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ON THE AVERAGE LOWER 2-DOMINATION NUMBER OF A GRAPH

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Abstract. Computer scientists and network scientists want a speedy, reliable, and non-stop communication. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. The average lower 2-domination number of a graph G relative to a vertex v is the cardinality of a minimum 2-dominating set in G containing v. Consider the graph G modeling a network. The average lower 2-domination number of G, denoted as $\gamma_{2av}(G)$, is a new measure of the network vulnerability, given by $\gamma_{2av}(G) = \frac{1}{|V(G)|}\sum_{v\in V(G)} \gamma_{2v}(G)$. In this paper, above mentioned new parameter is defined and examined, also the average lower 2-domination number of well known graph families are calculated. Then upper and lower bounds are determined and exact formulas are found for the average lower 2-domination number of any graph G.

Keywords: Graph vulnerability; Connectivity; Network design and communication; Domination number; Average lower 2-domination number

AMS Subject Classification: 05C40, 05C69, 68M10, 68R10.

1. Introduction

Networks are important structures and appear in many different applications and settings. The most common networks are telecommunication networks, computer networks, the internet, road and rail networks and other logistic networks [20]. In a communication network, the measures of vulnerability are essential to guide the designers in choosing a suitable network topology. They have an impact on solving difficult optimization problems for networks [20]. Furthermore, the fundamental component of a distributed system is the interconnection network.

The network topology is significant since the communication between processors is derived via message exchange in distributed systems. For this reason, a communication network is modeled by a graph to measure the vulnerability, where stations correspond to vertices and communication links correspond to edges. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. As the networks begin losing connection lines or centers, eventually, there is a loss of efficiency [2, 3, 12, 19].

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In the literature, various measures have been defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. The graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph, also it is denoted by $k(G)$ ($k'(G)$) [14]. Then the toughness [12], the integrity [7], the residual closeness [22], the domination number [8], the bondage number [4, 5], the 2-domination number [18], the 2-bondage number [18], etc. have been proposed for measuring the vulnerability of networks. Recently, some average vulnerability parameters such as the average lower independence number [3, 10, 16], the average lower domination number [6, 16, 21], the average connectivity number [9], the average lower connectivity number [1] and the average lower bondage number [23], etc. have been defined. The average parameters have been found to be more useful in some circumstance than the corresponding measures based on worst-case situation [15].

Let $G = (V(G), E(G))$ be a simple undirected graph of order n. For any vertex $v \in V(G)$, the open neighborhood of v is $N_G(v) = \{u \in V(G)|uv \in E(G)\}$ and closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex v in G, denoted by $d_G(v)$, is the size of its open neighborhood [13]. The minimum degree of the graph G is denoted by $\delta(G)$. A vertex v with degree one is called a leaf vertex (or a pendant vertex) and a vertex adjacent to a leaf vertex is called a support vertex. The graph G is called r-regular graph if $d_G(v) = r$ for every vertex $v \in V(G)$. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The diameter of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. A set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets of G is called the domination number of G and it is denoted by $\gamma(G)$ [13]. Another domination concept is 2-domination number. A 2-dominating set of a graph G is a set $D \subseteq V(G)$ of vertices of G such that every vertex of $V(G) \setminus D$ has at least two neighbors in D. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of the graph G [11, 18].

In 2004, Henning introduced the concept of average domination and average independence [16]. Finding maximum dominating sets and maximum independent sets in graphs are the problems closely related to the concept of average domination and average independence. Also, the average lower domination and the average lower independence number are the theoretical vulnerability parameters for a network which have been modeled by a graph [3, 6]. The average lower domination number of a graph G, denoted by $\gamma_{av}(G)$, is defined as: $\gamma_{av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_v(G)$, where the lower domination number, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [10, 16].

The aim of this paper is to define a new vulnerability parameter, so called average lower 2-domination number. In the Section 2, well-known basic results have been given for the average lower domination number and the 2-domination number, respectively. In the Section 3, the average lower 2-domination number denoted by $\gamma_{2av}(G)$ has been defined. In the Section 4, the upper and lower bounds and exact solutions of the average lower 2-domination numbers for any graph G have been determined. Finally, the average lower 2-domination number of some graphs have been computed in the Section 5.

2. Basic Results

In this section, well known basic results with regard to the average lower domination number and the 2-domination number are given.

Theorem 2.1. [16] *If G is a graph of order n with the domination number $\gamma(G)$, then $\gamma_{av}(G) \leq \gamma(G) + 1 - \frac{\gamma(G)}{n}$, with equality if and only if G has a unique $\gamma(G)$-set.*

Theorem 2.2. [16] *If $K_{1,n-1}$ is a star graph of order n, where $n \geq 3$,then*

$$\gamma_{av}(K_{1,n-1}) = 2 - \frac{1}{n}.$$

Theorem 2.3. [16] *If P_n is a path graph of order n, then*

$$\gamma_{av}(P_n) = \begin{cases} \frac{n+2}{3} - \frac{2}{3n} & ,\text{if } n \equiv 2 \ (mod3); \\ \frac{n+2}{3} & ,\text{otherwise.} \end{cases}$$

Theorem 2.4. [16] *If C_n is a cycle graph of order n, then $\gamma_{av}(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 2.5. [16] *If K_n is a complete graph of order n, then $\gamma_{av}(K_n) = 1$.*

Observation 2.1. *If W_n is a wheel graph of order $n+1$, then $\gamma_{av}(W_n) = \frac{2n+1}{n+1}$.*

Theorem 2.6. [18] *Every leaf vertex of a graph G is in every $\gamma_2(G)$-set.*

Theorem 2.7. [18] *If K_n is a complete graph of order n, then $\gamma_2(K_n) = min\{2, n\}$.*

Theorem 2.8. [18] *If P_n is a path graph of order n, then $\gamma_2(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Theorem 2.9. [18] *If C_n is a cycle graph of order n, then $\gamma_2(C_n) = \lfloor \frac{n+1}{2} \rfloor$.*

Observation 2.2. *If W_n is a wheel graph of order $n+1$, where $n \geq 3$, then*

$$\gamma_2(W_n) = \begin{cases} 2 & ,\text{if } n=3,4; \\ 1 + \lceil \frac{n}{3} \rceil & ,\text{if } n \geq 5. \end{cases}$$

Theorem 2.10. [13] *If T is a tree of order n, then $\gamma_2(T) \geq \frac{n+1}{2}$.*

3. The Average Lower 2-domination Number

In this section, a new parameter of a graph has been introduced. It is called the average lower 2-domination number, defined by $\gamma_{2av}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} \gamma_{2v}(G)$, where $\gamma_{2v}(G)$ is the cardinality of the minimum 2-domination set in G that contains v, called lower 2-domination number of G relative to v.

When a graph is considered as modeling a network, the average lower 2-domination number may be more sensitive to the vulnerability of a graph than the other known vulnerability measures. For example, let G and H, presented in Figure 1, be two graphs, where $|V(G)| = |V(H)| = 10$ and $|E(G)| = |E(H)| = 17$. The graphs G and H have not only equal connectivity but also equal domination number, average lower domination number and 2-domination number, respectively, $k(G) = k(H) = 1$, $\gamma(G) = \gamma(H) = 1$, $\gamma_{av}(G) = \gamma_{av}(H) = 1.9$ and $\gamma_2(G) = \gamma_2(H) = 5$. These values can be checked by the readers. So, how can be distinguished between the graphs G and H?

To compute the $\gamma_{2av}(H)$ for the graph H, the 2-dominating sets that include every vertex $v_i \in V(H)$ must be found. Sets $\{v_1, v_3, v_5, v_7, v_9\}$, $\{v_2, v_1, v_4, v_6, v_7, v_9\}$, $\{v_3, v_1, v_5, v_7, v_9\}$, $\{v_4, v_1, v_6, v_8, v_{10}\}$, $\{v_5, v_1, v_3, v_7, v_9\}$, $\{v_6, v_1, v_4, v_8, v_{10}\}$, $\{v_7, v_1, v_3, v_5, v_9\}$, $\{v_8, v_1, v_4, v_6, v_{10}\}$, $\{v_9, v_1, v_3, v_5, v_7\}$ and $\{v_{10}, v_1, v_4, v_6, v_8\}$ are minimum 2- dominating sets that include every vertex $v_i \in V(H)$. Thus, the following results have been obtained:

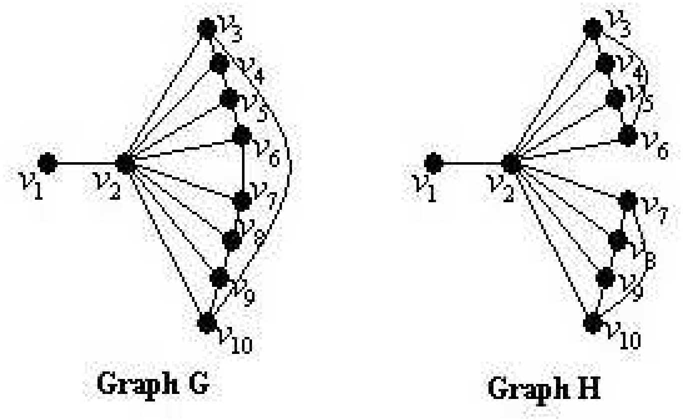

FIGURE 1. Graphs G and H

$\gamma_{2v_1}(H) = 5$, $\gamma_{2v_2}(H) = 6$, $\gamma_{2v_3}(H) = 5$, $\gamma_{2v_4}(H) = 5$, $\gamma_{2v_5}(H) = 5$, $\gamma_{2v_6}(H) = 5$, $\gamma_{2v_7}(H) = 5$, $\gamma_{2v_8}(H) = 5$, $\gamma_{2v_9}(H) = 5$ and $\gamma_{2v_{10}}(H) = 5$. As a result, $\gamma_{2av}(H) = 5.1$ is obtained. The reader can check that $\gamma_{2av}(G) = 5$.

Thus, the average lower 2-domination number may be a better parameter than the connectivity, the domination number, the average lower domination number and the 2-domination number to distinguish these two graphs G and H. Furthermore, it is not difficult to see that the graph H is more vulnerable than the graph G. Because, if an attack occurs on the links that are relevant to the vertex v_2, then two disconnected components would be remain in the graph G, while three disconnected components would be remain in the graph H. Due to $\gamma_{2av}(G) < \gamma_{2av}(H)$, it can be said that the graph H is more vulnerable than the graph G. In other words the graph G is tougher than the graph H.

4. THE UPPER AND LOWER BOUNDS AND EXACT FORMULAS

Theorem 4.1. *Let G be any connected graph of order n. If the 2-dominating set is unique, then*

$$\gamma_{2av}(G) = \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}.$$

Proof. Let D^* be unique 2-dominating set and let $v_1^*, v_2^*, ..., v_{|D^*|}^*$ be vertices of D^*. Then $\gamma_{2v_i^*}(G) = \gamma_2(G)$, where $i \in \{1, ..., |D^*|\}$ is obtained. It is clear that the lower 2-domination number is $\gamma_2(G) + 1$ for every vertex of $V(G) \setminus D^*$. Thus,

$$\gamma_{2av}(G) = \frac{1}{|V(G)|}(\textstyle\sum_{v_i^* \in D^*} \gamma_{2v_i^*}(G) + \sum_{v_i \in V(G)\setminus D^*} \gamma_{2v_i}(G))$$

$$= \frac{1}{|V(G)|}(|D^*|\gamma_2(G) + (\gamma_2(G) + 1)(|V(G)| - |D^*|))$$

$$= \gamma_2(G) + 1 - \frac{|D^*|}{|V(G)|}$$

is found.
Clearly, $|D^*| = \gamma_2(G)$ and $|V(G)| = n$. Then,

$$\gamma_{2av}(G) = \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}$$

is obtained. □

Theorem 4.2. *If G is a connected graph of order n, then*

$$\gamma_2(G) \leq \gamma_{2av}(G) \leq \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}.$$

Proof. Let D be a set including minimum 2-dominating sets. If the union of the minimum 2-dominating sets is equal to $V(G)$, then the lower 2-domination number is $\gamma_2(G)$ for every vertex of $V(G)$. Thus, the lower bound is $\gamma_{2av}(G) = \gamma_2(G)$. Furthermore, $\gamma_{2av}(G) = \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}$ which is obtained by the Theorem 4.1 is also upper bound. As a result, $\gamma_2(G) \leq \gamma_{2av}(G) \leq \gamma_2(G) + 1 - \frac{\gamma_2(G)}{n}$ is obtained. Hence the proof is completed. □

Theorem 4.3. *Let G be any connected graph of order n, where $n \geq 2$. Then,*

$$2 \leq \gamma_{2av}(G) \leq n - 1 + \frac{1}{n}.$$

Proof. The complete graph K_2 is the smallest connected graph providing that $n \geq 2$. Due to $\gamma_{2av}(K_2) = 2$, a fair lower bound is obtained. On the other hand, the maximum 2-domination number is $n-1$ for any connected graph G of order n. This value is obtained for the star graph $K_{1,n-1}$. Since the $\gamma_2(K_{1,n-1})$-set is unique for the graph $K_{1,n-1}$, so $\gamma_{2av}(K_{1,n-1}) = n - \frac{n-1}{n} = n - 1 + \frac{1}{n}$ is obtained by the Theorem 4.1. As a result, $2 \leq \gamma_{2av}(G) \leq n - 1 + \frac{1}{n}$ is obtained. □

Theorem 4.4. *If G is a connected graph of order n and e is an edge on the complement of G, then $\gamma_{2av}(G) \geq \gamma_{2av}(G+e)$.*

Proof. Consider the graph $G+e$, obtained adding to $E(G)$ any edge e from the complement of G. It is easy to see that $\gamma_{2v}(G) \geq \gamma_{2v}(G+e)$ for every vertex $v \in V(G)$ and $v \in V(G+e)$. It is clear that $(\frac{1}{n})\sum_{v \epsilon V(G)} \gamma_{2v}(G) \geq (\frac{1}{n})\sum_{v \epsilon V(G+e)} \gamma_{2v}(G+e)$ is known. As a result, $\gamma_{2av}(G) \geq \gamma_{2av}(G+e)$ is obtained. □

Definition 4.1. [12] *Consider two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$. The join graph $G = G_1 + G_2$ is the graph $G = (V(G), E(G))$, such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.*

Theorem 4.5. *Let G and H be two connected graphs of order n and m, respectively.*

(a) *If $\gamma(G) = \gamma(H) = 1$, then $2 \leq \gamma_{2av}(G+H) \leq 3 - \frac{2}{m+n}$.*

(b) *For any graphs G and H, $2 \leq \gamma_{2av}(G+H) \leq 4$.*

Proof. For (a): Due to $\gamma(G) = 1$ and $\gamma(H) = 1$, then $\gamma_2(G+H) = 2$ is obtained. Moreover, $\gamma_2(G+H)$- set can be unique. Thus, $\gamma_{2av}(G+H) \leq 3 - \frac{2}{m+n}$ is known by the Theorem 4.2. Let graphs G and H be complete graphs of order n and m, respectively. So, $\gamma_{2av}(G+H) = 2$ is obtained. As a result, $2 \leq \gamma_{2av}(G+H) \leq 3 - \frac{2}{m+n}$.

For (b): If $\gamma_2(G) > 2$, $\gamma_2(H) > 2$, $\gamma(G) > 2$ and $\gamma(H) > 2$, then the cardinality of every $\gamma_{2v}(G+H)$-set is 4. So, $\gamma_{2av}(G+H) = 4$ is obtained. It is known that $2 \leq \gamma_{2av}(G+H)$ from the (a). As a result, $2 \leq \gamma_{2av}(G+H) \leq 4$ is obtained. □

Theorem 4.6. *Let G and H be two connected graphs of order n and m, respectively. If $n \geq 2$ and $m \geq 2$, then $\gamma_{2av}(G) + \gamma_{2av}(H) \geq \gamma_{2av}(G+H)$.*

Proof. Clearly, $\gamma_{2av}(G) \geq 2$ and $\gamma_{2av}(H) \geq 2$ are known by the Theorem 4.3. Furthermore, clearly $\gamma_{2av}(G+H) \leq 4$ is known by the Theorem 4.5. As a result, $\gamma_{2av}(G) + \gamma_{2av}(H) \geq \gamma_{2av}(G+H)$ is obtained. □

Theorem 4.7. *Let T be any connected tree of order n. If the number of support vertices is s and the number of leaf vertices is $n-s$ in T, then $\gamma_{2av}(T) \geq n - s + \frac{s}{n}$.*

Proof. Consider a tree T of order n. Furthermore, a set called by $S_T(p)$ is expressed by $S_T(p) = \{v \in V(T) | pv \in E(T) \, and \, d_T(v) = 1\}$, where p is a support vertex. Clearly, if $|S_T(p)| \geq 2$ has been got for every support vertex p, then $\gamma_2(T)$-set will be an unique set. Since the number of support vertices is s, then $\gamma_2(T) = n - s$ is obtained. So $\gamma_{2av}(T) = n - s + \frac{s}{n}$, which this value is a lower bound, is found by the Theorem 4.1. If $|S_T(p)| < 2$ for any support vertex, then the vertex p is not 2-dominated by the vertices of degree 1 that are neighbors of the support vertex p. Thus, the cardinality of $\gamma_{2p}(T)$-set is increased. So, $\gamma_{2av}(T) > n - s + \frac{s}{n}$ is found. As a result, $\gamma_{2av}(T) \geq n - s + \frac{s}{n}$ is obtained. □

Theorem 4.8. *Let T be any connected tree of order n, then $\gamma_{2av}(T) \geq \frac{n+1}{2}$.*

Proof. The proof follows directly from the Theorem 2.10. □

5. The Average Lower 2-domination Number of Some Graphs

In this section, the average lower 2-domination numbers of the well known graph classes such as P_n, C_n, K_n and $K_{1,n-1}$ have been calculated.

Theorem 5.1. *If P_n is a path graph of order n, then*

$$\gamma_{2av}(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 2 - \frac{\lfloor \frac{n}{2} \rfloor + 1}{n} & \text{,if } n \text{ is odd;} \\ \frac{n}{2} + 1 & \text{,if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, $\gamma_2(P_n)$-set is unique in the graph P_n. Clearly, $\gamma_2(P_n) = \lfloor \frac{n}{2} \rfloor + 2 - \frac{\lfloor \frac{n}{2} \rfloor + 1}{n}$ is known by the Theorems 2.8 and 4.1, respectively. If n is even, then $\gamma_2(P_n) = \frac{n}{2} + 1$ can be written. It is clear that the lower 2-domination number is $\frac{n}{2} + 1$ for the every vertex $v \in V(P_n)$. Thus, $\gamma_{2av}(P_n) = \frac{n}{2} + 1$ is obtained.

□

Theorem 5.2. *If C_n is a cycle graph of order n, then $\gamma_{2av}(C_n) = \lfloor \frac{n+1}{2} \rfloor$.*

Proof. Clearly, $\gamma_2(C_n) = \lfloor \frac{n+1}{2} \rfloor$ is known by the Theorem 2.10. Since the graph C_n is regular graph, the cardinality of the $\gamma_{2v}(C_n)$-set is $\lfloor \frac{n+1}{2} \rfloor$ for every vertex $v \in V(C_n)$. Thus, $\gamma_{2av}(C_n) = \lfloor \frac{n+1}{2} \rfloor$ is obtained. □

Corollary 5.1. *If K_n is a complete graph of order $n \geq 2$, then $\gamma_{2av}(K_n) = 2$.*

Proof. Clearly, to obtain the 2-domination number of K_n any two vertices must be taken to $\gamma_2(K_n) - set$ (see Theorem 2.7.). Thus, the definition of the average lower 2-domination number, $\gamma_{2av}(K_n) = 2$ is obtained. □

Corollary 5.2. *If $K_{1,n-1}$ is a star graph of order n, then $\gamma_{2av}(K_{1,n-1}) = n - 1 + \frac{1}{n}$.*

Proof. Clearly, $\gamma_2(K_{1,n-1}) = n - 1$ is known by the Theorem 2.6, Moreover, the 2-dominating set is unique. Thus, by the Theorem 4.1

$$\gamma_{2av}(K_{1,n-1}) = \gamma_2(K_{1,n-1}) + 1 - \frac{\gamma_2(K_{1,n-1})}{n} = n - 1 + \frac{1}{n}$$

is obtained. □

6. Conclusion

In this study, a new graph theoretical parameter namely the average lower 2-domination number has been presented for the network vulnerability. The present parameter has been constructed by summing of the lower 2-domination number of every vertex of a graph divided by the number of vertices of the graph. Additionally, the stability of popular interconnection networks has been studied and their average lower 2-domination numbers have been computed. These networks have been modeled with the complete graphs, the path graphs, the cycle graphs, the star graphs. Furthermore, upper and lower bounds and exact formulas have been obtained for the average lower 2-domination number of any graph G, the join graph $G_1 + G_2$ and trees. As a further study, exact formulas or bounds may be obtained for graph operations and large network topologies.

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