence of the critical point theorem due to Ricceri, which in it the coercivity of the energy Euler functional was required and is important, we attempt the existence of multiplicity solutions for our problem under algebraic conditions on the nonlinear parts. We also give an explicit example to illustrate the obtained result.

Keywords: Multiplicity of weak solutions, perturbed Kirchhoff type elliptic problems,

Hardy potential, Critical points.

AMS Subject Classification: 35J35, 35J60

1. Introduction

In this article, we study the existence of three weak solutions for the fourth-order Kirchhoff type elliptic problems with Hardy potential and Navier boundary condition

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p\,dx\right)\Delta_p^2 u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega,\\ u=0, \quad \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin and with smooth boundary $\partial\Omega$, $1 < p < \frac{N}{2}, \Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is an operator of fourth order, the so-called p-biharmonic operator, λ, μ are two positive parameters, $M : [0, +\infty[\to \mathbb{R}$ is a

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continuous function, and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions. The problem (1) is related to the stationary problem

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the length and ρ_0 the initial axial tension. Kirchhoff [42] first introduced the model given by the equation (2), which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. Equation (2) was developed to form

$$u_{tt}(x) - M\left(\int_\Omega |\nabla u(x)|^2 \, dx \right) \Delta u(x) = f(x, u(x)).$$

After that, many authors studied the following nonlocal elliptic boundary value problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \, dx \right) \Delta u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Problems like (3) are called the Kirchhoff type problems that can be used for modeling several physical and biological systems where u describes a process which depends on the average of it self, such as the population density, see [1]. Such nonlinear Kirchhoff model can also be used for describing the dynamics of an axially moving string. In recent years, axially moving string-like continua such as wires, belts, chains, band-saws have been subjects of the study of researchers (see [59]) and also many interesting results for problem of Kirchhoff type were obtained [1, 23, 29, 31, 34, 45, 47, 62]. Similar nonlocal problems also model several physical and biological systems where u describes a process that depends on the average of itself, for example, the population density. We also refer the reader to [3, 4, 16, 27, 46, 52, 58] which discuss the historical development of the problems as well as describe situations that can be realistically modeled by Kirchhoff-type problems. Recently, using the variational methods, Graef, Heidarkhani and Kong [27] studied the existence of at least three weak solutions to the Kirchhoff-type problem

$$\begin{cases} -K\left(\int_\Omega |\nabla u|^2 \, dx \right) \Delta u(x) = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Precisely, they in the cited paper by using three critical points theorem due to Ricceri that follows from [57, Theorem 1] and is fundamental in your discussion established some new results for the existence of three weak solutions of (4). Also, in [63], the authors by using the mountain-pass techniques and the truncation method, studied the existence of nontrivial solutions for a class of fourth-order elliptic equations of Kirchhoff type. In [12], writers studied the problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \, dx \right) \Delta_p u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

in the semi-position case; i.e., $f(0) < 0$. In this paper ([12]) using Browder Theorem authors obtained the existence and uniqueness of solutions for (5). Fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. In

Lazer and Mckenna [43], have pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. Due to this, many authors have studied the existence of at least one solution, or multiple solutions, or even infinitely many solutions for fourth-order boundary value problems using lower and upper solution methods, Morse theory, the mountain-pass theorem, constrained minimization and concentration-compactness principle, fixed-point theorems and degree theory, and critical point theory and variational methods, and we refer the reader to the papers [10, 11, 14] and references therein. In recent years, the existence and multiplicity of stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) has been studied, via variational methods like the symmetric mountain pass theorem in [18] and via a three critical point theorem in [5]. Wang and An in [62], using the mountain-pass theorem established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem

$$\begin{cases} \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u(x) = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

We also refer the reader to the papers [22, 39, 48] for such other fourth-order problems. On the other hand, singular elliptic problems have been intensively studied in recent years, see for example, [25, 26, 44, 55] and the references. Recently, motivated by this large interest, the problem

$$\begin{cases} \Delta_p^2 u = \dfrac{|u|^{p-2}u}{|x|^{2p}} + g(\lambda, x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

where $g :]0, +\infty[\times \mathbb{R} \to \mathbb{R}$ is a suitable function, has been extensively investigated. For instance, when $p = 2$, Wang and Shen [64] considered the problem (7), assuming that the nonlinearity has the form $g(\lambda, x, u) = f(x, u)$. In this setting, the existence of non-trivial solutions by using variational methods is established. Successively, Berchio et al. [7] considered the case $g(\lambda, x, u) = (1 + u)^q$, are studied the behavior of extremal solutions to biharmonic Gelfand-type equations under Steklov boundary conditions. Also in [17, 51, 53], the authors are interested in the existence and multiplicity solutions for this kind of singular elliptic problems. Precisely, the existence of multiple solutions is proved by Chung [17] through a variant of the three critical point theorem by Bonanno [8]. Pérez-Llanos and Primo [53] studied the optimal exponent q to have solvability of problem with $g(\lambda, x, u) = u^q + cf$. Sign-changing solutions is investigated by Pei and Zhang [51]. Huang and Liu [41] studied the sign-changing solutions for the following p-biharmonic equations with Hardy potential

$$\begin{cases} \Delta_p^2 u - \dfrac{a}{|x|^{2p}}|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

by using the method of invariant sets of descending flow. Also, in [65], by using critical point theory, was looked for the existence of infinitely many solutions of the problem (1). Precisely, authors, under appropriate hypotheses on the nonlinear term f, g, the existence of two intervals Λ and J such that, for each $\lambda \in \Lambda$ and $\mu \in J$, the boundary value problem (1) admits a sequence of pairwise distinct solutions proved.

Our goal of this work is to establish some new criteria for the fourth-order Kirchhoff type elliptic problem with Hardy potential (1) to have at least three weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, by means of a very recent abstract critical points result of Ricceri [57] (see Lemma 2.1). More precisely, for differentiable functionals and assignable constants λ

and μ sufficiently small, the existence of three solutions for problem (1) will be obtained (see Theorem 3.1) requiring that the nonlinearity f has a growth condition and some other sufficient conditions in addition to a suitable oscillating behavior of the associated potential g (see condition (g_1)). So, under appropriate hypotheses on the nonlinear terms f, g, the existence of two open intervals involving parameters λ and μ such that, for each λ and μ belonging to those, the boundary value problem (1) admits at least three weak solutions is proved.

The plan of the paper is as follows: Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main result and its proof. A concrete example of an application is then presented (see Example 3.1).

Finally, we cite the manuscripts [15, 19, 20, 22, 30, 36, 39, 48], where the existence of multiple solutions for this type of nonlinear differential equations was studied.

2. Basic definitions and Preliminary results

Let X be the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm $\|u\| = \left(\int_\Omega |\Delta u|^p \, dx\right)^{\frac{1}{p}}$. We recall Rellich inequality [21], which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}} \, dx \leq \frac{1}{H} \int_\Omega |\Delta u|^p \, dx, \tag{8}$$

where the best constant is

$$H = \left(\frac{(p-1)N(N-2p)}{p^2}\right)^p. \tag{9}$$

In this article, we assume that the following condition holds,

$(H1)$ $M : [0, +\infty[\to \mathbb{R}$ is a continuous function. And there are two positive constants m_0, m_1 such that

$$m_0 \leq M(t) \leq m_1, \quad \forall\, t \geq 0. \tag{10}$$

Set $p^* = \frac{pN}{N-p}$. By the Sobolev embedding theorem there exist a positive constant c such that $\|u\|_{L^{p^*}(\Omega)} \leq c\|u\|$, $\forall\, u \in X$, where

$$c := \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right]^{\frac{1}{N}}, \tag{11}$$

see, for instance, [61]. Fixing $q \in [1, p^*)$, again from the Sobolev embedding theorem, there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \quad \forall\, u \in X. \tag{12}$$

Thus, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. By (11), as a simple consequence of Hölder's inequality, one has the upper bound

$$c_q \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left[\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right]^{\frac{1}{N}} |\Omega|^{\frac{p^*-q}{p^*q}}, \tag{13}$$

where $|\Omega|$ denotes the Lebesgue measure of the open set Ω.

In order to prove our main result, stated in Theorem 3.1, in the following we will perform the variational principle of Ricceri established in [57, Theorem 1]. For the sake of clarity, we recall it here below.

Lemma 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*, bounded on bounded subsets of X, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that*

(i) $\frac{\sup_{\Phi(x)\leq r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}\ ,\ \frac{r}{\sup_{\Phi(x)\leq r}\Psi(x)}\right[$, *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each compact interval $[a, b] \subseteq \Lambda_r$, there exists $\rho_0 > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation $\Phi'(x) - \lambda\Psi'(x) - \mu J'(x) = 0$, has at least three solutions in X whose norms are less than ρ_0.

Also, for a through on the subject we refer to the papers [9, 13, 24, 28, 32, 40, 49, 50, 56], and we refer to the recent papers [2, 6, 33, 35, 37, 38, 54] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

Definition 2.1. *We mean by a (weak) solution of the problem* (1)*, any function $u \in X$ such that*

$$\begin{aligned} M\left(\int_\Omega |\Delta u(x)|^p\,dx\right)\int_\Omega &|\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,dx \\ &- a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,dx \\ &- \lambda\int_\Omega f(x,u(x))v(x)\,dx - \mu\int_\Omega g(x,u(x))v(x)\,dx \\ &= 0 \end{aligned}$$

for every $v \in X$.

We need the following Proposition in the proof of Theorem 3.1. Throughout in paper suppose that $0 < a < m_0 H$.

Proposition 2.1. *Let $\Upsilon : X \to X^*$ be the operator defined by*

$$\begin{aligned} \Upsilon(u)(v) = M\left(\int_\Omega |\Delta u(x)|^p\,dx\right)\int_\Omega &|\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,dx \\ - a\int_\Omega &\frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,dx \end{aligned}$$

for every $u, v \in X$. Then Υ admits a continuous inverse on X^*.

Proof. Since $\Upsilon(u)u \geq \left(m_0 - \frac{a}{H}\right)\|u\|^p$, the operator Υ is coercive. Taking into account (2.2) of [60] for $p > 1$ there exists a positive constant C_p such that

$$\left\langle |x|^{p-2}x - |y|^{p-2}y, x - y\right\rangle \geq \begin{cases} C_p\,|x-y|^p, & \text{if } p \geq 2, \\ C_p\,\dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases}$$

where $< .,. >$ denotes the usual inner product in $\mathbb{R}^N$, for every $x, y \in \mathbb{R}^N$. Thus, for $1 < p < 2$, it is easy to see that

$$(\Upsilon(u) - \Upsilon(v))(u - v) \geq C_p M^- \int_\Omega \left[\frac{|\Delta u(x) - \Delta v(x)|^2}{(|\Delta u(x)| + |\Delta v(x)|)^{2-p}} + \frac{|u(x) - v(x)|^2}{|x|^{2p}\left(|u(x)| + |v(x)|\right)^{2-p}} \right] dx > 0$$

for every $u, v \in X$, $u \neq v$, where $M^- := \min\{a, m_0\}$, which means that Υ is strictly monotone. For $p \geq 2$, we also observe that $(\Upsilon(u) - \Upsilon(v))(u - v) \geq C_p M^- \int_\Omega \left[|\Delta u(x) - \Delta v(x)|^p + \frac{|u(x)-v(x)|^p}{|x|^{2p}} \right] dx > 0$, which means that Υ, in this case, is strictly monotone too. Moreover, we observe that Υ is the dual mapping on $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ corresponding to the gauge function $\phi_p(t) = t^{p-1}$. Hence, in the same way as in the proof [39, Proposition 2.3], we observe that the inverse operator Υ^{-1} of Υ exists and Υ^{-1} is continuous. □

3. Main results

In this section we establish the main abstract result of this paper, that is., here we deal with the existence of three weak solutions for the problem (1) when the nonlinear term satisfies a growth condition. Before introducing our result we observe that, putting $\vartheta(x) = \sup\{\vartheta > 0 : \ B(x, \vartheta) \subseteq \Omega\}$ for all $x \in \Omega$, one can prove that there exists $x_0 \in \Omega$ such that $B(x_0, D) \subseteq \Omega$, where $D = \sup_{x\in\Omega} \vartheta(x)$.

Theorem 3.1. *Suppose that (H_1) and $0 < a < m_0 H$ hold (with H is as in (9)). Also let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition*

(f$_1$) *there exist $a_1, a_2 \in [0, +\infty[$ and $q \in (1, p^*)$ such that*

$$|f(x,t)| \leq a_1 + a_2|t|^{q-1}$$

for each $(x, t) \in \Omega \times \mathbb{R}$, where $p^* := \frac{Np}{N-p}$;

(f_2) *$F(x, \xi) \geq 0$ for each $(x, \xi) \in \Omega \times \mathbb{R}^+$, where $F(x, \xi) := \int_0^\xi f(x,t)\,dt$ for each $(x, \xi) \in \Omega \times \mathbb{R}$;*

(f_3) *there exist $c \geq 0$ and $s \in (1, p)$ such that*

$$F(x,t) \leq c\,(1 + |t|^s)$$

for each $(x, t) \in \overline{\Omega} \times \mathbb{R}$;

(f_4) *there exist two positive constants r, d with $r < \left(\frac{m_0 H - a}{pH}\right)\frac{L}{D^p}$ such that*

$$\overline{\omega}_r := \frac{1}{r}\left\{ a_1 c_1 \left(\frac{pHr}{m_0 H - a}\right)^{\frac{1}{p}} + \frac{a_2}{q} c_q^q \left(\frac{pHr}{m_0 H - a}\right)^{\frac{q}{p}} \right\} < k_0 \inf_{x\in\Omega} F(x, d),$$

where $k_0 := \frac{\pi^{\frac{N}{2}}(\frac{D}{2})^N m_1 2^p L}{\frac{N}{2}\Gamma(\frac{N}{2}) p D^p}$, $L := \frac{\pi^{\frac{N}{2}}\left[D^N - (\frac{D}{2})^N\right]}{\frac{N}{2}\Gamma(\frac{N}{2})}\left(\frac{2d(N-1)}{D}\right)^p$ and Γ is the Gamma function.

Then, for every $\lambda \in \Lambda_{r,d} := \left]\frac{1}{k_0 \inf_{x\in\Omega} F(x,d)}, \frac{1}{\overline{\omega}_r}\right[$, and for every $g \in C(\overline{\omega} \times \mathbb{R})$ such that

(g_1) *for every $\rho > 0$, $\sup_{|t|\leq\rho} |g(x,t)| \in L^1(\Omega)$;*

there exist $\rho_0 > 0$ and $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem (1) admits at least three weak solutions in X whose norms are less than ρ_0.

Proof. Our aim is to apply Lemma 2.1 to the space $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and to the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined as

$$\Phi(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{a}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}\,dx, \quad \text{and} \quad \Psi(u) = \int_\Omega F(x, u(x))\,dx,$$

for all $u \in X$, where $\widehat{M}(t) := \int_0^t M(s)\,ds,\ t \geq 0$. It is easy to show that functionals Φ, Ψ are well defined and continuously Gâteaux differentiable functionals whose derivatives at the point $u \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\begin{aligned}\Phi'(u)(v) = {} & M\left(\int_\Omega |\Delta u(x)|^p\,dx\right)\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,dx \\ & - a\int_\Omega \frac{|u(x)|^{p-2}}{|x|^{2p}}u(x)v(x)\,dx,\end{aligned}$$

and $\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\,dx$, for every $u, v \in X$. By (8), it follows that

$$\frac{m_0H - a}{pH}\|u\|^p \leq \Phi(u) \leq \frac{m_1}{p}\|u\|^p, \quad u \in X \tag{14}$$

which implies that Φ is coercive and bounded on each bounded subset of X. Moreover, from the weakly lower semicontinuity of norm, and the monotonicity and continuity of $\widehat{M}$, we known that Φ is sequentially weakly lower semicontinuous. Furthermore, while Proposition 2.1 gives that $\Phi' : X \to X^*$ admits a continuous inverse on X^*. Thanks to condition (f_1) and to the compact embedding $X \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*)$, the functional $\Psi' : X \to X^*$ is in $C^1(X, \mathbb{R})$ and compact operator. Moreover, we have $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Now, let $w_d \in X$ be the function defined by

$$w_d(x) = \begin{cases} 0, & x \in \Omega \backslash B(x_0, D), \\ \dfrac{2d}{D}(D - |x - x_0|), & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \\ d, & x \in B(x_0, \frac{D}{2}), \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. We have

$$\frac{\partial w_d(x)}{\partial x_i} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ \dfrac{-2d}{D}\dfrac{(x_i - x_{0,i})}{|x - x_0|}, & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

$$\frac{\partial^2 w_d(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ \dfrac{-2d}{D}\left[\dfrac{|x - x_0|^2 - (x_i - x_{0,i})^2}{|x - x_0|^3}\right], & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}), \end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 w_d(x)}{\partial x_i^2} = \begin{cases} 0, & x \in (\Omega\backslash B(x_0, D)) \cup B(x_0, \frac{D}{2}), \\ \dfrac{-2d(N-1)}{D|x - x_0|}, & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}). \end{cases}$$

So, one has $\|w_d\|^p = \int_\Omega |\Delta w_d(x)|^p\,dx = \int_{B(x_0,D)\backslash B(x_0,\frac{D}{2})} \frac{(\frac{2d}{D})^p(N-1)^p}{|x-x_0|^p}\,dx$, and in particular, one has

$$\frac{L}{D^p} \leq \|w_d\|^p \leq (\frac{2}{D})^p L. \tag{15}$$

Here, we obtain from (14) and (15) that

$$\frac{L}{D^p}\left(\frac{m_0H - a}{pH}\right) \leq \Phi(w_d) \leq \frac{m_1}{p}(\frac{2}{D})^p L. \tag{16}$$

By the assumption $r < \left(\frac{m_0H-a}{pH}\right)\frac{L}{D^p}$, one has $r < \Phi(w_d)$. Thus, the functionals Φ and Ψ satisfy the regularity assumptions requested in Lemma 2.1. So, we will only verify the conditions (i) and (ii). Thanks to (f_2) and this fact that $0 \leq w_d(x) \leq d$ for each $x \in \Omega$, one has

$$\begin{aligned}\Psi(w_d) = \int_\Omega F(x, w_d(x))\,dx &\geq \int_{B(x_0,\frac{D}{2})} F(x,d)\,dx \\ &\geq \frac{\pi^{\frac{N}{2}}(\frac{D}{2})^N}{\frac{N}{2}\Gamma(\frac{N}{2})} \inf_{x\in\Omega} F(x,d). \end{aligned} \tag{17}$$

So, from (16) and (17) we observe that

$$\begin{aligned}\frac{\Psi(w_d)}{\Phi(w_d)} &\geq \frac{\frac{\pi^{\frac{N}{2}}(\frac{D}{2})}{\frac{N}{2}\Gamma(\frac{N}{2})}\inf_{x\in\Omega}F(x,d)}{\frac{pD^p}{m_1L2^p}} \\ &= k_0 \inf_{x\in\Omega} F(x,d). \end{aligned} \tag{18}$$

The compact embedding $X \hookrightarrow L^q(\Omega)$ for each $q \in [1, p^*), (f_1)$ and (14) imply that, for each $u \in \Phi^{-1}(]-\infty, r])$, we have

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,u(x))\,dx &\leq a_1 \int_\Omega |u(x)|\,dx + \frac{a_2}{q}\int_\Omega |u(x)|^q\,dx \\ &\leq a_1c_1\|u\| + \frac{a_2}{q}c_q^q\|u\|^q \\ &\leq a_1c_1\left(\frac{pHr}{m_0H-a}\right)^{\frac{1}{p}} + \frac{a_2}{q}c_q^q\left(\frac{pHr}{m_0H-a}\right)^{\frac{q}{p}}, \end{aligned}$$

and so

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} \leq \overline{\omega}_r. \tag{19}$$

Therefore, from the conditions (f_4), (18) and (19), one has

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(w_d)}{\Phi(w_d)},$$

and so condition (i) of Lemma 2.1 is verified. By argument similar to those used before, we obtain

$$\int_\Omega |u(x)|^s\,dx \leq c_s^s\|u\|^s, \tag{20}$$

and so, for each $u \in X$ with $\|u\| \geq \max\{1, \frac{1}{c_s}\}$, from (f_3) and (20) one has

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,u(x))\,dx &\leq \int_\Omega c\,(1+|u(x)|^s)\,dx \\ &\leq c\,\{meas(\Omega) + c_s^s\|u\|^s\}, \end{aligned}$$

where $meas(\Omega)$ denotes the Lebesgue measure of the open set Ω. This and (14) lead to $I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \geq \left(\frac{m_0H-a}{pH}\right)\|u\|^p - \lambda c\,\{meas(\Omega) + c_s^s\|u\|^s\}$ and, since $s < p$, coercivity of I_λ is obtained. In addition, since $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is in $C(\Omega\times\mathbb{R})$ and satisfying condition (g_1), the functional $J(u) := \int_\Omega G(x,u(x))\,dx$, where $G(x,t) := \int_0^t g(x,s)\,ds$ for each $(x,t) \in \Omega\times\mathbb{R}$, is well defined and continuously Gâteaux differentiable on X with a compact derivative, and $J'(u)(v) = \int_\Omega g(x,u(x))v(x)\,dx$ for all $u, v \in X$. Thus, all the hypotheses of Lemma 2.1 are satisfied. Also, note that the solutions of the equation $\Phi'(u)-$

$\lambda\Psi'(u) - \mu J'(u) = 0$ are exactly the weak solutions of the problem (1). So, taking into account that $\Lambda_{r,d} \subseteq \left]\frac{\Phi(w_d)}{\Psi(w_d)}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right[$, the conclusion follows from Lemma 2.1. $\square$

Remark 3.1. When $r = 1$ condition (f_4) of Theorem 3.1 becomes

(f_4') there exists $d > 0$ with $p < \left(\frac{m_0H-a}{H}\right)\frac{L}{D^p}$ such that $\{a_1c_1\left(\frac{pH}{m_0H-a}\right)^{\frac{1}{p}} + \frac{a_2}{q}c_q^q\left(\frac{m_0H-a}{H}\right)^{\frac{q}{p}}\} < k_0\inf_{x\in\Omega}F(x,d)$, where $k_0 := \frac{(\pi^{\frac{N}{2}}\frac{D}{2})^N m_1 2^p L}{\frac{N}{2}\Gamma(\frac{N}{2})pD^p}$ and $L := \frac{\pi^{\frac{N}{2}}\left[D^N - (\frac{D}{2})^N\right]}{\frac{N}{2}\Gamma(\frac{N}{2})}\left(\frac{2d(N-1)}{D}\right)^p$.

Remark 3.2. We observe that, if $f(x,0) \neq 0$, then, by Theorem 3.1, we obtain the existence of at least three non-zero weak solutions.

We present an example to illustrate Theorem 3.1 as follows in which the nonlinearity $f(t,x)$ verifies the hypotheses of Theorem 3.1 and the construction of the nonlinear function is partly motivated by [9, Example 5.1].

Example 3.1. *The following function verifies the assumptions requested in Theorem* 3.1. Let $r > 1$, $q \in (1,p^*)$, $s \in (1,p)$. We consider the function $f : \Omega\times\mathbb{R} \to \mathbb{R}$ defined as $f(x,t) = \begin{cases} 1+|t|^{q-1}, & x \in \Omega,\ t \le r, \\ 1 + r^{q-s}t^{s-1}, & x \in \Omega,\ t > r. \end{cases}$ Condition (f_1) is easily verified. Taking into account that $F(x,t) = \begin{cases} \le 0, & x \in\Omega,\ t \le 0, \\ t + \dfrac{t^q}{q}, & x \in \Omega,\ 0 < t \le r, \\ t + r^{q-s}\dfrac{t^s}{s}, & x\in\Omega,\ t > r, \end{cases}$ one has $F(x,t) \ge 0$ for each $(x,t) \in \Omega\times[0,+\infty[$ and the condition (f_2) is verified. Finally, we observe that $F(x,t) = \begin{cases} \le 0, & x\in\Omega,\ t\le 0, \\ \le r + \dfrac{r^q}{s}, & x \in\Omega,\ 0 < t \le r, \\ \le (r + \dfrac{r^q}{s})t^s, & x\in\Omega,\ t > r, \end{cases}$ and since $F(x,t) \le (r + \frac{r^q}{s})(1+|t|^s)$ for each $(x,t) \in \Omega\times\mathbb{R}$, the condition (f_4) is verified.

If the function f is dependent on u only, we can get better result than Theorem 3.1. For simplicity, fixing $\Omega \subseteq \mathbb{R}^N$, $p = 2$, consider the following equation

$$\begin{cases} M\left(\displaystyle\int_\Omega |\Delta u|^2\,dx\right)\Delta(\Delta u) - \dfrac{a}{|x|^4}u = \lambda f(u) + \mu g(x,u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Now, we present another result, which is a immediate consequence of Theorem 3.1, in the following Theorem. Note that, for the case $p = 2$, the constant H is also accountable (see [7]).

Theorem 3.2. *Suppose that* (H_1) *and* $0 < a < m_0H$ *hold (with* H *is as in* (9)*). Also let* $f : \mathbb{R} \to \mathbb{R}$ *be a positive continuous function satisfying the following condition*

(f_1') *there exist* $b_1, b_2 \in [0,+\infty[$ *and* $q \in (1, 2^*)$ *such that* $|f(t)| \le b_1 + b_2|t|^{q-1}$ *for each* $t \in \mathbb{R}$, *where* $2^* := \frac{2N}{N-2}$;

(f_3') *there exist* $c \ge 0$ *and* $s \in (1,2)$ *such that* $F(t) := \int_0^t f(s)\,ds \le c\,(1+|t|^s)$, *for each* $t \in \mathbb{R}$.

Moreover, assume that there exist two positive constants r, d' *with* $r < \left(\frac{m_0H-a}{2H}\right)\frac{L'}{D^2}$ *such that* $\overline{\omega'}_r := \frac{1}{r}\left\{b_1c_1\sqrt{\frac{2Hr}{m_0H-a}} + \frac{b_2}{q}c_q^q\sqrt{\left(\frac{2Hr}{m_0H-a}\right)^q}\right\} < k_0'F(d')$, *where* $k_0' := \frac{4\pi^{\frac{N}{2}}(\frac{D}{2})^N m_1 L'}{N\Gamma(\frac{N}{2})D^2}$, $L' :=$

$\frac{4\pi^{\frac{N}{2}}\left[D^N-(\frac{D}{2})^N\right]}{\frac{N}{2}\Gamma(\frac{N}{2})}\left(\frac{d'(N-1)}{D}\right)^2$, *and* Γ *is the Gamma function. Then, for every* $\lambda \in \Lambda_{r,d'} := \left]\frac{1}{k'_0 F(d')}, \frac{1}{\omega'_r}\right[$, *and for every* $g \in C(\overline{\Omega} \times \mathbb{R})$ *satisfying in the condition* (g_1), *there exist* $\overline{\rho} > 0$ *and* $\overline{\delta} > 0$ *such that, for each* $\mu \in [0, \overline{\delta}]$, *the problem* (21) *admits at least three weak solutions in* X *whose norms are less than* $\overline{\rho}$.

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