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EXISTENCE OF THREE SOLUTIONS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL SYSTEMS THROUGH VARIATIONAL METHODS

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Abstract. This paper is devoted to the study of the multiplicity results of existence of solutions for a class of impulsive fractional differential systems. Indeed, we will use variational methods for smooth functionals, defined on the reflexive Banach spaces in order to achieve the existence of at least three solutions for these systems. In particular, in the scalar case, we will prove that the impulsive fractional differential problem has three non-negative solutions. Finally, by presenting two examples, we will ensure the applicability of our results.

Keywords: Three solutions, Fractional differential equation, Impulsive effect, Variational methods, Critical point theory.

AMS Subject Classification: 26A33, 34B15, 35A15, 34B15, 34K45, 58E05.

1. Introduction

In this paper we study the following perturbed impulsive fractional differential system

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + \mu G_{u_i}(t,u) + h_i(u_i), & t \in (0,T),\ t \neq t_j, \\ \Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), & j = 1,2,\dots,m, \\ u_i(0) = u_i(T) = 0 \end{cases} \qquad (P_{\lambda,\mu}^{F,G})$$

for $i = 1,\dots,n$, where $n \geq 1$, $u = (u_1,\dots,u_n)$, $0 < \alpha_i \leq 1$ for $i = 1,\dots,n$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $a_i \in \mathrm{L}^\infty([0,T])$, $\bar{a}_i = \operatorname{ess\,inf}_{t\in[0,T]} a_i(t) > 0$ for $i = 1,\dots,n$, ${}_0D_t^{\varsigma}$ and ${}_tD_T^{\varsigma}$ denote the left and right Riemann-Liouville fractional derivatives of order ς, respectively, $F, G : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are measurable with respect to t, for all $u \in \mathbb{R}^n$, continuously differentiable in u, for almost every $t \in [0,T]$ such that $F(t,0,\dots,0) = G(t,0,\dots,0) = 0$ for every $t \in [0,T]$ and satisfy in the following standard summability condition:

$$\sup_{|\xi|\leq\varrho_1} (\max\{|F(.,\xi)|,\ |G(.,\xi)|,\ |F_{\xi_i}(.,\xi)|,\ |G_{\xi_i}(.,\xi)|,\ i = 1,\dots,n\}) \in \mathrm{L}^1([0,T]) \qquad (1)$$

for any $\varrho_1 > 0$ with $\xi = (\xi_1,\dots,\xi_n)$ and $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$, $h_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L_i > 0$, i.e.,$|h_i(\xi_1) - h_i(\xi_2)| \leq$

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$L_i|\xi_1 - \xi_2|$ for every $\xi_1, \xi_2 \in \mathbb{R}$, satisfying $h_i(0) = 0$ for $i = 1, \dots, n$, $I_{ij} \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ for $i = 1, \dots, n$, $j = 1, \dots, m$, $m \geq 1$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$, the operator Δ is defined as $\Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u))(t_j) = {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^+) - {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^-)$ where ${}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^+) = \lim_{t \to t_j^+} {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t)$ and so on ${}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t_j^-) = \lim_{t \to t_j^-} {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u)(t)$ and ${}_0^cD_t^{\alpha_i}$ is the left Caputo fractional derivatives of order α_i. Here, F_{u_i} and G_{u_i} denote respectively the partial derivatives of F and G with respect to u_i for $i = 1, \dots, n$.

Fractional differential equations (FDEs) are generalization of ordinary differential equations and integration to arbitrary non-integer orders. FDEs both ordinary and partial ones form a very important and significant part of mathematical analysis and its applications to real-world problems, see for details [5, 6, 11, 14, 13] and references therein.

From [18, 19, 20], we know that the coupled system of differential equations of fractional order is also important and several authors have done a lot of work in this topic. We also refer to the paper [10] in which based on variational methods for smooth functionals defined on reflexive Banach spaces the existence of one weak solution for a class of fractional differential systems was investigated.

On the other hand, impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The governing equations of such phenomena may be modeled as impulsive differential equations (see [1, 3]). Due to the great development in the theory of fractional calculus and impulsive differential equations as well as having wide applications in several fields. See [2, 7, 9, 17] and the references therein for detailed discussions.

Motivated by the above works, in this paper we look for the existence of at least three weak solutions for the system $(P_{\lambda,\mu}^{F,G})$ for appropriate values of the parameters λ and μ belonging to real intervals. Our approach is variational methods and a three critical points theorem due to Ricceri [15]. As an application of our result, we study the scaler case of the system and we establish the existence of at least three weak solutions for the problem, and assuming that the nonlinear terms are non-negative we show that the solutions are non-negative. Some examples are presented to illustrate our main results.

2. Preliminaries

In this section, we will introduce some basic definitions, notations, lemmas and propositions which are used throughout this paper.

Definition 2.1. *Let $0 < \alpha_i \leq 1$ for $i = 1, \dots, n$. The fractional derivative space $\mathrm{E}_0^{\alpha_i}$ is defined by the closure $\mathrm{C}_0^\infty([0,T], \mathbb{R})$, that is $\mathrm{E}_0^{\alpha_i} = \overline{\mathrm{C}_0^\infty([0,\mathrm{T}], \mathbb{R})}$ with respect to the weighted norm $\|u_i\|_{\alpha_i} = \big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2\mathrm{d}t + \int_0^T |u_i(t)|^2\mathrm{d}t\big)^{\frac{1}{2}}$ for every $u_i \in \mathrm{E}_0^{\alpha_i}$ and for $i = 1, \dots, n$.*

Remark 2.1. *It is obvious that $\mathrm{E}_0^{\alpha_\mathrm{i}}$ is the space of functions $u_i \in \mathrm{L}^2([0,\mathrm{T}], \mathbb{R})$ having an α_i-order Riemann-Loiuville fractional derivative ${}_0D_t^{\alpha_i}u_i \in \mathrm{L}^2([0,\mathrm{T}], \mathbb{R})$ and $u_i(0) = u_i(T) = 0$ for $i = 1, \dots, n$. From* [12, Propostion 3.1]*, we know for $0 < \alpha_i \leq 1$, the space $\mathrm{E}_0^{\alpha_\mathrm{i}}$ is a reflexive and separable Banach space for $i = 1, \dots, n$.*

Lemma 2.1 ([19])**.** *Let $0 < \alpha_i \leq 1$ for $i = 1, \dots, n$. We can consider $\mathrm{E}_0^{\alpha_i}$ with respect to the norm $\|u_i\|_{\alpha_i} = \big(\int_0^T a_i(t)|{}_0D_t^{\alpha_i}u_i(t)|^2\mathrm{d}t\big)^{\frac{1}{2}}$ for every $u_i \in \mathrm{E}_0^{\alpha_i}$ and for $i = 1, \dots, n$, which* *is equivalent to the norm of definition 2.1. Then we have* $\sum_{i=1}^{n}\|u_i\|_{\mathrm{L}^2}^2 \leq S\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2$ *and if* $\alpha_i > \frac{1}{2}$,

$$\sum_{i=1}^{n}\|u_i\|_{\infty}^2 \leq M\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 \tag{2}$$

with $S = \max\{\frac{T^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\bar{a}_i}, i = 1,\dots,n\}$ *and* $M = \max\{\frac{T^{2\alpha_i-1}}{\Gamma^2(\alpha_i)\bar{a}_i(2\alpha_i-1)}, i = 1,\dots,n\}$.

Now, we let E be the Cartesian product of n Sobolev spaces $\mathrm{E}_0^{\alpha_1}$, $\mathrm{E}_0^{\alpha_2}$,..., and $\mathrm{E}_0^{\alpha_n}$, i.e., $\mathrm{E} = \mathrm{E}_0^{\alpha_1}\times\mathrm{E}_0^{\alpha_2}\times\cdots\times\mathrm{E}_0^{\alpha_n}$, which is uniformly convex and reflexive Banach space endowed with the norm $\|(u_1,\dots,u_n)\| = \sum_{i=1}^{n}\|u_i\|_{\alpha_i}$. Obviously, E is compactly embedded in $(\mathrm{C}^0([0,T]))^n$. Corresponding to the function h_i, we introduce the function $H_i : \mathbb{R} \longrightarrow \mathbb{R}$ by $H_i(x) = \int_0^x h_i(\xi)\mathrm{d}\xi$ for all $x \in \mathbb{R}$ and $i = 1,\dots,n$.

Definition 2.2. *We mean by a (weak) solution of the system* $(P_{\lambda,\mu}^{F,G})$, *any function* $u = (u_1,\dots,u_n) \in \mathrm{E}$ *such that*

$$\sum_{i=1}^{n}\Big(\int_0^T a_i(t)\,{}_0D_t^{\alpha_i}u_i(t)\,{}_0D_t^{\alpha_i}v_i(t)\mathrm{d}t\Big) - \sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)\mathrm{d}t$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j) - \lambda\sum_{i=1}^{n}\int_0^T F_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)\mathrm{d}t$$
$$-\mu\sum_{i=1}^{n}\int_0^T G_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)\mathrm{d}t = 0$$

for every $v = (v_1,\dots,v_n) \in \mathrm{E}$.

We assume throughout and without further mention, that the following conditions hold:

$(\mathcal{H}_1)$ $\frac{1}{2} < \alpha_i \leq 1$ for $i = 1,\dots,n$;

$(\mathcal{H}_2)$ $I_{ij}(0) = 0$ and there exists a constant $L_{ij} > 0$ such that
$|I_{ij}(s_1) - I_{ij}(s_2)| \leq L_{ij}|s_1 - s_2|$ for any $s_1, s_2 \in \mathbb{R}$ $i = 1,\dots,n,\ 1 \leq j \leq m$;

$(\mathcal{H}_3)$ $\sum_{i=1}^{n}\frac{L_iT^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\bar{a}_i} + MCm\|\tilde{a}\|_\infty < 1$ where $C = \max_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}L_{ij}$ and $\tilde{a} = \max\{a_i(t), t \in [0,T], i = 1,\dots,n\}$.

Put $\sigma = \min\{1 - \frac{L_iT^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\bar{a}_i}, i = 1,\dots n\}$ and $\rho = \max\{1 + \frac{L_iT^{2\alpha_i}}{\Gamma^2(\alpha_i+1)\bar{a}_i}, i = 1,\dots n\}$.

Our main tool is Theorem 2.1 which has been obtained by Ricceri([15, Theorem 2]). It is as follows:

If X is a real Banach space, denoted by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For example, if X is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.1. *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0}\frac{J(u)}{\Phi(u)}\right\},$$

$\sigma = \sup_{u\in\Phi^{-1}(]0,+\infty[)}\frac{J(u)}{\Phi(u)}$, *assume that* $\rho < \sigma$. *Then for each compact interval* $[c,d] \subset (\frac{1}{\sigma},\frac{1}{\rho})$ *(with the conventions* $\frac{1}{0} = +\infty, \frac{1}{+\infty} = 0$*), there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every* C^1 *functional* $\Psi : X \to \mathbb{R}$ *with compact derivative,* *there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, $\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$ has at least three solutions in X whose norms are less than R.*

We refer the reader to the papers [4, 8, 16] in which Theorem 2.1 was successfully employed to ensure the existence of at least three solutions for BVPs.

Now for every $u \in \mathrm{E}$, we define

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n} \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^{n} \int_0^T H_i(u_i(t))\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)\int_0^{u_i(t_j)} I_{ij}(s)\mathrm{d}s. \tag{3}$$

Moreover, for every $u \in \mathrm{E}$, we set

$$J(u) = \int_0^T F(t, u_1(t)\dots, u_n(t))\mathrm{d}t \ \text{ and } \ \Psi(u) = \int_0^T G(t, u_1(t)\dots, u_n(t))\mathrm{d}t. \tag{4}$$

Standard arguments show that $\Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in \mathrm{E}$ given by

$$\begin{aligned}(\Phi' - \mu\Psi' - \lambda J')(u)(v) &= \sum_{i=1}^{n}\Big(\int_0^T a_i(t)_0D_t^{\alpha_i}u_i(t)_0D_t^{\alpha_i}v_i(t)\mathrm{d}t\Big)\\ &- \sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j)\\ &-\lambda\int_0^T \big(\sum_{i=1}^{n} F_{u_i}(t, u_1(t), \dots, u_n(t))\big)v_i(t)\mathrm{d}t\\ &-\mu\big(\sum_{i=1}^{n} G_{u_i}(t, u_1(t), \dots, u_n(t))\big)v_i(t)\mathrm{d}t\end{aligned}$$

for all $u, v \in X$. Hence, a critical point of the functional $\Phi - \mu\Psi - \lambda J$, gives us a weak solution of $(P_{\lambda,\mu}^{F,G})$.

We need the following proposition in the proof of our main result.

Proposition 2.1 ([9, Proposition 2.6]). *Let $J : X \to X^\star$ be the operator for every $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in X$, defined by*

$$\begin{aligned}J(u)(v) = \ &\sum_{i=1}^{n}\Big(\int_0^T a_i(t)_0D_t^{\alpha_i}u_i(t)_0D_t^{\alpha_i}v_i(t)\mathrm{d}t\Big)\\ &+\sum_{i=1}^{n}\sum_{j=1}^{m} a_i(t_j)I_{ij}(u_i(t_j))v_i(t_j) - \sum_{i=1}^{n}\int_0^T h_i(u_i(t))v_i(t)\mathrm{d}t.\end{aligned}$$

Then, J admits a continuous inverse on $X^\star$.

3. Main results

In this section, we formulate our main results.

Let us denote by $\mathcal{F}$ the class of all functions $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ measurable with respect to t, for all $\xi \in \mathbb{R}^n$, continuously differentiable in ξ, for almost every $t \in [0, T]$, satisfying the standard summability condition (1). For each $0 < \kappa < \frac{1}{2}$ set

$$\begin{aligned}P_i(\alpha_i, \kappa) = \frac{1}{2\kappa^2T^2}\Big(&\int_0^T a_i(t)t^{2(1-\alpha_i)}\mathrm{d}t + \int_{\kappa T}^T a_i(t)(t-\kappa T)^{2(1-\alpha_i)}\mathrm{d}t\\ &+\int_{(1-\kappa)T}^T a_i(t)(t-(1-\kappa)T)^{2(1-\alpha_i)}\mathrm{d}t - 2\int_{\kappa T}^T a_i(t)(t^2-\kappa Tt)^{1-\alpha_i}\mathrm{d}t\\ &+2\int_{(1-\kappa)T}^T a_i(t)(t^2-(1-\kappa)Tt)^{1-\alpha_i}\mathrm{d}t\\ &+2\int_{(1-\kappa)T}^T a_i(t)(t^2-\kappa Tt+\kappa(1-\kappa)T^2)^{1-\alpha_i}\mathrm{d}t\Big)\end{aligned}$$

for $i = 1, \dots, n$. Let

$$\lambda_1 = \inf\left\{\frac{\sum_{i=1}^{n}\|u_i\|_{\alpha_i}^2 - 2\mathcal{J}(u)}{2\int_0^T F(t, u(t))\mathrm{d}t} : \ u \in \mathrm{E}, \int_0^T F(t, u(t))\mathrm{d}t > 0\right\}$$

with

$$\mathcal{J}(u) = \sum_{i=1}^n \int_0^T H_i(u_i(t))\mathrm{d}t - \sum_{i=1}^n \sum_{j=1}^m a_i(t_j) \int_0^{u_i(t_j)} I_{ij}(s)\mathrm{d}s \tag{5}$$

and $\lambda_2 = \frac{1}{\max\{0,\lambda_0,\lambda_\infty\}}$, where

$$\lambda_0 = \limsup_{|u|\to 0} \Big(\frac{2\int_0^T F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - 2\mathcal{J}(u)} \Big)$$

and

$$\lambda_\infty = \limsup_{\|u\|\to +\infty} \Big(\frac{2\int_0^T F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - 2\mathcal{J}(u)} \Big)$$

with $\mathcal{J}(u)$ given by (5) and $u = (u_1, \dots, u_n)$.

Theorem 3.1. *Suppose that $F \in \mathcal{F}$. Assume that*

($\mathcal{A}_1$) *there exists a constant $\varepsilon > 0$ such that*

$$\max\Big\{\limsup_{\xi\to(0,\dots,0)} \frac{\max_{t\in[0,T]} F(t,\xi)}{|\xi|^2}, \limsup_{|\xi|\to\infty} \frac{\max_{t\in[0,T]} F(t,\xi)}{|\xi|^2}\Big\} < \varepsilon$$

where $\xi = (\xi_1, \dots, \xi_n)$ with $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$;

($\mathcal{A}_2$) *there exists a function $w \in \mathrm{E}$ such that*

$$\sum_{i=1}^n \|w_i\|_{\alpha_i}^2 - 2\sum_{i=1}^n \int_0^T H_i(w_i(t))\mathrm{d}t + 2\sum_{i=1}^n \sum_{j=1}^m a_i(t_j) \int_0^{w_i(t_j)} I_{ij}(s)\mathrm{d}s \neq 0$$

and $\varepsilon < \frac{(\frac{\sigma}{M} - Cm\|\tilde{a}\|_\infty)\int_0^T F(t,w(t))\mathrm{d}t}{\sum_{i=1}^n \|w_i\|_{\alpha_i}^2 - 2\sum_{i=1}^n \int_0^T H_i(w_i(t))\mathrm{d}t + 2\sum_{i=1}^n \sum_{j=1}^m a_i(t_j)\int_0^{w_i(t_j)} I_{ij}(s)\mathrm{d}s}$.

Then, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(P_{\lambda,\mu}^{F,G})$ has at least three weak solutions whose norms in E *are less than R.*

Proof. Take $X = \mathrm{E}$. Clearly, X is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (3) and (4), respectively. The functional Φ is C^1, and due to Proposition 2.1 its derivative admits a continuous inverse on X^*. Moreover, by the sequentially weakly lower semicontinuity of $\sum_{i=1}^n \|u_i\|_{\alpha_i}^2$ and the continuity of H_i, $i = 1, \dots, n$ and I_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, Φ is sequentially weakly lower semicontinuous in X. Since $h_i(0) = 0$ one has $|h_i(x_i)| \le L_i|x_i|$ for $i = 1, \dots, n$, from (3) and the condition ($\mathcal{H}_2$) we see that

$$\frac{\sigma - MCm\|\tilde{a}\|_\infty}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 \le \Phi(u) \le \frac{\rho + MCm\|\tilde{a}\|_\infty}{2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 \tag{6}$$

and bearing the condition ($\mathcal{H}_3$) in mind, it follows $\lim_{\|u\|\to+\infty} \Phi(u) = +\infty$, namely Φ is coercive. Moreover, let A be a bounded subset of X. That is, there exist constants $c_i > 0$, $i = 1, \dots, n$ such that $\|u_i\|_{\alpha_i} \le c_i$ for each $u \in A$. Then, by (6) we have $|\Phi(u)| \le \frac{\rho + MCm\|\tilde{a}\|_\infty \sum_{i=1}^n c_i^2}{2}$. Hence Φ is bounded on each bounded subset of X. Furthermore, $\Phi \in \mathcal{W}_X$. Indeed, let $\{u_k\}_{k=1}^\infty = \{(u_{k1}, \dots, u_{kn})\}_{k=1}^\infty \subset X$, $u = (u_1, \dots, u_n) \subset X$, $u_k \rightharpoonup u$ and $\liminf_{k\to\infty} \Phi(u_k) \le \Phi(u)$. Since the functions H_i and I_{ij} are continuous, one has $\liminf_{n\to\infty} \sum_{i=1}^n \frac{\|u_{ki}\|_{\alpha_i}^2}{2} \le \sum_{i=1}^n \frac{\|u_i\|_{\alpha_i}^2}{2}$. Thus, $\{u_k\}_{k=1}^\infty$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1 functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of ($\mathcal{A}_1$), there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that $F(t,u) \le \varepsilon \sum_{i=1}^n |u_i|^2$ for every $t \in [0,T]$ and every $u = (u_1, \dots, u_n)$ with $|u| \in [0,\tau_1) \cup (\tau_2, +\infty)$. By (1), $F(t,u)$

is bounded on $t \in [0,T]$ and $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > 2$ such that $F(t,u) \le \varepsilon \sum_{i=1}^n |u_i|^2 + \eta \sum_{i=1}^n |u_i|^\upsilon$ for all $(t,u) \in [0,T] \times \mathbb{R}^n$. So, by (2), we have $J(u) \le M\varepsilon \sum_{i=1}^n \|u_i\|_{\alpha_i}^2 + \eta M^{\upsilon/2} \sum_{i=1}^n \|u_i\|_{\alpha_i}^\upsilon$ for all $u \in X$. Hence, we have

$$\limsup_{|u|\to 0} \frac{J(u)}{\Phi(u)} \le \frac{2M\varepsilon}{\sigma - MCm\|\tilde{a}\|_\infty}. \tag{7}$$

Moreover, by using the inequality $F(t,u) \le \varepsilon \sum_{i=1}^n |u_i|^2$, for each $u \in X \setminus \{0\}$, we obtain

$$\begin{aligned} \frac{J(u)}{\Phi(u)} &= \frac{\int_{|u|\le\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} + \frac{\int_{|u|>\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} \\ &\le \frac{2\sup_{t\in[0,T],|u|\in[0,\tau_2]} F(t,u)}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2} + \frac{2M\varepsilon}{\sigma - MCm\|\tilde{a}\|_\infty}. \end{aligned}$$

So, we get

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)} \le \frac{2M\varepsilon}{\sigma - MCm\|\tilde{a}\|_\infty}. \tag{8}$$

In view of (7) and (8), we have

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to(0,\ldots,0)} \frac{J(u)}{\Phi(u)}\right\} \le \frac{2M\varepsilon}{\sigma - MCm\|\tilde{a}\|_\infty}. \tag{9}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (9) yields

$$\sigma \;= \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)} = \sup_{X\setminus\{0\}} \frac{J(u)}{\Phi(u)} \ge \frac{\int_0^T F(t,w(t))\mathrm{d}t}{\Phi(w(t))} > \frac{2M\varepsilon}{\sigma - MCm\|\tilde{a}\|_\infty} \ge \rho.$$

Thus, all the hypotheses of Theorem 2.1 are satisfied. Clearly, $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \frac{1}{\rho}$. Then, using Theorem 2.1, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(P_{\lambda,\mu}^{F,G})$ has at least three weak solutions whose norms in X are less than R. □

The another announced application of Theorem 2.1 reads as follows:

Theorem 3.2. *Suppose that $F \in \mathcal{F}$. Assume that*

$$\max\left\{\limsup_{\xi\to(0,\ldots,0)} \frac{\max_{t\in[0,T]} F(t,\xi)}{|\xi|^2}, \limsup_{|\xi|\to\infty} \frac{\max_{t\in[0,T]} F(t,\xi)}{|\xi|^2}\right\} \le 0 \tag{10}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ with $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$, and

$$\sup_{u\in \mathrm{E}} \frac{2\int_0^T F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n \|u_i\|_{\alpha_i}^2 - 2\sum_{i=1}^n \int_0^T H_i(u_i(t))\mathrm{d}t + 2\sum_{i=1}^n \sum_{j=1}^m a_i(t_j) \int_0^{u_i(t_j)} I_{ij}(s)\mathrm{d}s} > 0. \tag{11}$$

Then for each compact interval $[c,d] \subset (\lambda_1, +\infty)$ there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$ there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the system $(P_{\lambda,\mu}^{F,G})$ has at least three weak solutions whose norms in E *are less than R.*

Proof. In view of (10), there exist an arbitrary $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that $F(t,u) \le \varepsilon \sum_{i=1}^n |u_i|^2$ for every $t \in [0,T]$ and every $u = (u_1, \ldots, u_n)$ with $|u| \in [0,\tau_1) \cup (\tau_2, +\infty)$. By (1), $F(t,u)$ is bounded on $t \in [0,T]$ and $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > 2$ in a manner that $F(t,u) \le \varepsilon \sum_{i=1}^n |u_i|^2 + \eta \sum_{i=1}^n |u_i|^\upsilon$ for all $(t,u) \in [0,T] \times \mathbb{R}^n$. So, by the same process in proof of Theorem 3.1 we have Relations (7) and (8). Since ε is arbitrary, (7) and (8) gives $\max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to(0,\ldots,0)} \frac{J(u)}{\Phi(u)}\right\} \le 0$. Then, with the notation of Theorem 2.1, we have $\rho = 0$. By (11), we also have $\sigma > 0$. In this case clearly $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = +\infty$. Thus, by using Theorem 2.1 result is achieved. □

Remark 3.1. *In Assumption* $(\mathcal{A}_2)$ *if we choose* $w(t) = (w_1(t), \ldots, w_n(t))$ *with*

$$w_i(t) = \begin{cases} \frac{\Gamma(2-\alpha_i)\delta}{\kappa T} t, & t \in [0, \kappa T[, \\ \Gamma(2-\alpha_i)\delta, & t \in [\kappa T, (1-\kappa)T], \\ \frac{\Gamma(2-\alpha_i)\delta}{\kappa T}(T-t), & t \in](1-\kappa)T, T], \end{cases} \tag{12}$$

where $0 < \kappa < \frac{1}{2}$, *for* $i = 1, \ldots, n$, *then it becomes to the following form:*

$(\mathcal{A}_2')$ *there exists a positive constant* κ *with* $0 < \kappa < \frac{1}{2}$ *such that* $\min\{P_i(\alpha_i, \kappa), i = 1, \ldots, n\} \neq 0$ *and there exists a positive constant* δ *such that*

$$\varepsilon < \frac{\int_0^T F(t, w(t)) \mathrm{d}t}{2Mn\delta^2 \min\{P_i(\alpha_i, \kappa), i = 1, \ldots, n\}}.$$

Clearly $\omega_i(0) = \omega_i(1) = 0$ *and* $\omega_i \in \mathrm{L}^2[0, T]$ *for* $i = 1, \ldots, n$. *A direct calculation shows that*

$$_0D_t^{\alpha_i}\omega_i(t) =$$

$$\begin{cases} \frac{\delta}{\kappa T} t^{1-\alpha_i}, & t \in [0, \kappa T[, \\ \frac{\delta}{\kappa T}(t^{1-\alpha_i} - (t-\kappa T)^{1-\alpha_i}), & t \in [\kappa T, (1-\kappa)T], \\ \frac{\delta}{\kappa T}(t^{1-\alpha_i} - (t-\kappa T)^{1-\alpha_i} - (t-(1-\kappa)T)^{1-\alpha_i}), & t \in](1-\kappa)T, T] \end{cases}$$

for $1 \leq i = 1, \ldots, n$. *Furthermore,* $\|\omega_i\|_{\alpha_i}^2 = \int_0^T a(t)|_0D_t^{\alpha_i}\omega_i(t)|^2 \mathrm{d}t = 2P(\alpha_i, \kappa)\delta^2$. *Thus,* $w \in \mathrm{E}$ *and* $\Phi(w) \geq n\delta^2 (\sigma - MCm\|\tilde{a}\|_\infty) \min\{P_i(\alpha_i, \kappa), i = 1, \ldots, n\}$.

Now, we point out some results in which the function F has separated variables. To be precise, consider the following system

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t)_0D_t^{\alpha_i}u_i(t)) = \lambda\theta(t)F_{u_i}(u_1, \ldots, u_n) & \\ +\mu G_{u_i}(t, u_1, \ldots, u_n) + h_i(u_i), & t \in (0, T),\ t \neq t_j, \\ \Delta({}_tD_T^{\alpha_i - 1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), & j = 1, 2, \ldots, m, \\ u_i(0) = u_i(T) = 0 & \end{cases} \tag{$P_{\lambda,\mu}^{F,G,\theta}$}$$

where $\theta : [0, T] \to \mathbb{R}$ is a non-zero function such that $\theta \in \mathrm{L}^1([0, T])$ and $F : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function and $G : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is as introduced for the system $(P_{\lambda,\mu}^{F,G})$ in Introduction.

Set $F(t, x_1, \ldots, x_n) = \theta(t)F(x_1, \ldots, x_n)$ for every $(t, x_1, \ldots, x_n) \in [0, T] \times \mathbb{R}^n$. The following existence results are consequences of Theorem 3.1.

Theorem 3.3. *Assume that*

$(\mathcal{A}_1')$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\sup_{t \in [0,T]} \theta(t). \max\left\{\limsup_{\xi \to (0,\ldots,0)} \frac{F(\xi)}{|\xi|^2}, \limsup_{|\xi| \to \infty} \frac{F(\xi)}{|\xi|^2}\right\} < \varepsilon$$

where $\xi = (\xi_1, \ldots, \xi_n)$ *with* $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$;

$(\mathcal{A}_2'')$ *there exist two positive constants* δ *and* κ *with* $0 < \kappa < \frac{1}{2}$ *such that* $\min\{P_i(\alpha_i, \kappa), i = 1, \ldots, n\} \neq 0$ *and* $\varepsilon < \frac{\int_0^T F(t, w(t)) \mathrm{d}t}{2Mn\delta^2 \min\{P_i(\alpha_i, \kappa), i=1,\ldots,n\}}$ *where* $w = (w_1, \ldots, w_n)$ *and* w_i, $i = 1, \ldots, n$ *are given by* (12).

Then, for each compact interval $[c, d] \subset (\lambda_3, \lambda_4)$ *where* λ_3 *and* λ_4 *are the same as* λ_1 *and* λ_2, *but* $\int_0^T F(t, u(t)) \mathrm{d}t$ *replaced by* $\int_0^T \theta(t)F(u(t)) \mathrm{d}t$, *respectively, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c, d]$ *and every* $G \in \mathcal{F}$ *there exists* $\gamma > 0$ *such that for each* $\mu \in [0, \gamma]$, *the system* $(P_{\lambda,\mu}^{F,G,\theta})$ *has at least three weak solutions whose norms in* E *are less than* R.

Theorem 3.4. *Assume that there exist two positive constants δ and κ with $0<\kappa<\frac{1}{2}$ such that*

$$\min\{P_i(\alpha_i,\kappa), i=1,\dots,n\}>0 \text{ and } \int_0^T \theta(t)F(w(t))\mathrm{d}t>0 \tag{13}$$

where $w=(w_1,\dots,w_n)$ and w_i, $i=1,\dots,n$ are given by (12). Moreover, suppose that

$$\limsup_{\xi\to(0,\dots,0)}\frac{F(\xi)}{|\xi|^2}=\limsup_{|\xi|\to\infty}\frac{F(\xi)}{|\xi|^2}=0, \tag{14}$$

where $\xi=(\xi_1,\dots,\xi_n)$ with $|\xi|=\sqrt{\sum_{i=1}^n\xi_i^2}$. Then, for each compact interval $[c,d]\subset(\lambda_3,+\infty)$ where λ_3 is the same as λ_1 but $\int_0^T F(t,u(t))\mathrm{d}t$ replaced by $\int_0^T\theta(t)F(u(t))\mathrm{d}t$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every $G\in\mathcal{F}$ there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the system $(P_{\lambda,\mu}^{F,G,\theta})$ has at least three weak solutions whose norms in E are less than R.

Proof. We easily observe that from (14) the assumption $(\mathcal{A}_1')$ is satisfied for every $\varepsilon>0$. Moreover, using (13), by choosing $\varepsilon>0$ small enough one can drive the assumption $(\mathcal{A}_2'')$. Hence, the conclusion follows from Theorem 3.3. □

Now, we exhibit an example in which the hypotheses of Theorem 3.4 are satisfied.

Example 3.1. *Let $\alpha_1=0.75$, $\alpha_2=0.8$, $T=1$, $m=n=2$, $\theta(t)=e^t$ for all $t\in[0,1]$, $a_i(t)=1$ for all $t\in[0,1]$ and $i=1,2$,*

$$F(x_1,x_2)=\begin{cases}(x_1^2+x_2^2)\sin(\frac{\pi x_1^2+\pi x_2^2}{2}), & \text{if } x_1^2+x_2^2<1,\\ 1, & \text{if } x_1^2+x_2^2\ge 1,\end{cases}$$

$h_1(x_1)=\frac{1}{10}\ln(\frac{1}{\cosh x_1})$ and $h_2(x_2)=\frac{1}{100}\arctan x_2$ for all $x_1,x_2\in\mathbb{R}$. Thus $L_1=\frac{1}{10}$, $L_2=\frac{1}{100}$. Now by choosing $\delta=1$ and $\kappa=\frac{1}{3}$, we have $w(t)=(w_1(t),w_2(t))$ with

$$w_1(t)=\begin{cases}3\Gamma(1.25)t, & t\in[0,\frac{1}{3}[,\\ \Gamma(1.25), & t\in[\frac{1}{3},\frac{2}{3}],\\ 3\Gamma(1.25)(1-t), & t\in]\frac{2}{3},1],\end{cases}\qquad w_2(t)=\begin{cases}3\Gamma(1.2)t, & t\in[0,\frac{1}{3}[,\\ \Gamma(1.2), & t\in[\frac{1}{3},\frac{2}{3}],\\ 3\Gamma(1.2)(1-t), & t\in]\frac{2}{3},1],\end{cases}$$

then we have $P_1(\alpha_1,\kappa)>0$, $P_2(\alpha_2,\kappa)>0$, $\int_0^1\theta(t)F(w(t))\mathrm{d}t>0$,

$$\lim_{(\xi_1,\xi_2)\to(0,0)}\frac{F(\xi_1,\xi_2)}{\xi_1^2+\xi_2^2}=0 \quad \text{and} \quad \lim_{|\xi|\to\infty}\frac{F(\xi_1,\xi_2)}{\xi_1^2+\xi_2^2}=0$$

where $\xi=(\xi_1,\xi_2)$ with $|\xi|=\sqrt{\xi_1^2+\xi_2^2}$. It is clear that by choosing $m=2$, $t_1=\frac{1}{3}$, $t_2=\frac{2}{3}$, $I_{11}(s)=\frac{1}{100}s$, $I_{12}(s)=\frac{1}{100}\sin s$, $I_{21}(s)=\frac{1}{10}\arctan s$ and $I_{22}(s)=\frac{1}{20}\ln(1+s^2)$ for all $s\in\mathbb{R}$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied with $L_{11}=L_{12}=\frac{1}{10}$ and $L_{21}=L_{22}=\frac{1}{100}$. Hence, by applying Theorem 3.4 for each compact interval $[c,d]\subset(0,+\infty)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every $G\in\mathcal{F}$ there exists $\gamma>0$ such that, for each $\mu\in[0,\gamma]$, the system $(P_{\lambda,\mu}^{F,G,\theta})$, in this case, has at least three weak solutions whose norms in the space $\mathrm{E}_0^{0.75}\times\mathrm{E}_0^{0.8}$ are less than R.

Remark 3.2. *It is obvious that impulsive problems are more general than non-impulsive ones. Moreover, impulsive effects are common phenomena due to the short-term perturbations the duration of which is negligible in comparison with the total duration of the original process. On the other hand, by putting $I_{ij}=0$ for all $i=1,\dots,n$ and $j=1,\dots,m$ all over of this article, we can have similar results in this regard for the system $(P_{\lambda,\mu}^{F,G})$ without impulsive terms. Readers can see another multiplicity result for the system $(P_{\lambda,\mu}^{F,G})$ without impulsive terms in [19] in the case $n=2$.*

4. Scalar Case

As an application of the results from Section 3, we consider the problem

$$\begin{cases} {}_tD_T^{\alpha}(a(t){}_0D_t^{\alpha}u(t)) = \lambda f(t,u) + \mu g(t,u) + h(u), & t \in (0,T),\, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = I_j(u(t_j)), & j = 1,2,\dots,m, \\ u(0) = u(T) = 0 \end{cases} \qquad (P_{\lambda,\mu}^{f,g})$$

where $\frac{1}{2} < \alpha \leq 1$, $\lambda > 0$, $\mu \geq 0$, $T > 0$, $a_0 = \operatorname{ess\,inf}_{t\in[0,T]} a(t) > 0$, ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ denote the left and right Riemann-Liouville fractional derivatives of order α, respectively, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $h : \mathbb{R} \to [0,+\infty)$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e., $|h(\xi_1) - h(\xi_2)| \leq L|\xi_1 - \xi_2|$ for every $\xi_1,\ \xi_2 \in \mathbb{R}$, satisfying $h(0) = 0$, $I_j \in \mathrm{C}(\mathbb{R},\ \mathbb{R})$ for $j = 1, 2, ..., m$, such that $I_j(0) = 0$ and there exists a constant $L_j > 0$ such that $|I_j(s_1) - I_j(s_2)| \leq L_j|s_1 - s_2|$ for any $s_1, s_2 \in \mathbb{R}$ for $j = 1,\dots,m$, $m \geq 1$, $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = T$.

From now, by x we mean a real number.

Put $F(t,x) = \int_0^x f(t,\xi)\mathrm{d}\xi \quad G(t,x) = \int_0^x g(t,\xi)\mathrm{d}\xi$ for every $(t,x) \in [0,T] \times \mathbb{R}$ and $H(x) = \int_0^x h(\xi)\mathrm{d}\xi$ for every $x \in \mathbb{R}$. Set $\bar{\sigma} = 1 - \frac{LT^{2\alpha}}{\Gamma^2(\alpha+1)a_0}$, $\bar{\rho} = 1 + \frac{LT^{2\alpha}}{\Gamma^2(\alpha+1)a_0}$, $\bar{M} = \frac{T^{2\alpha-1}}{a_0(2\alpha-1)\Gamma^2(\alpha)}$ and

$$\begin{aligned} P(\alpha,\kappa) &= \tfrac{1}{2\kappa^2T^2}\Big(\int_0^T a(t)t^{2(1-\alpha)}\mathrm{d}t + \int_{\kappa T}^T a(t)(t-\kappa T)^{2(1-\alpha)}\mathrm{d}t \\ &+ \int_{(1-\kappa)T}^T a(t)(t-(1-\kappa)T)^{2(1-\alpha)}\mathrm{d}t - 2\int_{\kappa T}^T a(t)(t^2-\kappa Tt)^{1-\alpha}\mathrm{d}t \\ &-2\int_{(1-\kappa)T}^T a(t)(t^2-(1-\kappa)Tt)^{1-\alpha}\mathrm{d}t \\ &+2\int_{(1-\kappa)T}^T a(t)(t^2-\kappa Tt+\kappa(1-\kappa)T^2)^{1-\alpha}\mathrm{d}t\Big) \end{aligned}$$

where $0 < \kappa < \frac{1}{2}$. We assume in the rest of the paper and without further mention, that the following conditions hold:

($\mathcal{H}_4$) $\frac{1}{2} < \alpha \leq 1$;

($\mathcal{H}_5$) $\bar{K} = \frac{LT^{2\alpha}}{(\Gamma(\alpha+1))^2a_0} + \bar{M}\bar{C}m\|a\|_\infty < 1$ where $\bar{C} = \max_{j\in\{1,\dots,m\}} L_j$.

The following results are consequences of Theorems 3.1 and 3.2, respectively.

Theorem 4.1. *Assume that*

($\mathcal{B}_1$) *there exists a constant $\varepsilon > 0$ such that*

$$\max\left\{\limsup_{\xi\to0}\frac{\max_{t\in[0,T]}F(t,\xi)}{|\xi|^2}, \limsup_{|\xi|\to\infty}\frac{\max_{t\in[0,T]}F(t,\xi)}{|\xi|^2}\right\} < \varepsilon;$$

($\mathcal{B}_2$) *there exists a function $w \in \mathrm{E}_0^{\alpha}$ such that*

$$\|w\|_\alpha^2 - 2\int_0^T H_i(w_i(t))\mathrm{d}t + 2\sum_{j=1}^m a(t_j)\int_0^{w(t_j)} I_j(s)\mathrm{d}s \neq 0$$

and $\varepsilon < \frac{(\frac{\bar{\sigma}}{\bar{M}} - \bar{C}m\|a\|_\infty)\int_0^T F(t,w(t))\mathrm{d}t}{\|w\|_\alpha^2 - 2\mathcal{J}_1(w)}$ *where*

$$\mathcal{J}_1(w) = \int_0^T H(w(t))\mathrm{d}t - \sum_{j=1}^m a(t_j)\int_0^{w(t_j)} I_j(s)\mathrm{d}s. \qquad (15)$$

Then, for each compact interval $[c,d] \subset (\bar{\lambda}_1, \bar{\lambda}_2)$ where

$$\bar{\lambda}_1 = \inf\left\{\frac{\|u\|_\alpha^2 - 2\mathcal{J}_1(u)}{2\int_0^T F(t,u(t))\mathrm{d}t} : u \in \mathrm{E}_0^{\alpha},\ \int_0^T F(t,u(t))\mathrm{d}t > 0\right\}$$

and

$$\bar{\lambda}_2 = \max\left\{0, \limsup_{|u|\to0}\frac{\int_0^T F(t,u(t))\mathrm{d}t}{\frac{1}{2}\|u\|_\alpha^2 - \mathcal{J}_1(u)}, \limsup_{\|u\|\to+\infty}\frac{\int_0^T F(t,u(t))\mathrm{d}t}{\frac{1}{2}\|u\|_\alpha^2 - \mathcal{J}_1(u)}\right\}$$

where $\mathcal{J}_1(u)$ is given by (15), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every L^1-Carathéodory function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0, \gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E^α_0 are less than R.

Theorem 4.2. *Assume that*

$$\max\left\{\limsup_{\xi\to 0} \frac{\sup_{t\in[0,T]} F(t,\xi)}{|\xi|^2}, \limsup_{|\xi|\to+\infty} \frac{\sup_{t\in[0,T]} F(t,\xi)}{|\xi|^2}\right\} \leq 0$$

and $\sup_{u\in \mathrm{E}^\alpha_0} \frac{2\int_0^T F(t,u(t))\mathrm{d}t}{\|u\|^2_\alpha - 2\mathcal{J}_1(u)} > 0$ where $\mathcal{J}_1(u)$ is given by (15). Then for each compact interval $[c, d] \subset (\lambda_1, +\infty)$ there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every L^1-Carathéodory function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0, \gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E^α_0 are less than R.

Remark 4.1. *If $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are non-negative functions, the weak solutions ensured by Theorems* 4.1 *and* 4.2 *are non-negative. Indeed, suppose that $u_0 \in \mathrm{E}^\alpha_0$ is a non-trivial weak solution of the problem $(P^{f,g}_{\lambda,\mu})$, then u_0 is positive. Arguing by a contradiction, assume that the set $\mathcal{N} = \{t \in]0, T] : u_0(t) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(t) = \min\{0, u_0(t)\}$ for all $t \in [0, T]$. Clearly, $\bar{v} \in \mathrm{E}^\alpha_0$ and one has*

$$\begin{aligned}&\int_0^T a(t){}_0D^\alpha_t u_0(t){}_0D^\alpha_t \bar{v}(t)\mathrm{d}t - \int_0^T h(u_0(t))\bar{v}(t)\mathrm{d}t + \sum_{j=1}^m a(t_j)I_j(u_0(t_j))\bar{v}(t_j)\\ &-\lambda \int_0^T f(t, u_0(t))\bar{v}(t)\mathrm{d}t - \mu \int_0^T g(t, u_0(t))\bar{v}(t)\mathrm{d}t = 0.\end{aligned}$$

Thus, from our sign assumptions on the data we have

$$\begin{aligned}0 &\leq (1 - \bar{K}) \int_{\mathcal{A}} a(t)|{}_0D^\alpha_t u_0(t)|^2\mathrm{d}t \leq \int_{\mathcal{A}} a(t)|{}_0D^\alpha_t u_0(t)|^2\mathrm{d}t - \int_{\mathcal{A}} h(u_0(t))u_0(t)\mathrm{d}t\\ &+ \sum_{j=1}^m a(t_j)I_j(u_0(t_j))u_0(t_j) \leq 0.\end{aligned}$$

Hence, by $(\mathcal{H}_6)$, $u_0 = 0$ in $\mathcal{N}$ and this contradicts with this fact that u_0 is a non-trivial weak solution. Hence, the set $\mathcal{N}$ is empty, and u_0 is positive. In addition, if either $f(t, 0) \neq 0$ for all $t \in [0, T]$ or $g(t, 0) \neq 0$ for all $t \in [0, T]$ or both are true, the solutions are positive.

Now we consider the following problem

$$\begin{cases} {}_tD^\alpha_T(a(t){}_0D^\alpha_t u(t)) = \lambda\theta(t)f(u) + \mu g(t, u) + h(u), & t \in (0, T), t \neq t_j,\\ \Delta({}_tD^{\alpha-1}_T({}^c_0D^\alpha_t u))(t_j) = I_j(u(t_j)), & j = 1, 2, \ldots, m,\\ u(0) = u(T) = 0 \end{cases} \tag{16}$$

for $i = 1, \ldots, n$, where $\theta : [0, T] \to \mathbb{R}$ is a non-negative and non-zero function such that $\theta \in \mathrm{L}^1([0, T])$, $f : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a non-negative L^1-Carathéodory function.

Put $F(x) = \int_0^x f(\xi)\mathrm{d}\xi$ for all $x \in \mathbb{R}$. Taking Remark 4.1 in to account, the following theorems are immediate consequences of Theorems 3.3 and 3.4, respectively.

Theorem 4.3. *Assume that there exists a constant $\varepsilon > 0$ such that*

$$\sup_{t\in[0,T]} \theta(t).\left\{\limsup_{\xi\to 0} \frac{F(\xi)}{|\xi|^2}, \limsup_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|^2}\right\} < \varepsilon$$

and there exist two positive constants δ and κ with $0 < \kappa < \frac{1}{2}$ such that δ such that $P(\alpha, \kappa) \neq 0$ and $2\bar{M}n\delta^2 P(\alpha, \kappa)\varepsilon < \int_0^T \theta(t)F(\bar{w}(t))\mathrm{d}t$ where

$$\bar{w}(t) = \begin{cases} \frac{\Gamma(2-\alpha)\delta}{\kappa T} t, & t \in [0, \kappa T[,\\ \Gamma(2 - \alpha)\delta, & t \in [\kappa T, (1 - \kappa)T],\\ \frac{\Gamma(2-\alpha)\eta}{\kappa T}(T - t), & t \in](1 - \kappa)T, T]. \end{cases} \tag{17}$$

Then, for each compact interval $[c,d] \subset (\bar{\lambda}_3, \bar{\lambda}_4)$ *where* $\bar{\lambda}_3$ *and* $\bar{\lambda}_4$ *are the same as* $\bar{\lambda}_1$ *and* $\bar{\lambda}_2$*, but* $\int_0^T F(t,u(t))\mathrm{d}t$ *replaced by* $\int_0^T \theta(t)F(u(t))\mathrm{d}t$*, respectively, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every non-negative* L^1*-Carathéodory function* $g : [0,T] \times \mathbb{R} \to \mathbb{R}$*, there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem* (16) *has at least three non-negative weak solutions whose norms in* E_0^α *are less than* R*.*

Theorem 4.4. *Assume that there exist two positive constants* δ *and* κ *with* $0 < \kappa < \frac{1}{2}$ *such that* $P(\alpha,\kappa) > 0$ *and* $\int_0^T F(\bar{w}(t))\mathrm{d}t > 0$ *where* $\bar{w}$ *is given by* (17)*. Moreover, suppose that* $\limsup_{\xi\to 0} \frac{f(\xi)}{|\xi|} = \limsup_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = 0$*. Then, for each compact interval* $[c,d] \subset (\bar{\lambda}_3, +\infty)$ *where* $\bar{\lambda}_3$ *is the same as* $\bar{\lambda}_1$*, but* $\int_0^T F(t,u(t))\mathrm{d}t$ *replaced by* $\int_0^T F(u(t))\mathrm{d}t$*, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every non-negative continuous function* $g : \mathbb{R} \to \mathbb{R}$*, there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$*, the problem*

$$\begin{cases} {}_tD_T^\alpha(a(t)\,{}_0D_t^\alpha u(t)) = \lambda f(u) + \mu g(u) + h(u), & t \in (0,T),\, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = I_j(u(t_j)), & j = 1,2,\ldots,m, \\ u(0) = u(T) = 0 \end{cases} \tag{18}$$

has at least three non-negative weak solutions whose norms in E_0^α *are less than* R*.*

Finally, we present the following example to illustrate Theorem 4.4.

Example 4.1. *Let* $\alpha = 0.75$*,* $T = 1$*,* $a(t) = 1 + \frac{2}{1+t^2}$ *for all* $t \in [0,1]$*,*

$$f(x) = \begin{cases} x^2 \sin^2 x, & \text{if } x < 0, \\ \sin^2 x, & \text{if } x \geq 0 \end{cases}$$

and $h(x) = \frac{1}{200}\arctan(e^x)$ *for all* $x \in \mathbb{R}$*. Thus* $a_0 = 1$ *and* $L = \frac{1}{100}$*. By choosing* $\delta = \frac{1}{\Gamma(1.25)}$ *and* $\kappa = \frac{1}{3}$*, we have* $\bar{w}(t) \geq 0$ *for all* $t \in [0,1]$*,* $P(\alpha,\kappa) > 0$*,*

$$\int_0^1 F(\bar{w}(t))\mathrm{d}t = \int_0^1 F(\bar{w}(t))\mathrm{d}t = \int_0^1 \int_0^{\bar{w}(t)} \sin^2 x\,\mathrm{d}x\mathrm{d}t = \frac{1}{12}(4 - \sin 2 + \cos 2) > 0$$

and $\lim_{\xi\to 0} \frac{f(\xi)}{|\xi|} = \lim_{\xi\to\infty} \frac{f(\xi)}{|\xi|} = 0$*. It is clear that by choosing* $m = 1$*,* $t_1 = \frac{1}{2}$ *and* $I_1(s) = \frac{1}{10}\ln(\frac{1}{\cosh s})$ *for all* $s \in \mathbb{R}$*, the assumptions* $(\mathcal{H}_5)$ *and* $(\mathcal{H}_6)$ *are satisfied with* $L_1 = \frac{1}{10}$*. Hence, by applying Theorem 4.4 for each compact interval* $[c,d] \subset (0,+\infty)$*, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every non-negative continuous function* $g : \mathbb{R} \to \mathbb{R}$*, there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$*, the problem* (18) *in this case has at least three non-negative weak solutions in* $\mathrm{E}_0^{0.75}$*.*

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