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EXISTENCE OF NONOSCILLATORY SOLUTIONS FOR SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

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Abstract. In this work, an attempt is made to discuss the existence of nonoscillatory solutions of second order nonlinear neutral differential equations with variable delays. The main tools are Lebesgue's dominated convergence theorem and Banach contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions. This problem is considered in various ranges of the neutral coefficient. Further, two illustrative examples showing applicability of the new results are included.

Keywords: Existence of positive solutions, neutral, delay, non-linear, Lebesgue's dominated convergence theorem, Banach's fixed point theorem.

AMS Subject Classification: 34C10, 34C15, 34K40.

1. Introduction

This article is concerned with sufficient conditions for existence of positive solutions of a nonlinear neutral second-order delay differential equation

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\big[x(t)+p(t)x(\tau(t))\big]\right]+q(t)G\big(x(\sigma(t))\big)=0. \qquad (1)$$

Suppose that the following assumptions hold.

(A1) $r\in C([t_0,\infty),(0,\infty))$, $p\in C([t_0,\infty),\mathbb{R})$ and $q\in C([t_0,\infty),(0,\infty))$, where q is not identically zero;

(A2) $G\in C(\mathbb{R},\mathbb{R})$ is nondecreasing such that $xG(x)>0$ for $x\neq 0$;

(A3) $\tau,\sigma\in C([t_0,\infty),(0,\infty))$ such that $\sigma(t),\tau(t)\leq t$ for $t\geq t_0,\sigma(t),\tau(t)\to\infty$ as $t\to\infty$ with invertible τ when necessary.

Culakova *et al.* [1] considered (1) and studied existence of nonoscillatory solutions when $p\in C([t_0,\infty),(-\infty,0))$. In [7], under various ranges of p, Santra studied oscillatory behaviour of the solutions of the following neutral differential equations

$$\frac{d}{dt}\big[x(t)+p(t)x(t-\tau)\big]+q(t)G\big(x(t-\sigma)\big)=0$$

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and

$$\frac{d}{dt}\big[x(t)+p(t)x(t-\tau)\big]+q(t)G\big(x(t-\sigma)\big)=f(t) \tag{2}$$

Also, sufficient conditions are obtained for existence of bounded nonoscillatory solutions of (2). The motivation of the present work come from the above studies. Hence, the objective of this work is to study existence of positive solutions of (1) for any $|p(t)|<+\infty$.

Oscillation and nonoscillation of functional differential equations have been studied in recent years. For further work to this type of equations, one may follow [1, 2, 11, 12] and the references cited therein. The existence of nonoscillatory solution of functional differential equations received much less attention, which is mainly due to the technical difficulties arising in its analysis.

By a solution of (1) we mean a continuously differentiable function $x(t)$ which is defined for $t\geq T^*=\min\{\tau(t_0),\sigma(t_0)\}$ such that $x(t)$ satisfies (1) for all $t\geq t_0$. In the sequel, it will always be assumed that the solutions of (1) exist on some half line $[t_1,\infty), t_1\geq t_0$. A solution of (1) is said to be oscillatory, if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory, if all its solutions are oscillatory.

2. Main Results

Theorem 2.1. *Assume that* $(A1)$–$(A3)$ *hold and* $p\in C(\mathbb{R}_+,[0,1))$. *Furthermore assume that* G *is Lipschitzian on the intervals of the form* $[a,b]$, $0<a<b<\infty$. *If*

$(A4)$ $\int_0^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta<+\infty,$

then (1) *has a bounded nonoscillatory solution.*

Proof. Let $0\leq p(t)\leq p<1$, $t\in\mathbb{R}_+$ and $p>0$. Due to $(A4)$, it is possible to find $T>T^*$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta<\frac{1-p}{5L},$$

where $L=\max\{L_1,G(1)\}$, L_1 is the Lipschitz constant of G on $\left[\frac{7(1-p)}{10},1\right]$ for $t\geq t_0$. Let $Y=BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions on $[t_0,\infty]$. Indeed, Y is a Banach space with respect to supremum norm defined by

$$\|x\|=\sup\{|x(t)|:t\geq t_0\}.$$

Define

$$S=\left\{v\in Y:\frac{7(1-p)}{10}\leq v(t)\leq 1,\ t\geq t_0\right\}.$$

Note that S is a closed and convex subspace of Y. Let $\Phi:S\to S$ be such that

$$(\Phi x)(t)=\begin{cases}(\Phi x)(T), & t\in[t_0,T]\\ -p(t)x(\tau(t))+\frac{9+p}{10}-\int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G(x(\sigma(\zeta)))d\zeta\right]d\eta, & t\geq T.\end{cases}$$

For every $x\in S$, $(\Phi x)(t)\leq \frac{9+p}{10}<1$ and

$$(\Phi x)(t)\geq -p+\frac{9+p}{10}-\frac{1-p}{5}=\frac{7}{10}(1-p)$$

implies that $\Phi x \in S$. For $x_1, x_2 \in S$, it follows that

$$\begin{aligned}
|(\Phi x_1)(t) - (\Phi x_2)(t)| &\le p|x_1(\tau(t)) - x_2(\tau(t))| \\
&\quad + \int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)|G\big(x_1(\sigma(\zeta))\big) - G\big(x_2(\sigma(\zeta))\big)|d\zeta\right]d\eta, \\
&\le p\|x_1 - x_2\| + \|x_1 - x_2\|L_1 \int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \\
&\le \left(p + \frac{1-p}{5}\right)\|x_1 - x_2\| \\
&= \frac{4p+1}{5}\|x_1 - x_2\|.
\end{aligned}$$

Therefore, $\|\Phi x_1 - \Phi x_2\| \le \frac{4p+1}{5}\|x_1 - x_2\|$ implies that Φ is a contraction. By using Banach's fixed point theorem, it follows that Φ has a unique fixed point $x(t)$ in $\left[\frac{7(1-p)}{10}, 1\right]$. This completes the proof of the theorem. □

Theorem 2.2. *Assume that (A_1)–(A_4) hold and $1 < p_1 \le p(t) \le p_2 < \infty$, where $p_1^2 \ge p_2$ for $t \in \mathbb{R}_+$. Furthermore assume that G be Lipschitzian on the intervals of the form $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. Then (1) has a bounded nonoscillatory solution.*

Proof. Due to $(A4)$, it is possible to find $T > T^*$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta < \frac{p_1 - 1}{3L},$$

where $L = \max\{L_1, L_2\}$, L_1 is the Lipschitz constant of G on $[\alpha, \beta]$, $L_2 = G(\beta)$ with

$$\alpha = \frac{3\lambda(p_1^2 - p_2) - p_2(p_1 - 1)}{3{p_1}^2 p_2}$$

$$\beta = \frac{p_1 - 1 + 3\lambda}{3p_1}, \qquad \lambda > \frac{p_2(p_1 - 1)}{3({p_1}^2 - p_2)} > 0.$$

Let $Y = BC([t_0, \infty), \mathbb{R})$ be the space of real valued functions defined on $[t_0, \infty)$. Indeed, Y is a Banach space with respect to supremum norm defined by

$$||x|| = \sup\{|x(t)| : t \ge t_0\}.$$

Define

$$S = \Big\{u \in Y : \ \alpha \le u(t) \le \beta, \ t \ge t_0\Big\}.$$

Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0, T] \\ -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{\lambda}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta, & t \ge T. \end{cases}$$

For every $x \in S$,

$$\begin{aligned}
(\Phi x)(t) &\le \frac{G(\beta)}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta + \frac{\lambda}{p(\tau^{-1}(t))} \\
&\le \frac{G(\beta)}{p(\tau^{-1}(t))}\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta + \frac{\lambda}{p(\tau^{-1}(t))} \\
&\le \frac{1}{p_1}\left[\frac{p_1 - 1}{3} + \lambda\right] = \beta
\end{aligned}$$

and

$$(\Phi x)(t) \geq -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{\lambda}{p(\tau^{-1}(t))} - \frac{\beta}{p_1} + \frac{\lambda}{p_2} = \alpha$$

implies that $\Phi x \in S$. For $x_1, x_2 \in S$, it follows that

$$\begin{aligned}|(\Phi x_1)(t) - (\Phi x_2)(t)| &\leq \frac{1}{|p(\tau^{-1}(t))|}|x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))| \\ &+ \frac{L_1}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\bigg[\int_\eta^\infty q(\zeta)|x_1(\sigma(\zeta)) - x_2(\sigma(\zeta))|d\zeta\bigg]d\eta, \\ &\leq \frac{1}{p_1}||x_1 - x_2|| + \frac{L_1}{p_1}||x_1 - x_2||\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\bigg[\int_\eta^\infty q(\zeta)d\zeta\bigg]d\eta \\ &< \frac{1}{p_1}||x_1 - x_2||\bigg(1 + \frac{p_1 - 1}{3}\bigg),\end{aligned}$$

that is,

$$||\Phi x_1 - \Phi x_2|| \leq \bigg(\frac{1}{p_1} + \frac{p_1 - 1}{3p_1}\bigg)||x_1 - x_2||.$$

Since $\bigg(\frac{1}{p_1} + \frac{p_1-1}{3p_1}\bigg) < 1$, then Φ is a contraction mapping of S into S. Note that S is a closed convex subset of Y and hence we apply Banach's fixed point to S. So, we conclude that Φ has a unique fixed point on $[\alpha, \beta]$. It is easy to verify that

$$x(t) = \begin{cases}(\Phi x)(T), & t \in [t_0, T] \\ -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{\lambda}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\bigg[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\bigg]d\eta, & t \geq T.\end{cases}$$

is a positive bounded solution of (1) on $[\alpha, \beta]$. This completes the proof of the theorem. □

Theorem 2.3. *Assume that* $(A1)$*–*$(A3)$ *hold and* $-1 < -p \leq p(t) \leq 0$*, where* $p > 0$*,* $t \in \mathbb{R}_+$*. Furthermore assume that*

- $(A5)$ $R(t) = \int_0^t \frac{d\eta}{r(\eta)}$ *and* $\lim_{t\to\infty} R(t) = +\infty$
- $(A6)$ $\int_0^\infty q(\eta)G(\varepsilon R(\sigma(\eta)))d\eta < +\infty$ *for every* $\varepsilon > 0$

hold, then (1) *has a unbounded nonoscillatory solution.*

Proof. Due to $(A6)$, it is possible to find $\varepsilon > 0$ such that

$$\int_T^\infty q(\eta)G(\varepsilon R(\sigma(\eta)))d\eta \leq \frac{\varepsilon}{3},\ T \geq T^*.$$

Let's consider

$$M = \bigg\{x : x \in C([t_0, +\infty), \mathbb{R}), x(t) = 0\ for\ t \in [t_0, T]\ and \\ \frac{\varepsilon}{3}[R(t) - R(T)] \leq x(t) \leq \varepsilon[R(t) - R(T)]\bigg\}$$

and define $\Phi : M \to C([t_0, +\infty), \mathbb{R})$ such that

$$(\Phi x)(t) = \begin{cases}0, & t \in [t_0, T) \\ -p(t)x(\tau(t)) + \int_T^t \frac{1}{r(\eta)}\bigg[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\bigg]d\eta, & t \geq T.\end{cases}$$

For every $x \in M$,

$$(\Phi x)(t) \geq \int_T^t \frac{1}{r(\eta)}\left[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta$$
$$\geq \frac{\varepsilon}{3}\int_T^t \frac{d\eta}{r(\eta)} = \frac{\varepsilon}{3}[R(t) - R(T)]$$

and $x(t) \leq \varepsilon R(t)$ implies that

$$\begin{aligned}(\Phi x)(t) &\leq -p(t)x(\tau(t)) + \frac{2\varepsilon}{3}\int_T^t \frac{d\eta}{r(\eta)}\\
&\leq p\varepsilon[R(\tau(t)) - R(T)] + \frac{2\varepsilon}{3}[R(t) - R(T)]\\
&\leq p\varepsilon[R(t) - R(T)] + \frac{2\varepsilon}{3}[R(t) - R(T)]\\
&= \left(p + \frac{2}{3}\right)\varepsilon[R(t) - R(T)]\\
&\leq \varepsilon[R(t) - R(T)]\end{aligned}$$

implies that $(\Phi x)(t) \in M$. Define $v_n : [T - \rho, +\infty) \to \mathbb{R}$ by the recursive formula

$$v_n(t) = (\Phi v_{n-1})(t), \quad n \geq 1,$$

with the initial condition

$$v_0(t) = \begin{cases} 0, & t \in [t_0, T) \\ \frac{\varepsilon}{3}[R(t) - R(T)], & t \geq T. \end{cases}$$

Inductively, it is easy to verify that

$$\frac{\varepsilon}{3}[R(t) - R(T)] \leq v_{n-1}(t) \leq v_n(t) \leq \varepsilon[R(t) - R(T)].$$

for $t \geq T$. Therefore, for $t \geq t_0$, $\lim_{n\to\infty} v_n(t)$ exists. Let $\lim_{n\to\infty} v_n(t) = v(t)$ for $t \geq t_0$. By the Lebesgue's dominated convergence theorem $v \in M$ and $(\Phi v)(t) = v(t)$, where $v(t)$ is a solution of (1) on $[t_0, \infty)$ such that $v(t) > 0$. Note that $\lim_{t\to\infty} \frac{z(t)}{R(t)} = \frac{\varepsilon}{3}$, where $z(t) = x(t) + p(t)x(\tau(t))$. This completes the proof. □

Theorem 2.4. *Assume that (A1) – (A4) hold and* $p \in C(\mathbb{R}_+, (-1, 0])$. *Then* (1) *admits a bounded nonoscillatory solutions.*

Proof. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$ for $t \in \mathbb{R}_+$. Due to (A4),

$$G(\varepsilon)\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \leq \frac{\varepsilon}{3},\ T \geq T^*,$$

where $\varepsilon > 0$ is a constant. Consider

$$M = \left\{x \in C([t_0, +\infty), \mathbb{R}) : x(t) = \frac{\varepsilon}{3},\ t \in [t_0, T]; \frac{\varepsilon}{3} \leq x(t) \leq \varepsilon \ for\ t \geq T\right\}$$

and let $\Phi : M \to M$ be defined by

$$(\Phi x)(t) = \begin{cases} \frac{\varepsilon}{3}, & [t_0, T] \\ -p(t)x(\tau(t)) + \frac{\varepsilon}{3} + \int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta, & t \geq T. \end{cases}$$

For every $x \in M$, $(\Phi x)(t) \geq \frac{\varepsilon}{3}$ and

$$
\begin{aligned}
(\Phi x)(t) &\leq p\varepsilon + \frac{\varepsilon}{3} + G(\varepsilon)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \\
&\leq p\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \left(p + \frac{2}{3}\right)\varepsilon \leq \varepsilon
\end{aligned}
$$

implies that $\Phi x \in M$. The rest of the proof follows from Theorem 2.3.

□

Theorem 2.5. *Assume that (A1) − (A4) hold and* $-\infty < -p_1 \leq p(t) \leq -p_2 < -1$, *where* $p_1, p_2 > 0$ *such that* $3p_2 > p_1$ *for* $t \in \mathbb{R}_+$. *Furthermore assume that* G *is Lipschitzian on the interval of the form* $[a,b]$, $0 < a < b < \infty$. *Then equation (1) has a bounded nonoscillatory solution.*

Proof. Due to (A4), it is possible to find $T > T^*$ such that

$$
\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta < \frac{p_2 - 1}{3L},
$$

where $L = \max\{L_1, G(1)\}$, L_1 is the Lipschitz constant of G on $(\alpha, 1)$, $\alpha = \frac{(p_2-1)(3p_2-p_1)}{3p_1p_2}$. Let $Y = BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions defined on $[t_0,\infty)$. Indeed, Y is a Banach space with the supremum norm defined by

$$
||x|| = \sup\{|x(t)| : t \geq t_0\}.
$$

Define

$$
S = \left\{v \in Y : \ \alpha \leq v(t) \leq 1, \ t \geq t_0\right\}.
$$

and note that S is a closed and convex subspace of Y. Let $\Psi : S \to S$ be such that

$$
(\Psi x)(t) = \begin{cases} (\Psi x)(T), & t \in [t_0, T] \\ -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{p_2-1}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta, & t \geq T. \end{cases}
$$

For every $x \in S$,

$$
(\Psi x)(t) \leq -\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{p_2 - 1}{p(\tau^{-1}(t))} \leq \frac{1}{p_2} + \frac{p_2 - 1}{p_2} = 1
$$

and

$$
\begin{aligned}
(\Psi x)(t) &\geq -\frac{p_2 - 1}{p(\tau^{-1}(t))} + \frac{1}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\sigma(\zeta))\big)d\zeta\right]d\eta \\
&\geq \frac{p_2 - 1}{p_1} + \frac{G(1)}{p(\tau^{-1}(t))}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \\
&\geq \frac{p_2 - 1}{p_1} - \frac{G(1)}{p_2}\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \\
&\geq \frac{p_2 - 1}{p_1} - \frac{p_2 - 1}{3p_2} = \alpha
\end{aligned}
$$

implies that $\Psi x \in S$. For $x_1, x_2 \in S$, it follows that

$$\begin{aligned}
|(\Psi x_1)(t) - (\Psi x_2)(t)| &\leq \frac{1}{|p(\tau^{-1}(t))|}|x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))| \\
&+ \frac{L_1}{|p(\tau^{-1}(t))|}\int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)|x_1(\sigma(\zeta)) - x_2(\sigma(\zeta))|d\zeta\right]d\eta, \\
&\leq \frac{1}{p_2}||x_1 - x_2|| + \frac{p_2 - 1}{3p_2}||x_1 - x_2|| \\
&= \mu||x_1 - x_2||,
\end{aligned}$$

where $\mu = \frac{1}{p_2}\left(1 + \frac{p_2-1}{3}\right) < 1$. Therefore, Ψ is a contraction. Hence, by Banach's fixed point theorem Ψ has a unique fixed point $x \in S$. It is easy to see that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

3. Examples and Open Problem

Example 3.1. *Consider the equation*

$$\frac{d}{dt}\left[e^t \frac{d}{dt}\big[x(t) - e^{-(t+1)}x(t-1)\big]\right] + \frac{2}{e}e^{-\frac{2}{3}t}\big(x(t-3)\big)^{\frac{1}{3}} = 0,\ t \geq 3, \tag{3}$$

and note that $r(t) = e^t$, $p(t) = -e^{-(t+1)}$, $q(t) = \frac{2}{e}e^{-\frac{2}{3}t}$, $\tau(t) = t-1$, $\sigma(t) = t-3$ *and* $G(x) = x^{\frac{1}{3}}$. *A straightforward verification yields that the conditions of Theorem 2.4 are satisfied. We note that* $x(t) = e^{-t}$ *is a nonoscillatory solution of* (3).

Example 3.2. *Consider the equation*

$$\frac{d}{dt}\left[e^{-3t} \frac{d}{dt}\big[x(t) - e^{-7t}x(t-2)\big]\right] + 28e^{-10t}\big(x(t-2)\big)^3 = 0,\ t \geq 2, \tag{4}$$

and note that $r(t) = e^{-3t}$, $p(t) = -e^{-7t}$, $q(t) = 28e^{-10t}$, $\tau(t) = t-2$, $\sigma(t) = t-2$ *and* $G(x) = x^3$. *Clearly all conditions of Theorem 2.3 are satisfied. In particular,* $x(t) = e^{3t}$ *is a positive unbounded solution of* (4).

Remark 3.1. *It is interesting to study the necessary and sufficient conditions for the oscillation of* (1) *for any* $|p(t)| < +\infty$.

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