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AN ALGORITHMIC APPROACH TO EQUITABLE EDGE CHROMATIC NUMBER OF GRAPHS

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Abstract. The equitable edge chromatic number is the minimum number of colors required to color the edges of graph G, for which G has a proper edge coloring and if the number of edges in any two color classes differ by at most one. In this paper, we obtain the equitable edge chromatic number of S_n, W_n, H_n and G_n.

Keywords: Equitable edge coloring, Wheel, Helm, Gear, Sunlet.

AMS Subject Classification: 05C15, 05C76

1. Introduction

Graphs considered in this paper are finite undirected graphs without loops. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the maximum degree of G by $\Delta(G)$. An edge coloring of G is an assignment of colors to the edges of G, such that no two adjacent edges receives the same color. Given an edge-coloring of G with k colors $1, 2, \ldots, k$ for all $v \in V(G)$, let $c_i(v)$ denote the number of edges incident with v colored i. The chromatic number of a graph G, denoted $\chi'(G)$, is the minimum number of different colors required for a proper edge coloring of G. The graph G is k-edge-chromatic if $\chi'(G) = k$.

The first paper on edge coloring was written by Tait in 1880 and he proved that, if the four color conjecture is true then the edges of every 3-connected planar graph can be properly colored using only three colors. Several years later, in 1891 Petersen pointed out that there are 3-connected cubic graphs which are not 3-colorable. Since all edges incident to the same vertex must be assigned different colors, obviously $\chi'(G) \geq \Delta(G)$. In 1916, Kőnig has proved that every bipartite graph can be edge colored with exactly $\Delta(G)$ colors, that is $\chi'(G) = \Delta(G)$. In 1949 Shannon proved that every graph can be edge colored with at most $\frac{3\Delta G}{2}$ colors, that is $\chi'(G) \leq \frac{3\Delta G}{2}$. In 1964, Vizing [7] proved that $\chi'(G) \leq \Delta(G) + 1$. In 1973, Meyer[5] presented the concept of equitable coloring and equitable chromatic number. The notion of equitable edge coloring was defined by Hilton and de Werra[3] in 1994.

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In our day to day life many problems on optimization, network designing, scheduling problems, timetabling and so on are related to edge coloring. In general, such problems are NP-complete and it is NP-Hard to decide the bound for these graphs is Δ or $\Delta + 1$. For example, consider the timetabling problem, the minimum number of rooms needed at any one time can be scheduled by equitable edge coloring. In this paper, we determine the equitable edge chromatic number for S_n, W_n, H_n and G_n.

2. Preliminaries

Definition 2.1. [4] *For any integer $n \geq 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_0 to each of the $n-1$ vertices $\{v_1, v_2, \ldots, v_n\}$ of the cycle graph C_{n-1}.*

Definition 2.2. *The Helm graph H_n is the graph obtained from a Wheel graph W_n by adjoining a pendant edge to each vertex of the $n-1$ cycle in W_n.*

Definition 2.3. *The Gear graph G_n is the graph obtained from a Wheel graph W_n by adding a vertex to each edge of the $n-1$ cycle in W_n.*

Definition 2.4. *The n- sunlet graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.*

Definition 2.5. [6] *For k-proper edge coloring f of graph G, if $||E_i| - |E_j|| \leq 1$, $i, j = 0, 1, 2, \ldots, k-1$, where $E_i(G)$ is the set of edges of color i in G, then f is called a k-equitable edge coloring of graph G, and*

$$\chi'_{=}(G) = \min\{k : \text{there exists a } k\text{-equitable edge coloring of graph } G\}$$

is called the equitable edge chromatic number of graph G.

Lemma 2.1. *For any complete graph K_p with order p,*

$$\chi'_{=}(K_p) = \begin{cases} p, & p \equiv 1 \,(\mathrm{mod}\ 2)\,, \\ p-1, & p \equiv 0 \,(\mathrm{mod}\ 2)\,, \end{cases}$$

Lemma 2.2. [1] *For any simple graph $G(V, E)$, $\chi'_{=}(G) \geq \Delta(G)$.*

Lemma 2.3. [1] *For any simple graph G and H,$\chi'_{=}(G) = \chi'(G)$ and if $H \subseteq G$ then $\chi'(H) \leq \chi'(G)$, where $\chi'(G)$ is the proper edge chromatic number of G.*

Theorem 2.1. [7] *Let G be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Lemma 2.4. [8] *Let G be a graph and let $k \geq 2$. If $k \nmid d(v)$ for each $v \in V(G)$, then G has an equitable edge-coloring with k colors.*

Lemma 2.5. [8] *Let G be a graph and let $k \geq 2$. If the k-core of G is a set of isolated vertices, then G has an equitable edge-coloring with k colors.*

For additional graph theory terminologies not defined in this paper can be found in [1, 2]. In the following section, the equitable edge chromatic number of S_n, W_n, H_n and G_n are determined.

3. Main Results

Theorem 3.1. *The equitable edge chromatic number of the Sunlet graph is $\chi'_{=}(S_n) = 3$, for $n \geq 3$.*

Proof. Let S_n be the sunlet graph on $2n$ vertices and $2n$ edges.

$$\text{Let } V(S_n) = \{v_1, v_2, v_3, \ldots, v_n\} \bigcup \{u_1, u_2, u_3, \ldots, u_n\}$$

$$\text{and } E(S_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e_n\} \bigcup \{e'_i : 1 \leq i \leq n\}$$

where e_i is the edge $v_i v_{i+1}\,(1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i$ $(1 \leq i \leq n)$.

We define an edge coloring f, such that $f : S \to C$, where $S = E(S_n)$ and $C = \{1, 2, 3\}$. In this edge coloration, $C(e_i)$ means the color of the i^{th} rim edge e_i. While coloring, when the value mod 3 is equal to 0 it should be replaced by 3. The order of coloring is done by coloring the edges corresponding to the cycle first and later the pendant edges.

Case 1: $n \equiv 0 (\text{mod } 3)$ and $n \equiv 2 (\text{mod } 3)$

$$f(e_i) = \begin{cases} 1, & \text{if } i \equiv 1 (\text{mod } 3) \\ 2, & \text{if } i \equiv 2 (\text{mod } 3) \\ 3, & \text{if } i \equiv 0 (\text{mod } 3) \end{cases} \quad \text{for } 1 \leq i \leq n$$

$$f(e'_i) = \begin{cases} 6 - \{C(e_i) + C(e_n)\}, & \text{for } i = 1 \\ 6 - \{C(e_{i-1}) + C(e_i)\}, & \text{for } 2 \leq i \leq n \end{cases}$$

Case 2: $n \equiv 1 (\text{mod } 3)$

$$f(e_i) = \begin{cases} 1, & \text{if } i \equiv 1 (\text{mod } 3) \\ 2, & \text{if } i \equiv 2 (\text{mod } 3) \\ 3, & \text{if } i \equiv 0 (\text{mod } 3) \end{cases} \quad \text{for } 1 \leq i \leq n-1$$

$$f(e_n) = 2$$

$$f(e'_i) = \begin{cases} 6 - \{C(e_i) + C(e_n)\}, & \text{for } i = 1 \\ 6 - \{C(e_{i-1}) + C(e_i)\}, & \text{for } 2 \leq i \leq n \end{cases}$$

We see that S_n is edge colorable with 3 colors . Let $E(S_n) = \{E_1, E_2, E_3\}$ such that the color classes of $E'_i s$ are independent sets with no edges in common. For example consider the case $n \equiv 0(\text{mod } 3)$ (See Figure 1), in which $|E_1| = |E_2| = |E_3| = 3$ and implies $||E_i| - |E_j|| \leq 1$ for $i \neq j$. Hence it is equitably edge colorable with 3 colors. Therefore $\chi'_{=}(S_n) \leq 3$. Since $\Delta = 3$ and by lemma 2.2, it follows that $\chi'_{=}(S_n) \geq \chi'(S_n) \geq \Delta$. This implies $\chi'_{=}(S_n) \geq 3$. Therefore $\chi'_{=}(S_n) = 3$. Similarly this is true for all other cases. Hence f is an equitable edge 3-coloring of S_n. □

Algorithm : Equitable edge coloring of Sunlet graph
Input: n, the number of vertices of S_n
Output: Equitably edge colored S_n

Initialize S_n with $2n$ vertices, the rim vertices by $v_1, v_2, v_3, \ldots, v_n$ and pendant vertices by $u_1, u_2, u_3, \ldots, u_n$.

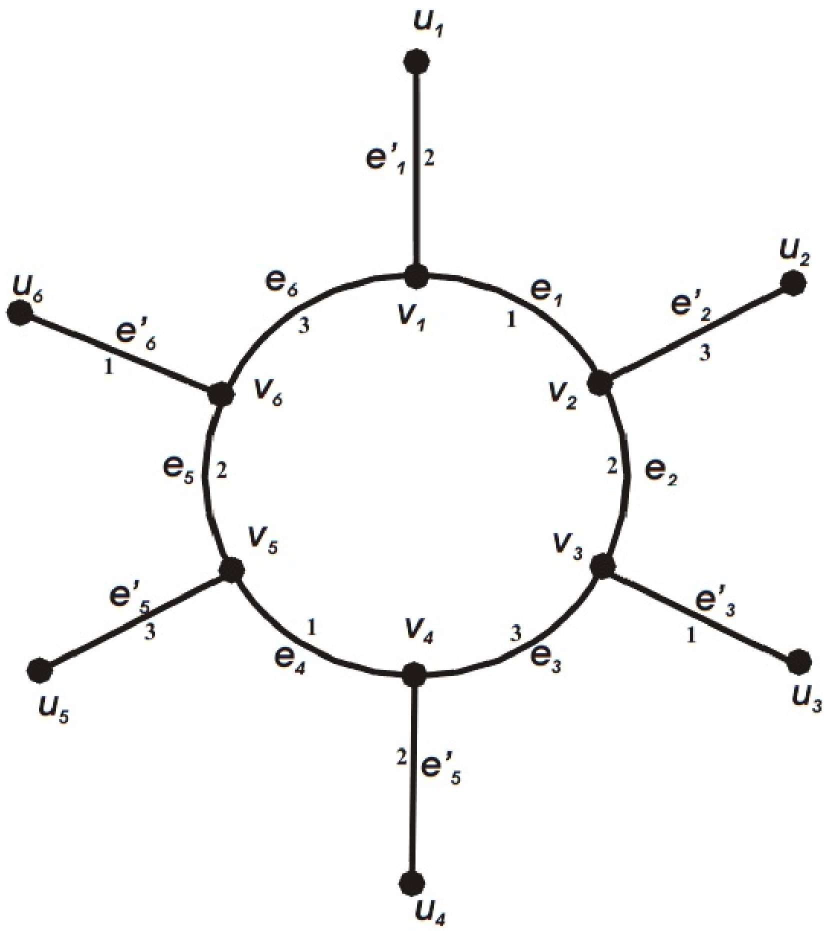

FIGURE 1. Sunlet S_6.

Initialize the adjacent edges on the rim by $e_1, e_2, e_3, \ldots, e_n$ and pendant edges by $e'_1, e'_2, e'_3, \ldots, e'_n$.

Let $f : E(G) \to \{1, 2, 3\}$ be the coloring of the edges in S_n.

Apply the coloring rules of Theorem 3.1 for each of the following cases

if ($n \equiv 0(\text{mod } 3)$ or $n \equiv 2(\text{mod } 3)$)
for $i = 1$ to n
{
$e_i = 1;\ i \equiv 1(\text{mod } 3);$
$e_i = 2;\ i \equiv 2(\text{mod } 3);$
$e_i = 3;\ i \equiv 0(\text{mod } 3);$
if ($i = 1$)
$e'_i = 6 - [C(e_i) + C(e_n)];$

else
$e'_i = 6 - [C(e_{i-1}) + C(e_n)];$
}
end for

if $(n \equiv 1 (\text{mod } 3))$
for $i = 1$ to n
{
if $(i < n)$
$e_i = 1;\ i \equiv 1(\text{mod } 3);$
$e_i = 2;\ i \equiv 2(\text{mod } 3);$
$e_i = 3;\ i \equiv 0(\text{mod } 3);$
if $(i = n)$
$e_n = 2;$
if $(i = 1)$
$e'_i = 6 - [C(e_i) + C(e_n)];$
else
$e'_i = 6 - [C(e_{i-1}) + C(e_n)];$
}
end for
return f ;

Theorem 3.2. *The equitable edge chromatic number of the Wheel graph is $\chi'_{=}(W_n) = n - 1$, for $n \geq 4$.*

Proof. The Wheel graph W_n consists of n vertices and $2(n-1)$ edges.

$$\text{Let } V(W_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n-1\} \text{ and}$$

$$E(W_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e'_i : 1 \leq i \leq n-1\}$$

where e_i is the edge $v_0v_i\,(1 \leq i \leq n-1)$ and e'_i is the edge $v_iv_{i+1}\,(1 \leq i \leq n-1)$.

Now define an edge coloring f, such that $f : S \to C$, where $S = E(W_n)$ and $C = \{1, 2, \ldots, n-1\}$. The equitable edge coloring is obtained by coloring the edges as follows:

$f(e_i) = i,$ for $1 \leq i \leq n-1$
$f(e'_1) = n-1$
$f(e'_i) = i-1,$ for $2 \leq i \leq n-1$

Clearly W_n is edge colorable with $n-1$ colors. Let $E(W_n) = \{E_1, E_2, \ldots, E_n\}$ such that the color classes of E_i's are independent sets with no edges in common. For example consider the case $n = 5$ (See Figure 2), which implies $|E_1| = |E_2| = |E_3| = |E_4| = 2$ and also satisfies $||E_i| - |E_j|| \leq 1$ for $i \neq j$. So it is equitably edge colorable with $n-1$ colors. Hence $\chi'_{=}(W_n) \leq n-1$. Since $\Delta = n-1$ and by lemma 2.2, it follows that $\chi'_{=}(W_n) \geq \chi'(W_n) \geq \Delta$. This implies $\chi'_{=}(W_n) \geq n-1$. Therefore $\chi'_{=}(W_n) = n-1$. Similarly this is true for all other values of $n \geq 4$. Hence $\chi'_{=}(W_n) = \Delta = n-1$. □

Algorithm : Equitable edge coloring of Wheel graph
Input: n, the number of vertices of W_n
Output: Equitably edge colored W_n

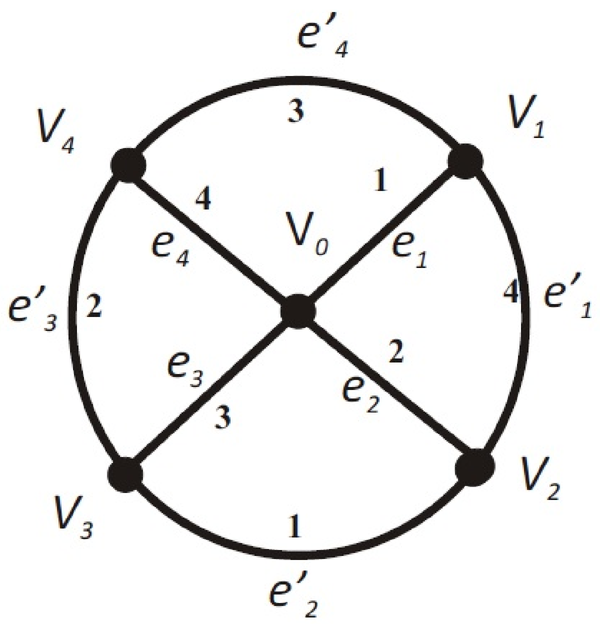

FIGURE 2. Wheel W_5.

Initialize W_n with n vertices, the center vertices by v_0 and rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$.

Initialize the adjacent edges on the center by $e_1, e_2, e_3, \ldots, e_{n-1}$ and adjacent edges on the rim by $e'_1, e'_2, e'_3, \ldots, e'_{n-1}$.

Let $f : E(G) \to \{1, 2, \ldots, n-1\}$ be the coloring of the edges in W_n.

Apply the coloring rules of Theorem 3.2

for $i = 1$ to $n-1$
{
$e_i = i$;
if $(i = 1)$
$e'_1 = n-1$;
else
$e'_i = i-1$;
}
end for
return f ;

Theorem 3.3. *The equitable edge chromatic number of the Helm graph is $\chi'_{=}(H_n) = n-1$, for $n \geq 4$.*

Proof. The Helm graph H_n consists of $2n-1$ vertices and $3(n-1)$ edges.

$$\text{Let } V(H_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n-1\} \bigcup \{u_i : 1 \leq i \leq n-1\}$$

and

$$E(H_n) = \{e_i : 1 \le i \le n-1\} \bigcup \{e'_i : 1 \le i \le n-2\} \bigcup \{e'_{n-1}\} \bigcup \{e''_i : 1 \le i \le n-1\}$$

where e_i is the edge $v_0 v_i \, (1 \le i \le n-1)$, e'_i is the edge $v_0 v_{i+1} \, (1 \le i \le n-2)$, e'_{n-1} is the edge $v_{n-1} v_1$ and e''_i is the edge $v_i u_i \, (1 \le i \le n-1)$.

Define a function $f : S \to C$ where $S = E(H_n)$ and $C = \{1, 2, \ldots, n-1\}$. The coloring pattern is as follows:

$f(e_i) = i$, for $1 \le i \le n-1$

$$f(e'_i) = \begin{cases} n-1, & \text{for } i = 1 \\ i-1, & \text{for } 2 \le i \le n-1 \end{cases}$$

$$f(e''_i) = \begin{cases} i+1, & \text{for } 1 \le i \le n-2 \\ 1, & \text{for } i = n-1 \end{cases}$$

With this pattern we can edge color H_n with $n-1$ colors. Let $E(H_n) = \{E_1, E_2, \ldots, E_n\}$ such that the color classes of E_i's are independent sets with no edges in common. For example consider the case $n = 7$ (See Figure 3), in which $|E_1| = |E_2| = |E_3| = |E_4| = |E_5| = |E_6| = 3$ and also satisfies $||E_i| - |E_j|| \le 1$ for $i \ne j$. So it is equitably edge colorable with $n-1$ colors. Hence $\chi'_=(H_n) \le n-1$. Since $\Delta = n-1$ and by lemma 2.2, $\chi'_=(H_n) \ge \chi'(H_n) \ge \Delta$. This implies $\chi'_=(H_n) \ge n-1$. Therefore $\chi'_=(H_n) = n-1$. Similarly this is true for all other values of $n \ge 4$. Hence $\chi'_=(H_n) = \Delta = n-1$. □

Algorithm : Equitable edge coloring of Helm graph
Input: n, the number of vertices of H_n
Output: Equitably edge colored H_n

Initialize H_n with $2n-1$ vertices, the center vertices by v_0, the rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$ and the pendant vertices by $u_1, u_2, u_3, \ldots, u_{n-1}$.

Initialize the $3(n-1)$edges, the adjacent edges on the center by $e_1, e_2, e_3, \ldots, e_{n-1}$, the adjacent edges on the rim by $e'_1, e'_2, e'_3, \ldots, e'_{n-1}$ and the pendant edges by $e''_1, e''_2, e''_3, \ldots, e''_{n-1}$.

Let $f : E(G) \to \{1, 2, \ldots, n-1\}$ be the edge coloring of H_n.

Apply the coloring rules of Theorem 3.3

for $i = 1$ to $n-1$
{
$e_i = i$;
if $(i = 1)$
$e'_1 = n-1$;
else
$e'_i = i-1$;
if $(i = n-1)$
$e''_{n-1} = 1$;
else
$e''_i = i+1$;
}
end for
return f ;

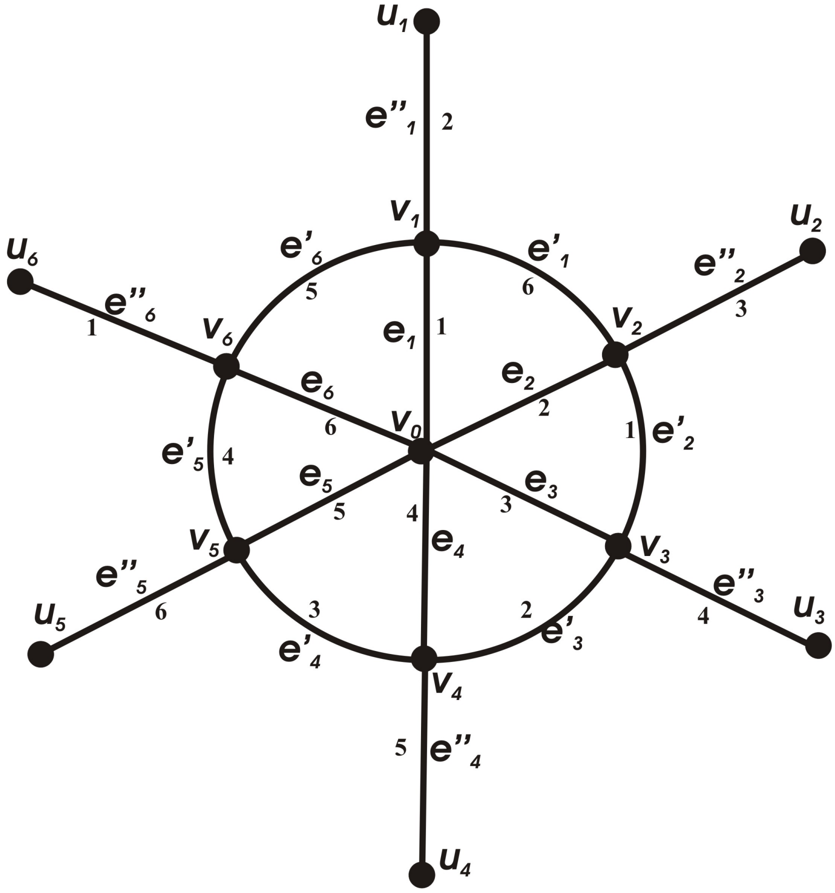

FIGURE 3. Helm H_7.

Theorem 3.4. *The equitable edge chromatic number of the Gear graph is* $\chi'_{=}(G_n) = n-1$, *for* $n \geq 4$.

Proof. The Gear graph G_n consists of $2n-1$ vertices and $3(n-1)$ edges.

$$\text{Let } V(G_n) = \{v_0\} \bigcup \{v_i : 1 \leq i \leq n-1\} \bigcup \{v'_i : 1 \leq i \leq n-1\} \text{ and}$$

$$E(G_n) = \{e_i : 1 \leq i \leq n-1\} \bigcup \{e'_i : 1 \leq i \leq n-1\} \bigcup \{e''_i : 1 \leq i \leq n-2\} \bigcup \{e''_{n-1}\}$$

where e_i is the edge $v_0v_i\,(1 \leq i \leq n-1)$, e'_i is the edge $v_iv'_i\,(1 \leq i \leq n-1)$, e''_i is the edge $v'_iv_{i+1}\,(1 \leq i \leq n-2)$ and e'_{n-1} is the edge $v'_{n-1}v_1$.

Define a function $f : S \to C$, where $S = E(G_n)$ and $C = \{1, 2, \ldots, n-1\}$. The coloring pattern is as follows:

$f(e_i) = i,$ for $1 \leq i \leq n-1$
$f(e'_i) = i+1,$ for $1 \leq i \leq n-1$
$f(e''_i) = i,\ 1 \leq i \leq n-1$

The graph G_n is edge colored with $n-1$ colors by sustituting different values for n, it is inferred that no adjacent edges receives the same color. Let $E(G_n) = \{E_1, E_2, \ldots, E_n\}$ such that the color classes of E_i 's are independent sets and they have no edges in common.

For example consider the case $n = 7$ (See Figure 4), which has $|E_1| = |E_2| = |E_3| = |E_4| = |E_5| = |E_6| = 3$ and satisfies the condition $||E_i| - |E_j|| \leq 1$ for $i \neq j$. Hence it is equitably edge colorable with $n-1$ colors. Therefore $\chi'_{=}(G_n) \leq n-1$. Since $\Delta = n-1$ and by lemma 2.2, $\chi'_{=}(G_n) \geq \chi'(G_n) \geq \Delta$. This implies $\chi'_{=}(G_n) \geq n-1$. Therefore $\chi'_{=}(G_n) = n-1$. $\square$

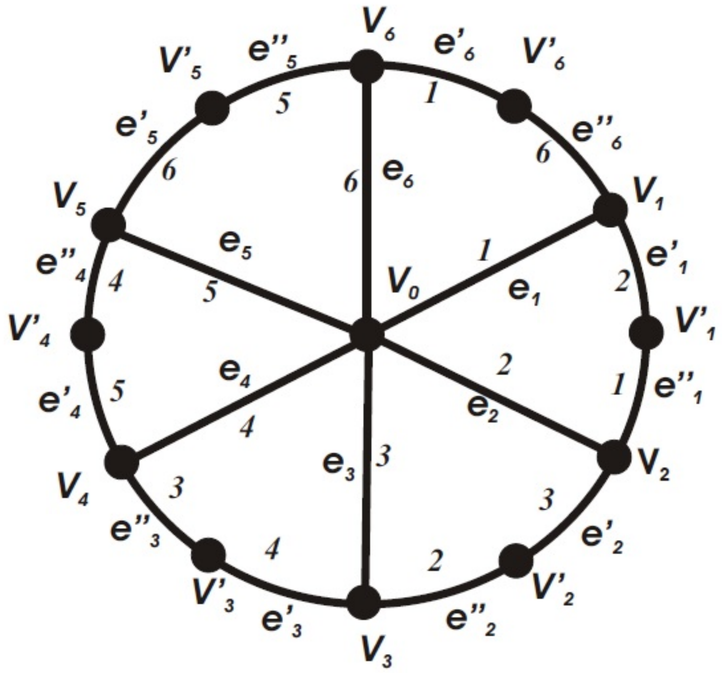

FIGURE 4. Gear G_7.

Algorithm : Equitable edge coloring of Gear graph
Input: n, the number of vertices of G_n
Output: Equitably edge colored G_n

Initialize G_n with $2n-1$ vertices, the center vertices by v_0, the rim vertices by $v_1, v_2, v_3, \ldots, v_{n-1}$ and $v'_1, v'_2, v'_3, \ldots, v'_{n-1}$.

Initialize the $3(n-1)$edges, the adjacent edges on the center by $e_1, e_2, e_3 \ldots, e_{n-1}$, the adjacent edges on the rim by $e'_1, e'_2, e'_3, \ldots, e'_{n-1}$ and $e''_1, e''_2, e''_3, \ldots, e''_{n-1}$.

Let $f : E(G) \to \{1, 2, \ldots, n-1\}$ be the edge coloring of G_n.

Apply the coloring rules of Theorem 3.4

for $i = 1$ to $n - 1$
{
$e_i = i$;
$e'_i = i + 1$;
$e''_i = i$;
}

end for
return f ;

4. Conclusion

In this paper, the equitable edge chromatic number of Sunlet S_n, Wheel W_n, Helm H_n and Gear graph G_n are obtained. The proofs establish an optimal solution to the equitable edge coloration of these graph families and are supported by algorithms. The field of equitable edge coloring of graphs is wide open. It would be further interesting to determine the bounds of equitably edge coloring of other families of graphs.

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